Refining threshold resummations
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We describe some recent refinements of the techniques of threshold resummation, with emphasis on the usefulness of dimensional regularization when applied to nonabelian exponentiation. Threshold resummation is now under theoretical control for DIS and electroweak annihilation cross sections all the way to the fourth tower of logarithms and up to corrections suppressed by powers of the threshold variable.

1. INTRODUCTION

Soft and collinear gluon radiation in perturbative QCD is well-known to have special properties of universality and factorization, which are related to its semiclassical nature \cite{12} and strongly tied to gauge invariance \cite{9}. These properties are at the heart of our understanding of strong interactions at high energies, as they allow us to define infrared and collinear safe quantities to all orders in perturbation theory, and lead to factorization of long distance contributions in hadronic processes. Understanding soft and collinear gluons is however not merely a theoretical exercise: it has important phenomenological consequences. Even when dealing with observables which are finite order by order in perturbation theory, one still finds in many cases that the cancellation of long distance singularities leaves behind finite but large contributions, typically logarithms of large ratios of kinematic scales. These contributions need to be resummed to all orders to have reliable predictions, in some cases even to get just a qualitative agreement with experimental data. Resummation is a well developed technology \cite{13}, deeply connected to the universality of soft and collinear singularities.

In perturbative QCD, the use of dimensional regularization to regularize mass singularities is not required in theory, but is in practice. Here we would like to discuss some recent developments concerning the resummation of threshold logarithms, while emphasizing the simplicity and elegance which can be achieved by making extensive use of dimensional regularization in dealing with the underlying pattern of soft and collinear divergences. In essence, resummation of threshold logarithms is always a consequence of exponentiation of soft and collinear singularities. Nonabelian exponentiation implies that double poles of infrared and collinear origin can be organized in a predictive manner in terms of an exponent containing only simple poles,

$$
\sum_{k} \alpha_{s}^{k} \sum_{p} c_{kp} \epsilon^{-p} \rightarrow \exp \left[ \sum_{k} \alpha_{s}^{k} \sum_{p} d_{kp} \epsilon^{-p} \right]. \quad (1)
$$

After the cancellation of singularities between real and virtual contributions has taken place, each pole leaves behind a logarithm $L$, with argument for example the Mellin variable in DIS, where $L = \log N$. A similar pattern of exponentiation then applies to these logarithms, which are organized as

$$
\sum_{k} \alpha_{s}^{k} \sum_{p} \hat{c}_{kp} L^{p} \rightarrow \exp \left[ L g_{1}(\alpha_{s} L) + g_{2}(\alpha_{s} L) \right]
$$
\[ + \alpha_s g_3(\alpha_s L) + \ldots \] (2)

We take then the viewpoint that in order to resum threshold logarithms we first need to resum infrared and collinear poles. Since all such resummations involve the coupling evaluated at a soft scale, the first tool must be a dimensionally regularized version of the running coupling. As is well known, in \( d = 4 - 2\epsilon \), with \( \epsilon < 0 \) for infrared regularization, this coupling must satisfy the equation

\[ \mu \frac{\partial \hat{\alpha}(s)}{\partial \mu} = \beta(\hat{\alpha}(s)) = -2 \epsilon \hat{\alpha}(s) + \hat{\beta}(\hat{\alpha}(s)), \]


\[ \hat{\beta}(\hat{\alpha}(s)) = -\frac{\hat{\alpha}(s)^2}{2\pi} \sum_{n=0}^{\infty} b_n \left( \frac{\hat{\alpha}(s)}{\pi} \right)^n, \] (3)

where \( \hat{\beta}(\hat{\alpha}(s)) \) is the ordinary 4-dimensional \( \beta \) function, with \( b_0 = (11C_A - 2n_f)/3 \). At one loop, the solution is

\[ \hat{\alpha}(s) = \left[ \frac{\beta_0 S(\hat{\alpha}(s))}{\hat{\alpha}(s)} \right]^{\epsilon} \]

which reduces to the well-known limit as \( \epsilon \to 0 \).

Working with eq. (3) leads to compact resummed expressions for both amplitudes and cross sections, which can be readily compared with Feynman diagram calculations. This was first done for the Sudakov form factor in [4], then applied to threshold resummations in [7]. More recently, it was shown [8] that eq. (3) has the added virtue of providing a regularization for the Landau pole, which moves off the real axis for \( \epsilon < -b_0 \alpha_s/(4\pi) \), leading to resummed expressions which are explicit analytic functions of the coupling and \( \epsilon \).

At the level of amplitudes, a generalization of the work of [8] to multiparton configurations has been provided in [9], proving an earlier statement [10] on the structure of single poles in QCD amplitudes at two loops. This was in turn an important ingredient in the formulation of a striking conjecture on the all-order structure of multileg amplitudes in \( N = 4 \) supersymmetric Yang-Mills theory, for which strong evidence was provided in [11,12]. A two-loop analysis of the soft functions appearing in the exponentiation of multiparton amplitudes was very recently started along these lines in [13].

In the context of resummations, a key feature of the usage of the \( d \)-dimensional running coupling is the fact that all infrared and collinear singularities arise by integrating the scale of the coupling itself, while all functions appearing in resummed exponents are finite. To illustrate how poles arise, consider the expansion of the coupling at the three-loop level,

\[ \hat{\alpha}(\xi^2, \alpha_s, \epsilon) = \alpha_s \xi^{-2\epsilon} + \alpha_s^2 \xi^{-4\epsilon} \frac{b_0}{4\pi^\epsilon} (1 - \xi^{2\epsilon}) + \alpha_s^3 \xi^{-6\epsilon} \frac{1}{8\pi^2\epsilon} \left[ \frac{b_0^2}{2\epsilon} (1 - \xi^{2\epsilon})^2 + b_1 (1 - \xi^{4\epsilon}) \right], \] (5)

where \( \xi \) is now a ratio of scales, while \( \alpha_s \) is evaluated at the lower (reference) scale. Clearly, eq. (5) is finite as \( \epsilon \to 0 \), but will generate poles when integrated over \( \xi \).

We will now proceed to further illustrate these ideas by describing two recent applications to resummation for processes of electroweak (EW) annihilation, such as Drell-Yan and Z-boson production, or Higgs production via gluon fusion.

2. N-INDEPENDENTTERMS

Using factorization techniques based on dimensional regularization, it is possible to show [14] that for simple processes, such as DIS or EW annihilation, threshold resummation can be extended, so that all \( N \)-independent terms in the cross section exponentiate together with logarithms. Consider for example the Drell-Yan cross section.

The key step is to recognize, as suggested already in [2], that at parton level the (collinear divergent) annihilation cross section can be factorized as

\[ \omega(N, \epsilon) = \left| \Gamma(Q^2, \epsilon) \right|^2 \psi_R(N, \epsilon)^2 U_R(N, \epsilon), \] (6)

where \( \Gamma(Q^2, \epsilon) \) is the quark form factor, \( \psi_R \) is the real emission contribution to a special quark distribution [2], and similarly \( U_R \) is the real emission contribution to an eikonal function describing wide-angle soft gluon radiation. Corrections to eq. (6) are suppressed by powers of \( N \). The key feature of eq. (6) is that real and virtual contributions are explicitly factorized, with all virtual poles collected in the form factor \( \Gamma \). Furthermore,
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all functions involved exponentiate up to 1/N corrections. The exponentiation of the form factor is described in [6], while the parton distribution \( \psi_R \) is given by an expression of the form

\[
\psi_R(N, \epsilon) = \exp \left[ \int_0^1 d\zeta \frac{z^{N-1}}{1-z} \int_0^1 d\eta \frac{1}{1-\eta} \kappa_\psi(\tau (1-\eta)^2 Q^2) \right].
\]

(7)

Similarly, the eikonal function \( U_R \) can be written as

\[
U_R(N, \epsilon) = \exp \left[ -\int_0^1 d\zeta \frac{z^{N-1}}{1-z} \right. \left. g_U(\tau (1-z)^2 Q^2, \epsilon) \right].
\]

(8)

Notice that the functions \( \kappa_\psi \) and \( g_U \) appearing in the exponents have their own Feynman rules and can be independently computed. At one loop, for example, one finds [4] \( \kappa_\psi(\alpha_s) = 2C_F(\alpha_s/\pi)\Gamma(2-\epsilon)/\Gamma(2-2\epsilon) \) and \( g_U = -2C_F(\alpha_s/\pi)\Gamma(1-\epsilon)/\Gamma(2-2\epsilon) \). In order to derive the finite Drell-Yan partonic hard cross section, one needs to divide eq. (5) by the square of a suitable quark-in-quark density. In the \( \overline{\text{MS}} \) scheme, one may use an explicit exponential representation of the \( \overline{\text{MS}} \) density, valid up to 1/N corrections, and containing only poles generated by the running coupling. It is

\[
\phi_{\overline{\text{MS}}}(N, \epsilon) = \exp \left[ \int_0^{Q^2} \frac{d\xi^2}{\xi^2} \left\{ B_\delta(\tau (\xi^2)) \right. \right. \\
\left. \left. \int_0^1 d\zeta \frac{z^{N-1}}{1-z} A(\tau (\xi^2)) \right\} \right].
\]

(9)

where \( A(\alpha_s) \) and \( B_\delta(\alpha_s) \) are, respectively, the coefficients of the plus distribution \( 1/(1-z) \), and of \( \delta(1-z) \) in the Altarelli-Parisi splitting function. Clearly, eq. (5) is simply designed to satisfy the Altarelli-Parisi equation and to be constructed of poles only. A key result of [13] is the fact that it is possible to further factor eq. (5) in a unique way by isolating pure pole terms associated with virtual contributions in a single factor. Virtual contributions to \( \phi_{\overline{\text{MS}}}(N, \epsilon) \) take the form

\[
\phi_V(\epsilon) = \exp \left\{ \frac{1}{2} \int_0^{Q^2} \frac{d\xi^2}{\xi^2} \left[ K(\alpha_s, \epsilon) + \tilde{G}(\tau(\xi^2)) \right] \right. \right. \\
\left. \left. + \frac{1}{2} \int_{\xi^2}^\alpha \frac{d\lambda^2}{\lambda^2} \gamma_K(\tau(\lambda^2)) \right\},
\]

(10)

which mimicks the exponentiation of the Sudakov form factor. In fact, \( \gamma_K(\alpha_s) \) and \( \tilde{K}(\alpha_s, \epsilon) \) are, respectively, the cusp anomalous dimension and the counterterm function featuring in the form factor resummation, while \( \tilde{G}(\tau) \), independent of \( \epsilon \), is constructed recursively from the analogous function present in \( \Gamma(Q^2, \epsilon) \). \( \phi_V(\epsilon) \) is designed to cancel exactly the virtual poles associated with the form factor. As a consequence, the Drell-Yan hard part \( \bar{\omega}(N) \equiv \omega(N, \epsilon)/(\phi_{\overline{\text{MS}}}(N, \epsilon))^2 \) can be written as the product of virtual and real contributions, which are separately finite. They are

\[
\bar{\omega}_V(Q^2) = \frac{[\Gamma(Q^2, \epsilon)]^2}{(\phi_V(\epsilon))^2},
\]

\[
\bar{\omega}_R(N) = \left[ U_R(N, \epsilon) \left( \phi_R(N, \epsilon) \right)^2 \right],
\]

(11)

where \( \phi_R \) is simply defined as \( \phi_{\overline{\text{MS}}}/\phi_V \). Taking the limit \( \epsilon \to 0 \), one recovers the usual resummation formula for the Drell-Yan cross section in the \( \overline{\text{MS}} \) scheme, where however all \( N \)-independent terms are now exponentiating, along with threshold logarithms. The final expression is

\[
\bar{\omega}_{\overline{\text{MS}}}(N) = \frac{[\Gamma(Q^2, \epsilon)]^2}{(\phi_V(Q^2, \epsilon))^2} \exp \left[ F_{\overline{\text{MS}}}(\alpha_s) \right. \right. \\
\left. \left. + \int_0^1 d\zeta \frac{z^{N-1}}{1-z} \left\{ D(\alpha_s (1-z)^2 Q^2) \right. \right. \\
\left. \left. + 2 \int_{Q^2}^{(1-z)^2 Q^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) \right\} \right],
\]

(12)

where the limit \( \epsilon \to 0 \) is understood in the virtual terms. Analogous formulas hold for the DIS cross section, for the Drell-Yan cross section in the DIS scheme, and for Higgs production via gluon fusion (where the gluon form factor replaces the quark form factor). Clearly, the exponentiation of \( N \)-independent terms does not have the predictive
power of the resummation of towers of logarithms, since, for example, the function \( F_{\text{MS}} (\alpha_s) \) receives novel contributions at every order. It should be emphasized, however, that \( F_{\text{MS}} \) has a precise definition of its own, and can be computed in principle without resorting to finite order results for the cross section. Furthermore, a large fraction of constant terms arising in ordinary perturbation theory are actually generated through exponentiation at lower orders, and, as we will see shortly, having defined \( F_{\text{MS}} \) helps uncover new structure to all orders.

3. EW ANNIHILATION AND DIS

Our second application is the derivation of a general relation expressing the coefficients of the resummation for EW annihilation processes in terms of data computed in DIS [15]. Within this formalism, the result can be extracted making use of the finiteness of equations like eq. (11), which follows from factorization. Consider specifically the real part of the cross section, \( \omega_R (N) \). While the numerator contains genuine information associated with Drell-Yan kinematics, the denominator is completely determined by knowledge which can be extracted from DIS, specifically form factor and splitting function information. Imposing the cancellation of all infrared and collinear poles yields relations between resummation coefficients for the two processes. It is straightforward to evaluate all functions involved at one loop: at this level, taking the limit \( \epsilon \to 0 \) and comparing the result with the real contribution to eq. (12) one finds immediately

\[ D^{(1)} = 4B_\delta^{(1)} - 2\tilde{G}^{(1)} = 0, \tag{13} \]

a well-known result.

At two loops, the relevant DIS information can be extracted from [16,17], while the second order coefficient of the function \( \tilde{G}(\alpha_s) \) was computed in [14]. The finiteness of eq. (14) leads to

\[ D^{(2)} = 4B_\delta^{(2)} - 2\tilde{G}^{(2)} - \frac{b_0 F_{\text{MS}}^{(1)}}{2} \]

\[ = \left(-\frac{101}{27} + \frac{11}{3}\zeta(2) + \frac{7}{2}\zeta(3)\right) C_AC_F \]

\[ + \left(\frac{14}{27} - \frac{2}{3}\zeta(2)\right)n_f C_F. \tag{14} \]

This result was previously obtained in [21,22], by comparing the results of resummation with the two-loop calculation of Ref. [19], along the lines of [20]. One can, finally, push the calculation to the three-loop level, thanks to the remarkable results of [21,22], where the three-loop nonsinglet splitting functions and form factor were explicitly computed. The result is

\[ D^{(3)} = 4B_\delta^{(3)} - 2\tilde{G}^{(3)} - b_0 F_{\text{MS}}^{(2)} - \frac{b_1 F_{\text{MS}}^{(1)}}{2} \]

\[ = \left(-\frac{297029}{23328} + \frac{6139}{324}\zeta(2) - \frac{187}{60}\zeta^2(2) \right. \]

\[ + \frac{2509}{108}\zeta(3) - \frac{11}{6}\zeta(2)\zeta(3) - 6\zeta(5) \right) C_A^2 C_F \]

\[ + \left(\frac{31313}{11664} - \frac{1837}{324}\zeta(2) \right. \]

\[ + \frac{23}{30}\zeta^2(2) - \frac{155}{36}\zeta(3) \right) n_f C_A C_F \]

\[ + \left(\frac{1711}{864} - \frac{1}{2}\zeta(2) - \frac{1}{5}\zeta^2(2) - \frac{19}{18}\zeta(3) \right) n_f C_F^2 \]

\[ + \left(-\frac{58}{729} + \frac{10}{27}\zeta(2) + \frac{5}{27}\zeta(3) \right) n_f^2 C_F. \tag{15} \]

The coefficient \( D^{(3)} \) was computed simultaneously and independently by [23], and the result later checked with different methods in [24,25].

The expressions we have derived for the perturbative coefficients of the function \( D(\alpha_s) \) up to three loops, in terms of \( B_\delta(\alpha_s) \), \( \tilde{G}(\alpha_s) \) and \( F_{\text{MS}} (\alpha_s) \) are strongly suggestive of a simple all-order relation. Since it is well-known that the function \( A(\alpha_s) \) coincides with (one half of) the cusp anomalous dimension \( \gamma_K (\alpha_s) \), one finds that in fact threshold resummation for the Drell-Yan process, at \( g \) loops in the exponent, is completely determined by DIS data at \( g \) loops, plus the knowledge of \( N \)-independent terms for Drell-Yan at \( g - 1 \) loops. The all-order results are

\[ A(\alpha_s) = \gamma_K (\alpha_s)/2, \]

\[ D(\alpha_s) = 4 B_\delta (\alpha_s) - 2 \tilde{G}(\alpha_s) \]

\[ + \beta(\alpha_s) \frac{d}{d\alpha_s} F_{\text{MS}} (\alpha_s). \tag{16} \]
A formally identical relation ties together the gluon annihilation cross section with the singular terms in the gluon splitting function and with the gluon form factor. Remarkably, up to three loops, the perturbative coefficients of the functions $A$ and $D$ in the two cases differ only by the replacement of the overall factor of $C_F$ for quarks with a factor $C_A$ for gluons. In other words, to this order the resummation is only sensitive to the color representation of the Wilson lines replacing the hard annihilating partons in the soft approximation. This simple replacement rule, however, is not expected to hold at yet higher orders.

4. PERSPECTIVE

We have given a short review of some of the results that can be obtained tackling threshold resummation with the tools provided by factorization and dimensional regularization. From a phenomenological viewpoint, perhaps the most interesting result is the calculation of $D(3)$, which contributes to resummation for EW annihilation at the N$^\text{3}L$L level. In fact, the only missing contribution in order to resum exactly to that accuracy is the four-loop cusp anomalous dimension $\gamma_K^{(4)}$, which in principle lies close the current boundaries of computability. In any case it can be convincingly shown that $\gamma_K^{(4)}$ makes a numerically negligible contribution to the cross section. Having at our disposal, with a good approximation, four towers of logarithms for both DIS and EW annihilation, we can stringently test the level of convergence of the perturbative expansion, both with and without resummation. Preliminary tests show that resummed perturbation theory converges well across much of the kinematical range relevant for Tevatron and the LHC.

REFERENCES

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