The Weyl curvature conjecture and black hole entropy

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The universe today, with structure such as stars, galaxies and black holes, seems to have evolved from a very homogeneous initial state. From this it appears as if the entropy of the universe is decreasing in violation of the second law of thermodynamics. It has been suggested by Roger Penrose \footnote{Electronic address: oysteir@lin.k.uio.no} that this inconsistency can be solved if one assigns an entropy to the spacetime geometry. He also pointed out that the Weyl tensor has the properties one would expect to find in a description of a gravitational entropy. In this article we make an attempt to use this so-called Weyl curvature conjecture to describe the Hawking-Bekenstein entropy of black holes and the entropy of horizons due to a cosmological constant.

I. GRAVITATIONAL ENTROPY

The second law of thermodynamics is well known as one of the most fundamental laws of natural sciences. It captures most of our understanding of how macroscopic systems evolve over time. It can be stated as the entropy of a closed system never decreases, or in a more mathematical form:

\[ \frac{dS}{dt} \geq 0. \] (1)

This law introduces the concept of entropy, \( S \). When it was introduced in the mid-nineteenth century it was initially considered as an abstract thermodynamical quantity, but Ludwig Boltzmann later provided it with a physical interpretation. The entropy of a macroscopic state can be expressed from the multiplicity as

\[ S = k_B \ln W \] (2)

where \( k_B = 1.381 \times 10^{-23} \text{J/K} \) is Boltzmann’s constant and the multiplicity, \( W \), is the number of different configurations of microstates that results in the given macrostate. Together with \( S \) this means that a closed macroscopic system will evolve towards the physically allowed state that can be represented by the largest possible number of microstates. This is a state of maximum entropy.

For systems of everyday scales, this works just fine. A common example is that of an ideal gas trapped in a bottle. Once the bottle is opened, the gas will spread out and very soon fill all the available space. A state of maximum entropy has been reached.

A gas of cosmological proportions, on the other hand, does not fit very well to this picture. Today most cosmologists agree that the universe started out in a very homogeneous state. Later gravitation made small density perturbations grow and ultimately formed structures such as stars, galaxies and black holes. This evolution appears to be the opposite of what happened in the previous example. Instead of spreading out, the gas collects itself into clumps of matter.

It should be noted, however, that linear perturbation theory shows that to first order in density- and corresponding temperature perturbations the thermodynamic entropy increases due to a transition of gravitational energy to thermal energy \footnote{Electronic address: oyvind.gron@uio.no}, but this has not been shown for non-linear evolution from a nearly homogeneous initial state to a final inhomogeneous state with stars and galaxies. A gravity dominated evolution may represent a violation of the second law of thermodynamics as long as only thermodynamic entropy is taken into account.

It was suggested by Roger Penrose \footnote{Electronic address: oyvind.gron@uio.no} that this problem could be solved by assigning entropy to the gravitational field itself. This means that a macroscopic state of spacetime geometry can be represented by many gravitational microstates. Without a working theory for quantum gravitation, it is difficult to find a measure for this entropy. We will have to base our investigations on the behaviour of macroscopic systems.

Penrose \footnote{Electronic address: oyvind.gron@uio.no} also suggested that the Weyl curvature tensor could be used to quantify the gravitational entropy. The Weyl tensor is a rank four tensor that contains the independent components of the Riemann tensor not captured by the Ricci tensor. It can be considered as the traceless part of the Riemann tensor, and in four dimensions it can be expressed as

\[ C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - g_{\alpha\gamma} R_{\beta\delta} + g_{\beta\gamma} R_{\alpha\delta} + \frac{1}{3} R g_{\alpha\gamma} g_{\beta\delta}. \]

where the bracketed indices should be understood as antisymmetric combinations \( (A_{\mu\nu}) = \frac{1}{2} (A_{\mu\nu} - A_{\nu\mu}) \).

The Weyl tensor is zero in the Friedmann-Robertson-Walker models, which describe the early universe quite well on small scales as well as on large scales since the universe was nearly homogeneous on all macroscopic scales.
before the first stars formed. It is also large in the Schwarzschild spacetime, which describes the spacetime geometry around a spherically symmetric mass distribution such as a black hole and other phenomena of the late universe. This is the kind of behaviour we would expect from a description of gravitational entropy, since it would be increasing throughout the history of the universe, in agreement with the second law.

Penrose \cite{Penrose1970} conjectured that the evolution of the universe started from an initial singularity in which the Weyl tensor was much less than the Ricci part of the spacetime curvature. O. Gron and S. Hervik \cite{Gron2000} have later used a mini-superspace model to perform a quantum mechanical calculation of the probability that this so-called Weyl curvature conjecture is true. The result supported the conjecture. Also, J. D. Barrow and S. Hervik \cite{Barrow1987} have studied the evolution of the Weyl curvature invariant in isotropic, homogeneous universe models, and shown that the Weyl curvature invariant dominates the Ricci invariant at late times.

The Weyl curvature tensor is also locally independent of the energy-momentum tensor, which makes the hypothesis more convincing. If it were coupled to the matter and energy content of spacetime (like the Ricci curvature), we would have to account for a degeneracy between gravitational microstates and the common, "thermodynamical" microstates of the matter.

There is of course a question whether the Weyl curvature tensor can be considered as a direct measure of gravitational entropy or if its entropy-like behaviour is only an effect of another process more closely related to the entropy. Nevertheless, it seems to be a natural place to start our investigations if we seek to understand gravitational entropy.

First, in section \ref{sec:II} we will briefly review black holes and the entropy associated with them. In section \ref{sec:III} we investigate possibilities for defining an expression involving the Weyl curvature scalar which can be used in the construction of a quantity representing gravitational entropy. A guideline for this construction is that it shall provide a geometrical interpretation of the Hawking-Bekenstein entropy. In section \ref{sec:IV} we add a cosmological constant to the system and apply the model for gravitational entropy to the Schwarzschild-de Sitter spacetime. Finally, in section \ref{sec:V} we remove the black hole and take a look at the de Sitter spacetime.

\section{Entropy of Black Holes}

The no hair theorem \cite{Israel1976} states that a black hole is completely described by three characteristics: mass, charge and angular momentum. Classically, all other information is lost when something falls beyond the event horizon. Therefore, if one drops a package of entropy into a black hole, the entropy seems to disappear from the universe \cite{Gibbons1977}. Again, the second law appears to be violated.

However, Hawking and Christodoulou \cite{Hawking1975, Christodoulou1976} showed that the horizon area of a black hole (or indeed the total area of several black holes) never decreases, and is only constant for a special class of transformations. The parallel to the second law seems obvious, but it was Bekenstein \cite{Bekenstein1973, Bekenstein1974, Bekenstein1975} who made the bold proposition that the horizon area actually was a measure of the entropy of the black hole. This entropy, called the Hawking-Bekenstein entropy is expressed as

\begin{equation}
S_{HB} = \frac{k_B}{4A} = \frac{k_B c^3}{4G h} \quad (3)
\end{equation}

where $A$ is the area of the black hole horizon, $A = \frac{c^3}{4G h}$ is the Planck area, $G = 6.673 \cdot 10^{-11} \text{Nm}^2/\text{kg}^2$ is the constant of gravitation and $h = \frac{1}{2\pi} = 1.055 \cdot 10^{-34} \text{Js}$ where $h$ is Planck's constant.

The total entropy is then the sum of the Hawking-Bekenstein entropy and the common thermodynamical entropy:

\begin{equation}
S_{\text{tot}} = S_{\text{td}} + S_{\text{HB}}
\end{equation}

and this entropy never decreases, in agreement with the second law.

\section{Black Holes and Weyl Entropy}

We will now assume that the Hawking-Bekenstein entropy is of geometrical origin, and is therefore a special case of gravitational entropy. The total entropy of a system will then be the sum of thermodynamical entropy and gravitational entropy

\begin{equation}
S_{\text{tot}} = S_{\text{td}} + S_{\text{grav}}.
\end{equation}

We will use this assumption to develop a more general model for gravitational entropy for spherically symmetric spacetimes.

\subsection{Construction of an expression for gravitational entropy}

First we need to make a more general description of the Hawking-Bekenstein entropy. It is proportional to the horizon area, so we assume that it can be expressed as a surface integral of some vector field $\vec{\Psi}$ over the horizon:

\begin{equation}
S = k_S \int_{\sigma} \vec{\Psi} \cdot d\sigma \quad (4)
\end{equation}

where $k_S$ is an unknown constant. Our goal now, is to find a vector field, $\vec{\Psi}$ that gives a convincing description of entropy for a black hole. Then we demand that this is equal to the Hawking-Bekenstein entropy

\begin{equation}
S = S_{HB}
\end{equation}

and use this to determine $k_S$. 
By means of Gauss’ divergence theorem, we can rewrite the surface integral as a volume integral:

$$\int_{\sigma} \Psi \cdot d\sigma = \int_{V} (\nabla \cdot \Psi) \, dV.$$ 

It is then convenient to define an entropy density:

$$s = kS \nabla \cdot \Psi.$$ 

Now, we assume that the vector field is radial:

$$\Psi = P\vec{e}_r.$$ 

Since we intend to describe the entropy according to the Weyl curvature conjecture, we assume that $P$ is some scalar derived from the Weyl tensor. At this point we have no further clue about how $P$ should be found, so we start by trying some of the easiest ways to extract a scalar from the Weyl tensor.

### B. The Weyl scalar

Usually, the easiest way to get a scalar value from a tensor is find the trace, by contracting the indices. The Weyl tensor however, is traceless, so this always yields zero.

$$C_{\alpha\beta}^{\alpha\beta} = 0$$

The next easiest way to create a scalar is to “square” it. We therefore start by defining

$$P^2 = C_{\alpha\beta\gamma\delta}^{\alpha\beta\gamma\delta}.$$ 

We will refer to this quantity as the Weyl scalar.

For now, we will constrain ourselves to non-rotating electrically neutral black holes, and therefore we can describe the spacetime geometry with the Schwarzschild metric:

$$ds^2 = -e^{\nu(r)} c^2 dt^2 + e^{\lambda(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

where

$$e^{\nu(r)} = e^{-\lambda(r)} = 1 - \frac{R_S}{r},$$ 

and $R_S = \frac{2MG}{c^2}$ is the Schwarzschild radius.

The nonzero, independent components of the Weyl tensor in this spacetime is given as

$$C_{0012}^3 = C_{2012}^5 = -C_{1314}^1 = -C_{2142}^1 = \frac{1}{2} C_{3232}^5 = -\frac{1}{2} C_{0101}^1 = \frac{e^{-\lambda(r)}}{6} \alpha(r)$$

where $\alpha(r)$ is

$$\alpha(r) = \frac{e^{\lambda(r)}}{r^2} - \frac{1}{r^2} - \frac{\nu'(r)^2}{4} + \frac{\nu''(r)\lambda'(r) - \nu'(r)\lambda''(r) - \nu'(r)}{2r}.$$ 

The Weyl tensor inherits the symmetries of the Riemann tensor:

$$C_{\alpha\beta\gamma\delta}^{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta} = C_{\beta\gamma\alpha\delta} = -C_{\beta\delta\alpha\gamma}$$

$$C_{\alpha\beta\gamma\delta}^{\alpha\beta\gamma\delta} + C_{\alpha\delta\beta\gamma} + C_{\alpha\gamma\delta\beta} = 0.$$ 

Using eqs. (10)–(14) in the calculation of the Weyl scalar, we find

$$C_{\alpha\beta\gamma\delta}^{\alpha\beta\gamma\delta} = 12 \left( \frac{e^{\lambda(r)}}{3} \frac{\alpha(r)}{r^2} \right)^2.$$ 

For the Schwarzschild metric we use (9) and (11) to find

$$\alpha(r) = \frac{3R_S}{r^3} \frac{1}{1 - \frac{R_S}{r}},$$

resulting in the Weyl scalar in the Schwarzschild spacetime

$$C_{\alpha\beta\gamma\delta}^{\alpha\beta\gamma\delta} = 12 \left( \frac{R_S}{r^3} \right)^2 = \frac{48M^2G^2}{r^6c^4}.$$ 

Combining the Weyl scalar with (7) we find (assuming the positive solution) that

$$P = 2\sqrt{3} \frac{R_S}{r^3}.$$ 

We then use this with (11) to find the entropy. Since we are operating in 3-space, we must at this point find a spatial metric. We define the spatial metric $h_{ij}$ as

$$h_{ij} = g_{ij} = g_{00} g_{ij} \frac{g_{00}}{g_{00}}$$

where $g_{ij}$ is the 4-dimensional spacetime metric and Latin indices denote spatial components, $i, j = 1, 2, 3$. This yields

$$h_{ij} = \text{diag}(e^{\lambda(r)}, r^2, r^2 \sin \theta).$$

The infinitesimal surface element on the horizon surface is then

$$d\sigma = e^\nu(\sqrt{h_{rr}} d\theta d\phi$$

$$= e^\nu \sin \theta d\theta d\phi$$

Then we use (11), (12), (13) and (14) to find the entropy. Since there is a singularity at the origin, we must be careful about integrating around it. Therefore also we can integrate over a small sphere with radius $\varepsilon$ around the origin, and subtract this from the integral (14). When we let $\varepsilon \to 0$ this should result in the entropy. The result is

$$S = kS_1 (P(R_S)R_S - P(\varepsilon)\varepsilon) \int_0^{2\pi} \int_0^\pi \sin \theta d\theta d\phi$$

$$= 4\pi kS_1 \left( 2\sqrt{3} - 2\sqrt{2} \frac{R_S}{\varepsilon} \right).$$ 

Here, we see two things. First, when we let $\varepsilon \to 0$ this entropy diverges. Secondly, this result is not proportional to the area. We should therefore find a different scalar, $P$, from the Weyl tensor.
**C. The surface gravity**

Let us now redefine

\[ P^2 = \frac{C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}}{\kappa^4(r)} \]  

(21)

where \( \kappa(r) \) is the acceleration of gravity. In the Schwarzschild spacetime this is \[ \kappa(r) = -\frac{R_S c^2}{2r^2} \]  

(22)

From \( 21, 17 \) and \( 22 \) we find

\[ P = \frac{8\sqrt{3}}{R_S c^4} r \]  

(23)

Again, we calculate the entropy \( 21, 23, 13 \) and \( 19 \) using the same approach as in the previous section

\[ S = k_{S2} \left( P(R_S) R_s^3 - P(\varepsilon) \varepsilon^3 \right) \int_0^{2\pi} \int_0^\pi \sin \theta d\theta d\phi \]

\[ = 4\pi k_{S2} \left( \frac{8\sqrt{3}}{c^4} R_s^3 - \frac{8\sqrt{3}}{R_S c^4} \varepsilon^3 \right) \]  

(24)

If we let \( \varepsilon \to 0 \) here, the entropy of a black hole becomes

\[ S = k_{S2} \frac{32\sqrt{3}\pi}{c^4} R_s^3 = k_{S2} \frac{8\sqrt{3}}{c^4} A \]  

(25)

where \( A = 4\pi R_s^2 \) is the horizon area of a black hole. Comparing with the Hawking-Bekenstein entropy \( 21 \) and demanding \( S = S_{HB} \) yields

\[ k_{S2} = \frac{k_B c^7}{32\sqrt{3} G h} \]

Note that the entropy is proportional to the area only for the case of a black hole horizon. The gravitational entropy of a general spherical mass distribution with radius \( R \) yields

\[ S = k_{S2} \frac{32\sqrt{3}\pi R^3}{R_S c^4} = k_{S2} \frac{24\sqrt{3}}{R_S c^4} V \]  

(26)

where \( V = \frac{4}{3}\pi R^3 \) is the volume of the sphere.

Recall that we used Gauss’ integral theorem to define entropy density in \( 16 \). Calculating the divergence of \( \Psi \) we find

\[ s = k_{S2} \nabla \cdot \Psi = k_{S2} \frac{24\sqrt{3}}{R_S c^4} \sqrt{1 - \frac{R_S}{r}} \]  

(27)

We see that this quantity becomes imaginary inside the horizon \( (r < R_S) \) even if the entropy \( 23 \) is real. If we instead use \( 21 \) to define a “physical” entropy density, \( \hat{s} \), by simply factoring out the volume, \( V \), we find

\[ \hat{s} = k_{S2} \frac{24\sqrt{3}}{R_S c^4} \]  

(28)

The physical entropy density \( 28 \) is constant for a given mass of a black hole, and the “mathematical” entropy density \( 27 \) approaches a constant positive value for \( r \to R_S \). As the spacetime is asymptotically Minkowski far away from the black hole, we would expect the entropy density to vanish for \( r \to \infty \). What we have found however, is that the gravitational entropy is non-vanishing even in the Minkowski spacetime. Since this seems quite unphysical, we should look for another expression of gravitational entropy.

**D. The Ricci tensor**

A quantity that has been much discussed in relation to the Weyl curvature conjecture is the ratio between the Weyl scalar and the squared Ricci tensor. The Ricci tensor seems to have opposite properties to the Weyl tensor with respect to gravitational entropy, and therefore this ratio has been suggested as an alternative to using only the Weyl tensor. It has also been pointed out that this quantity may be more well behaved in an initial singularity \( 16, 13 \). Therefore we redefine \( P \) again:

\[ P^2 = \frac{C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}}{R^\mu\nu R_{\mu\nu}} \]  

(29)

The first problem we encounter is that the Ricci tensor is zero in vacuum and therefore also in the Schwarzschild spacetime. In an attempt to solve this, we invoke the Hawking radiation \( 18 \). One may hope that the radiation from the horizon will curve the spacetime slightly and prevent the Ricci tensor from vanishing. In order to describe this, we use the Vaidya spacetime

\[ ds^2 = -\left( 1 - \frac{2MG}{rc^2} \right) dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]

(30)

where \( v = t + r^* \) is an advanced time coordinate, \( r^* = r + \frac{2MG}{c^2} \ln \left( \frac{r^2}{2MG} - 1 \right) \) and \( M = M(v) \).

The Weyl scalar in this spacetime is \( 19 \)

\[ C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} = \frac{48M^2(v)G}{r^6 c^4} \]

(31)

To find the squared Ricci tensor we start with Einstein’s field equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \]

(32)

this gives

\[ R^{\mu\nu} R_{\mu\nu} = \left( \frac{8\pi G}{c^4} \right)^2 T^{\mu\nu} T_{\mu\nu} \]

The only nonzero covariant component of the energy momentum tensor for radiation is \( 19 \)

\[ T_{\nu\rho} = \frac{2c^2}{8\pi r^2} \frac{\partial M}{\partial v} \]

(33)
However, the only nonzero contravariant component of the energy momentum tensor is

\[ T^{rr} = \frac{2c^2}{8\pi^2} \frac{\partial M}{\partial v}. \]

Thus the product \( T^{\mu \nu} T_{\mu \nu} \) is zero, and

\[ R^{\mu \nu} R_{\mu \nu} = 0 \quad (34) \]

for the Vaidya spacetime. Hence \( P \) is non-defined, and again we must find another expression for \( P \).

E. The Riemann tensor

We will now investigate an expression for \( P \) where we use the Kretschmann scalar instead of the squared Ricci tensor:

\[ P^2 = \frac{C_{\alpha \beta \gamma \delta} C_{\alpha \beta \gamma \delta}}{R^{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta}} \quad (35) \]

For static spherically symmetric spacetimes, \( 0 \leq P^2 \leq 1 \) (see appendix). This may possibly be connected to states of zero gravitational entropy and maximum gravitational entropy respectively.

Since the Kretschmann scalar is not necessarily zero in vacuum, we return to the Schwarzschild spacetime \( ^{S} \). In this metric all curvature is Weyl curvature, so that

\[ R^{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta} = C^{\alpha \beta \gamma \delta} C_{\alpha \beta \gamma \delta} = 12 \left( \frac{R_S}{r^4} \right)^2 \]

and \( P^2 \) becomes simply

\[ P^2 = 1. \]

If our speculations are correct, this means that a Schwarzschild black hole is a configuration of maximum gravitational entropy.

We find the entropy of a black hole as before:

\[ S = k_{S3} \int_{\sigma} \bar{\Psi} \cdot d\sigma \]

\[ = k_{S3} \int_{\sigma} \left( R_S^2 - \varepsilon^2 \right) \sin \theta d\theta d\phi \]

\[ = k_{S3} 4\pi \left( R_S^2 - \varepsilon^2 \right). \]

Letting \( \varepsilon \to 0 \) we get

\[ S = k_{S3} 4\pi R_S^2 = k_{S3} A \quad (36) \]

where \( A = 4\pi R_S^2 \) is the horizon area of the black hole. Comparing with the Hawking-Bekenstein entropy and demanding \( S = S_{\text{HE}} \) yields

\[ k_{S3} = \frac{k_B}{4l_P} = \frac{k_B r^3}{4G\hbar} \quad (37) \]

Now, we proceed to find the entropy density in the Schwarzschild spacetime. From the definition of entropy density \( \bar{\Psi} \) we find

\[ s = k_{S3} \bar{\Psi} \]

\[ = k_{S3} \frac{1}{\sqrt{\hbar}} \frac{\partial}{\partial r} \left( \sqrt{\hbar} \frac{P}{r} \right) \]

\[ = \frac{2k_{S3}}{r} \sqrt{1 - \frac{R_S}{r}} \quad (38) \]

where we have used the covariant divergence, \( \nabla \cdot \bar{\Psi} = \frac{1}{\sqrt{\hbar}} \frac{\partial}{\partial r} \left( \sqrt{\hbar} \Psi \right) \).

This entropy density vanishes as \( r \to \infty \) and the spacetime becomes Minkowski. The entropy density reaches a maximum at \( r = \frac{2}{3} R_S \), and then vanishes at the horizon. Inside the horizon, it becomes imaginary.

It should be noted, that for such a generalization, the entropy of the entire space diverges as \( A \to \infty \) in eq. \( \bar{\Psi} \).

IV. THE SCHWARZSCHILD-DE SITTER SPACETIME

We will now see how this description of gravitational entropy applies to a black hole in de Sitter background. If we include a cosmological constant into the field equations

\[ R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} + \Lambda g_{\mu \nu} = \frac{8\pi G}{c^4} T_{\mu \nu} \]

and solve for a spherically symmetric mass distribution we get the Schwarzschild-de Sitter (SdS) spacetime. This metric can be written on the form \( ^{S} \) where

\[ e^\nu(r) = e^{-\lambda(r)} = 1 - \frac{\Lambda r^2}{3} - \frac{R_S}{r}. \quad (39) \]

The Weyl scalar in the SdS spacetime is the same as in the Schwarzschild spacetime

\[ C^{\alpha \beta \gamma \delta} C_{\alpha \beta \gamma \delta} = \frac{48M^2G^2}{r^6c^4} \quad (40) \]

and the Kretschmann scalar is

\[ R^{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta} = \frac{8(18M^2G^2 + \Lambda^2 r^6c^4)}{3r^6c^4}. \quad (41) \]

We apply the expression \( \bar{\Psi} \) for \( P \) from the previous section and find

\[ P = \frac{1}{\sqrt{1 + \frac{2\Lambda r^4}{18M^2G^2r^6}}}. \]

Together with \( \bar{\Psi} \) we use this to find the entropy of the
black hole at the center of this spacetime as before.

\[
S = k_{S3} \int_{\sigma} \vec{\Psi} \cdot d\sigma
\]

\[
= k_{S3} \int_{\sigma} \left( P(R_h)R_h^2 - P(\varepsilon)\varepsilon^2 \right) \sin \theta d\theta d\phi
\]

\[
= k_{S3} 4\pi \left( P(R_h)R_h^2 - P(\varepsilon)\varepsilon^2 \right)
\]

\[
= k_{S3} 4\pi \left[ \frac{R_h^2}{\sqrt{1 + \frac{\Lambda r^4}{18M^2G^2}} R_h^6} - \frac{\varepsilon^2}{\sqrt{1 + \frac{\Lambda r^4}{18M^2G^2}} R_h^6} \right]
\]

where

\[
R_h = -\frac{2}{\sqrt{\Lambda}} \cos \left[ \frac{\arccos \left( \frac{3MG\sqrt{\Lambda}}{c^2} + \pi \right)}{3} \right]
\]

(42)

is the radius of the black hole horizon and \(k_{S3}\) can be found from (37). We let \(\varepsilon \to 0\) and get the entropy of the black hole

\[
S = \frac{k_{S3} 4\pi R_h^2}{\sqrt{1 + \frac{\Lambda r^4}{18M^2G^2}} R_h^6}
\]

(43)

From this we see that the entropy decreases for larger values of \(\Lambda\), and if we let \(\Lambda \to 0\) the black hole entropy becomes the Hawking-Bekenstein entropy, \(S = k_{S3}A\), where \(A = 4\pi R_h^2\) is the black hole horizon area.

The entropy density in this spacetime can be found in the same way as in the previous section (11), using the definition (6), and it becomes

\[
s = \frac{k_{S3}}{r} \left( \frac{1 - \frac{\Lambda r^4}{3} - \frac{R_S}{r}}{R_S} \right) \left( \frac{2 - \frac{\Lambda r^4}{18M^2G^2} r^6}{(1 + \frac{\Lambda r^4}{18M^2G^2} r^6)^{3/2}} \right).
\]

(44)

Also here we see that if \(\Lambda \to 0\) this reduces to the same result as in the Schwarzschild spacetime (15). The density (11) is plotted in figure 1. Obviously this shows a region with negative entropy density. This seems quite unphysical, especially when comparing with Boltzmann's interpretation of entropy (2), then this would correspond to a multiplicity of less than one.

To solve this problem we demand that the entropy density is positive.

\[
s = k_{S3} \left| \nabla \cdot \vec{\Psi} \right|
\]

(45)

This can be done by choosing the sign of \(P\). It is always defined as a square, and then we choose the positive square root, \(P = +\sqrt{P^2}\). Instead we could always choose the sign that results in positive entropy density.

For the Schwarzschild-de Sitter space, the entropy density is zero at

\[
r_0 = \left( \frac{6MG}{\Lambda c^2} \right)^{1/3}.
\]

(46)

By demanding that it is positive also outside this radius, the entropy density behaves as plotted in figure 2.

FIG. 1: The entropy density of the Schwarzschild-de Sitter spacetime between the horizons. Here, \(c = G = k_{S3} = 1, M = 1\) and \(\Lambda = 0.01\).

FIG. 2: The entropy density of the Schwarzschild-de Sitter spacetime between the horizons according to eq. (10). Here, \(c = G = k_{S3} = 1, M = 1\) and \(\Lambda = 0.01\).

V. THE DE SITTER SPACETIME

The SdS spacetime has two obvious special cases. One is obtained by putting \(\Lambda = 0\) resulting in Schwarzschild spacetime. The other is known as the de Sitter spacetime and is obtained by putting \(M = 0\). The line element in
this spacetime may also be expressed as in [5], but with
\[ e^{\nu(r)} = e^{-\lambda(r)} = 1 - \frac{\Lambda r^2}{3}. \]  

This spacetime has no black hole, only a cosmological horizon at \( R_\Lambda = \frac{\Lambda}{\sqrt{\kappa}} \).

G. Gibbons and S. W. Hawking [20] found that this horizon has entropy similar to a black hole horizon,
\[ S_\Lambda = \frac{k_B r^3}{4G\hbar} A. \]  

where \( A = 4\pi R_\Lambda^2 = \frac{4\pi}{\kappa} \) is the area of the cosmological horizon.

The Weyl curvature tensor vanishes in de Sitter spacetime, and consequently the gravitational entropy in this spacetime is zero. This conflicts with the idea of reconciling horizon entropy with gravitational entropy. This could of course be solved by developing some model where the gravitational entropy is non-vanishing in de Sitter spacetime. It would give little satisfaction however, since the de Sitter spacetime gives a very good description of the early universe where the gravitational entropy is expected to be small (or zero). The fact that the Weyl tensor vanishes in this spacetime was one of the reasons for using it to describe gravitational entropy in the first place. Thus the gravitational entropy is expected to vanish in this spacetime, and therefore the entropy of the cosmological horizon must be of non-geometrical origin.

VI. SUMMARY AND CONCLUSIONS

We have now developed a possible description of gravitational entropy motivated by the possibility of reconciling it with the Hawking-Bekenstein entropy of black holes. We have assumed the entropy of a black hole to be described by the surface integral
\[ S = k_S \int_\sigma \psi \cdot d\sigma \]  

where \( \sigma \) is the horizon of the black hole and the vector field \( \psi \) is
\[ \psi = P \epsilon_\cdot \cdot \cdot. \]

Comparison with the Hawking-Bekenstein entropy led us to define \( P^2 \) as the ratio of the Weyl scalar and the Kretschmann scalar:
\[ P^2 = \frac{C_{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}}{R_{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}}. \]

This resulted in
\[ k_S = k_{S3} = \frac{k_B}{4l_p^3} = \frac{k_B r^3}{4G\hbar} \]

By means of Gauss’ divergence theorem [19] can be rewritten as a volume integral, and with this in mind we defined the entropy density
\[ s = k_S \left| \nabla \cdot \psi \right| \]

where the absolute value brackets was added to avoid negative entropy.

After a brief investigation of the de Sitter spacetime we concluded that the entropy of the cosmological horizon could not be gravitational entropy. This leads us to two possible interpretations. Either, horizon entropy in general is different from gravitational entropy or there is a thermodynamical factor involved when we introduce the cosmological constant. The Schwarzschild and the de Sitter spacetimes can be viewed as two opposite special cases of the SdS spacetime. It might be that the Schwarzschild spacetime has only gravitational entropy, and the de Sitter spacetime has only thermodynamical entropy. This is actually what we expected: large thermodynamical entropy in the early universe and large gravitational entropy around black holes.

In this paper we have approached the problem of gravitational entropy from a phenomenological point of view instead of developing a description from the more fundamental properties of gravitation. This is because the classical theory of gravitation (general relativity) does not include microscopic states of gravitation. A working theory of quantum gravity should give a description of gravitational microstates, and then the gravitational entropy can be found using Boltzmann’s formula [2]. Until then, phenomenological approaches such as this may hopefully give us a hint of what a theory of quantum gravity will say about the entropy of a gravitational field.

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APPENDIX: THE RELATION BETWEEN THE WEYL AND THE KRETSCHAMANN SCALARS

In this appendix we will show that for static spherically symmetric spacetimes,
\[ R_{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \geq C_{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}. \]

The Weyl tensor can be expressed by the Riemann tensor as
\[ C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - g_{\alpha[\gamma} R_{\beta]\delta] + g_{\beta[\gamma} R_{\delta]\alpha} + \frac{1}{3} R g_{\alpha[\gamma} g_{\beta]\delta]. \]

where \( R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \) is the Ricci tensor and \( R = R^\Lambda_{\Lambda} \) is the Ricci scalar.
From this it is straightforward to show that the Weyl scalar can be expressed as

\[ C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - 2R_{\alpha\beta}R_{\gamma\delta} + \frac{1}{3}R^2 \]

\[ = R_{\alpha\beta\gamma\delta} + \Delta \]  \hspace{1cm} (A.1)

where we have defined the difference \( \Delta \) between the Weyl and Kretschmann scalars

\[ \Delta = -2R_{\alpha\beta}R_{\alpha\beta} + \frac{1}{3}R^2. \]  \hspace{1cm} (A.2)

If \( \Delta < 0 \), then \( C_{\alpha\beta\gamma\delta} < R_{\alpha\beta\gamma\delta} \), and if \( \Delta > 0 \) then \( C_{\alpha\beta\gamma\delta} > R_{\alpha\beta\gamma\delta} \). If \( \Delta = 0 \) then the two scalars are equal.

We begin with the general metric for a static spherically symmetric spacetime:

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{\nu(r)}c^2dt^2 + e^{-\nu(r)}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2. \]

We also need the contravariant components of the metric:

\[ g^{\mu\nu} = \text{diag} \left(-e^{-\nu}, \frac{1}{r^2} \frac{1}{sin^2\theta} \right). \]

The components of the Ricci tensor in this metric is [21]:

\[ R_{tt} = e^{\nu(r)} \left( \frac{1}{4} \nu'' - \frac{1}{4} \nu' \lambda' + \frac{1}{4} \nu'' + \frac{1}{4} \nu' \right) \]

\[ R_{rr} = -\frac{1}{r^2} \nu'' + \frac{1}{4} \nu' \lambda' - \frac{1}{4} \nu'' + \frac{\lambda'}{r} \]

\[ R_{\theta\theta} = 1 - e^{-\nu(r)} \left( \frac{1}{2} \left( \nu' - \lambda' \right) \right) \]

\[ R_{\phi\phi} = R_{\theta\theta} \sin^2 \theta \]

\[ R_{\mu\nu} = 0 \hspace{1cm} \mu \neq \nu \]

Using these, we find the Ricci scalar:

\[ R = g^{\mu\nu}R_{\mu\nu} \]

\[ = -e^{-\nu(r)} \left[ e^{\nu(r)} \left( \frac{1}{4} \nu'' - \frac{1}{4} \nu' \lambda' + \frac{1}{4} \nu'' + \frac{1}{4} \nu' \right) \right] \]

\[ + \frac{2}{r^2} \left[ 1 - e^{-\nu(r)} \left( \frac{1}{2} \left( \nu' - \lambda' \right) \right) \right] \]

\[ = -e^{-\nu(r)} \left( \frac{1}{2} \nu'' + \nu' \lambda' - \frac{1}{2} \nu'' + \lambda' + \frac{1}{2} \nu' \right) \]

\[ \text{In the expression for } \Delta \text{, the Ricci scalar appears squared, hence} \]

\[ R^2 = e^{-2\nu(r)} \left( \frac{1}{2} \nu'' + \nu' \lambda' - \frac{1}{2} \nu'' + \frac{1}{2} \nu' \right)^2 \]

\[ = e^{-2\nu(r)} \left( \nu'' + \nu' \lambda' - \frac{1}{2} \nu'' + \frac{1}{2} \nu' \right)^2 \]

\[ \text{and if we look closer at the terms in the parenthesis we notice that for some of them we can factor out } (e^{\nu} - 1) \text{ again. This, and some more sorting of the other terms} \]

Next, we find the “squared” Ricci tensor, \( R_{\alpha\beta} \)

\[ R_{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}R_{\mu\nu} \]

\[ = (g^{tt})^2 (R_{tt})^2 + (g^{rr})^2 (R_{rr})^2 \]

\[ + (g^{\theta\theta})^2 (R_{\theta\theta})^2 + (g^{\phi\phi})^2 (R_{\phi\phi})^2 \]

\[ = e^{-2\nu} \left( \frac{1}{2} \nu'' + \nu' \lambda' + \frac{1}{2} \nu'' + \nu' \right) \]

\[ - \frac{1}{2} \nu' \lambda' \]

\[ + \frac{1}{8} \nu' \lambda^2 - \frac{1}{4} \nu' \lambda' + \nu' \lambda + \frac{1}{4} \nu' \lambda^2 \]

\[ \text{We insert } (A.3) \text{ into } (A.2) \text{ we find } \Delta \]

\[ \Delta = -2R_{\alpha\beta}R_{\alpha\beta} + \frac{1}{3}R^2 \]

\[ = -\frac{1}{6} \left[ 4r^2 \nu'' + 4r^2 \nu' \lambda' + 4r^2 \nu' \lambda + 8r^2 \nu'' \right] \]

\[ - 4r^2 \nu' \nu' \lambda' + 4r^2 \nu' \nu'' - 8r^2 \nu' \lambda + 4r^2 \nu'' \lambda + 2r^2 \nu'^2 - 2r^2 \nu' \lambda' \]

\[ + 2r^2 \nu' \nu' + 4r^2 \nu'' \lambda + 2r^2 \nu' \lambda' + 4r^2 \nu' \lambda + 8r^2 \nu' \lambda' + 8r^2 \nu'' \lambda' + 8r^2 \nu' \lambda + 4r^2 \nu'' \lambda' \]

\[ \text{If we sort the terms carefully we see that we can extract a factor } (e^\nu - 1) \text{ from many of them} \]

\[ \Delta = \frac{1}{6} (e^\nu - 1) \left[ 4r^2 \nu'' + 4r^2 \nu'^2 - 4r^2 \nu' \lambda' \right] \]

\[ - 16 + 16e^\nu - 8r^2 \nu' \lambda' \]

\[ + 10r^2 \nu'^2 - 4r^2 \nu' \lambda' + 4r^2 \nu' \nu'' - 4r^2 \nu' \nu' \lambda' \]

\[ + 4r^2 \nu' \nu' \lambda' + 4r^2 \nu' \nu'' \]

\[ + 2r^2 \nu' \lambda'^2 + 2r^2 \nu'' \lambda'^2 - 2r^4 \nu'^3 \lambda' \]

\[ - 4r^2 \nu' \lambda'^2 \lambda + 4r^2 \nu' \lambda'^2 + 10r^2 \nu'^2 \]

\[ \text{and if we look closer at the terms in the parenthesis we notice that for some of them we can factor out } (e^\nu - 1) \text{ again. This, and some more sorting of the other terms} \]
\[
\Delta = - \frac{1}{6 (r^2 e^\lambda)^2} \left\{ 4 (e^\lambda - 1) \\
+ \frac{1}{2} r \left( 2 \{ \lambda' - \nu' \} + r \left( 2 \nu'' + \nu^2 - \nu' \lambda' \right) \right)^2 \\
- \frac{1}{4} r^2 \left[ 2 (\lambda' - \nu') + r \left( 2 \nu'' + \nu^2 - \lambda' \nu' \right) \right]^2 \\
+ \frac{1}{4} r^4 \left[ 4 \nu'' + 4 \nu'' \nu^2 - 4 \nu'' \nu' \lambda' + 4 \nu^3 \lambda' + \nu^2 \lambda^2 \right] \\
- 2 r^3 \left[ 2 \nu'' \lambda' + 2 \nu^2 \lambda' - \nu'' \lambda^2 - 2 \nu'' \nu' - \nu^3 \lambda' \right] \\
+ 2 r^2 \left[ 5 \lambda^2 - 2 \lambda \nu' + 5 \nu^2 \right] \right\} \\
\]  
\text{(A.7)}

We split the fifth term and write it as a sum of two squares
\[
2 r^2 \left[ 5 \lambda^2 - 2 \lambda \nu' + 5 \nu^2 \right] \\
= 4 r^2 \left[ \lambda^2 - 2 \lambda \nu' + \nu^2 \right] + 6 r^2 \left[ \lambda^2 + 2 \lambda \nu' + \nu^2 \right] \\
= 4 r^2 \left[ \lambda' - \nu' \right]^2 + 6 r^2 \left[ \lambda' + \nu' \right]^2 \\
\]  
\text{(A.8)}

and if we look closer we also see that the third term of \(\Delta\) is a square
\[
\Delta = \frac{r^4 \left[ 4 \nu'' + 4 \nu'' \nu^2 - 4 \nu'' \nu' \lambda' + \nu^4 - 2 \nu^3 \lambda' + \nu^2 \lambda^2 \right]}{6 (r^2 e^\lambda)^2} \\
\]  
\text{(A.9)}

and the fourth term can be written as a product
\[
-2 r^3 \left[ 2 \nu'' \lambda' + 2 \nu^2 \lambda' - \nu'' \lambda^2 - 2 \nu'' \nu' - \nu^3 \lambda' \right] \\
= -2 r^3 \left[ \lambda' - \nu' \right] \left[ 2 \nu'' + \nu^2 - \nu' \lambda' \right] \\
\]  
\text{(A.10)}

Inserting (A.6), (A.9), (A.10) into (A.7) and writing out the second term we get
\[
\Delta = - \frac{1}{6 (r^2 e^\lambda)^2} \left\{ 4 (e^\lambda - 1) \\
+ \frac{1}{2} r \left( 2 \{ \lambda' - \nu' \} + r \left( 2 \nu'' + \nu^2 - \nu' \lambda' \right) \right)^2 \\
- r^2 \left[ \lambda' - \nu' \right]^2 - r^3 \left[ \lambda' - \nu' \right] \left[ 2 \nu'' + \nu^2 - \lambda' \nu' \right] \\
- \frac{1}{4} r^4 \left[ 2 \nu'' + \nu^2 - \lambda' \nu' \right]^2 + r^4 \left[ 2 \nu'' + \nu^2 - \nu' \lambda' \right]^2 \\
- 2 r^3 \left[ \lambda' - \nu' \right] \left[ 2 \nu'' + \nu^2 - \nu' \lambda' \right] + 4 r^2 \left[ \lambda' - \nu' \right]^2 \\
+ 6 r^2 \left[ \lambda' + \nu' \right]^2 \right\} \\
\]  
\text{(A.11)}

Once again, we recognize a square, and rewrite the second, third and fourth term to get the final expression
\[
\Delta = - \frac{1}{6 (r^2 e^\lambda)^2} \left\{ 4 (e^\lambda - 1) \\
+ \frac{1}{2} r \left( 2 \{ \lambda' - \nu' \} + r \left( 2 \nu'' + \nu^2 - \nu' \lambda' \right) \right)^2 \\
+ 3 r^2 \left[ \lambda' - \nu' \right]^2 - 3 r^3 \left[ \lambda' - \nu' \right] \left[ 2 \nu'' + \nu^2 - \lambda' \nu' \right] \\
+ \frac{3}{4} r^4 \left[ 2 \nu'' + \nu^2 - \lambda' \nu' \right]^2 + 6 r^2 \left[ \lambda' + \nu' \right]^2 \right\} \\
\]  
\text{(A.11)}

containing only square terms. Thus \(\Delta \leq 0\), and consequently
\[
R^{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta} \geq C^{\alpha \beta \gamma \delta} C_{\alpha \beta \gamma \delta}. \\
\]  
\text{(A.12)}
[22] This was first pointed out by J. A. Wheeler, as mentioned by J. Bekenstein in [11].