Fixed points of Higher Derivative Gravity

Alessandro Codella

Dipartimento di Fisica Teorica, Università di Trieste, Viale B. Marzolo, I-34140 Trieste, Italy

Roberto Percacci

SISSA, via Beirut 4, I-34140 Trieste, Italy, and INFN, Sezione di Trieste, Italy

We recalculate the beta functions of higher derivative gravity in four dimensions using the one-loop approximation to an Exact Renormalization Group Equation. We reproduce the beta functions of the dimensionless couplings that were known in the literature but we find new terms for the beta functions of Newton’s constant and of the cosmological constant. As a result, the theory appears to be asymptotically safe at a non-Gaussian Fixed Point, rather than perturbatively renormalizable and asymptotically free.

PACS numbers: 04.60.-m, 11.10.Hi

It has been proven long ago that a generalization of Einstein’s theory containing four derivatives of the metric is renormalizable in flat space perturbation theory [1]. It was then established in a series of papers [2-4] that the dimensionless couplings of this theory are asymptotically free; the beta functions of the dimensionful couplings (Newton’s constant G and the cosmological constant Λ) are gauge-dependent, but the beta function of the dimensionless product AG is not, and this variable has also been claimed to be asymptotically free, justifying the use of flat space perturbation theory. For a review of higher derivative gravity see [5]; for the state of the art see [6]. A perturbatively renormalizable and asymptotically free Quantum Field Theory (QFT) holds to arbitrarily high energy scales, and therefore this can be regarded as a serious candidate for a fundamental theory of quantum gravity. Unfortunately, it is not entirely free of problems. The most notorious is the apparent lack of unitarity: the “bare” action contains massive negative-norm states (ghosts) at tree level. It was pointed out in [2,7] that these ghosts may not correspond to physical particles when quantum effects are taken into account. However, there exists to date no convincing proof that this happens. Another, less well-known problem is that asymptotic freedom of AG requires the choice of the unstable fixed point -5.467 for the dimensionless coupling ω of eq. (1) below, see [5]. Pending progress on these issues, higher derivative gravity does not seem to have gained wide acceptance.

In a parallel development, it was observed that in order to be predictive and to hold to arbitrarily high energies, it is enough that a theory admits a Fixed Point (FP) with a finite number of UV-attractive directions. This behaviour is called “asymptotic safety” [8]. A perturbatively renormalizable and asymptotically free theory is a special case of asymptotically safe theory, where the FP is the Gaussian FP (a free theory). More general asymptotically safe theories will be based on nontrivial FPs.

The hypothesis that gravity could be asymptotically safe has been investigated thoroughly within Einstein’s theory. The first positive results were found in 2 + ⅇ dimensions [8,9]; more recently, using an Exact Renormalization Group Equation (ERGE), the existence of a nontrivial FP has been established in four dimensions for a truncation of the action containing the cosmological and Einstein–Hilbert terms [10,11], also in the presence of matter fields [12,13]. Independent evidence comes from Monte Carlo simulations [14,15]. So far only partial results are known for higher-derivative terms [16,13].

The behaviour of A and G in this approach is quite different from the one predicted in the literature on higher derivative gravity. We have recalculated the beta functions of higher derivative gravity, starting from the one-loop approximation of the ERGE. We find some important modifications in the beta functions of Newton’s constant and of the cosmological constant, in such a way that the theory appears to be asymptotically safe at a nontrivial FP, rather than at the Gaussian FP. We report here the main results; details will be given elsewhere.

A general (Euclidean) theory containing terms quadratic in curvature has an action of the form

\[
\int d^4x \sqrt{g} \left[ 2Z_\Lambda - Z R + \frac{1}{2\Lambda} C^2 - \frac{\omega}{3\Lambda} R^2 + \frac{\theta E}{\Lambda} \right] , \quad (1)
\]

where \(Z = 1/16\pi G\), \(C^2\) is the square of Weyl’s tensor, \(E\) is the integrand in Euler’s topological invariant \(\chi = \int dx \sqrt{g} E\). We neglect the total derivative \(\nabla^2 R\).

For a quantum treatment, this action has to be supplemented by the gauge-fixing term, which is chosen to be of the form

\[
S_{GF} = \int d^4x \sqrt{g} \chi_{\mu} Y^{\mu\nu} \chi_{\nu} \quad (2)
\]

where \(\chi_{\nu} = \nabla^\mu h_{\mu\nu} + \beta \nabla_{\nu} h\) (all covariant derivatives are with respect to the background metric) and \(Y^{\mu\nu} = \frac{\Lambda}{2\Lambda^2} \Lambda^{(\mu\nu)} - \frac{1}{2} \Lambda_{(\mu\nu)}\).
\[ \frac{1}{2} \left[ g^{\mu\nu} \nabla^2 + \gamma \nabla^\mu \nabla^\nu - \delta \nabla^\mu \nabla^\nu \right]. \]

The ghost action contains the term

\[ S_c = \int d^4x \sqrt{g} \left( \Delta_{gh} \right)^{\mu\nu} \partial_{\mu} \partial_{\nu} \]

where \((\Delta_{gh})^{\mu\nu} = -\delta^{\mu\nu} \Box - (1 + 2\beta) \nabla^\mu \nabla^\nu + R^\mu_{\nu}^H\) as well as a term

\[ S_b = \frac{1}{2} \int d^4x \sqrt{g} b^\mu b^\nu \]

due to the fact that the gauge averaging operator \(Y\) depends nontrivially on the metric. We follow earlier authors in choosing the gauge fixing parameters \(\alpha, \beta, \gamma\) and \(\delta\) in such a way that the quadratic part of the action is:

\[ (\Gamma_k + S_{GF})^{(2)} = \frac{1}{2} \int d^4x \sqrt{g} \delta K \Delta^{(4)} \delta g \]

where \(\Delta^{(4)} = 1 \Box^2 + V^{\rho\lambda} \nabla^\rho \nabla^\lambda + U\). For details of the operators \(K, V\) and \(U\) we refer the reader to [6], whose notation we mostly follow.

The main tool in deriving nonperturbative information about the theory is the gravitational ERGE [17]

\[ \partial_t \Gamma_k = \frac{1}{2} \text{Tr} \left( \frac{\delta^2 (\Gamma_k + S_{GF})}{\delta \delta g} + R_k^k \right)^{-1} \partial_t R_k^k - \frac{1}{2} \text{Tr} \frac{\delta^2 S_b}{\delta \delta b} R_k^k \]

where \(\Gamma_k\) is a coarse-grained effective action depending on a momentum scale \(k\) and the kernels \(R_k\) act as infrared cutoffs.

In order to derive the beta functions of the couplings \(\hat{\Lambda} = k^{-2} \Lambda, \hat{G} = k^2 G, \lambda, \omega\) and \(\theta\), we assume for \(\Gamma_k\) the form (1) and insert it, together with the gauge-fixing and ghost terms (2,3,4), into the ERGE. Then, to calculate the r.h.s. of the ERGE we choose the cutoffs as follows:

\[ R_k^k(\Delta^{(4)}) = KR_k^k(\Delta^{(4)}), \quad R_k^k(\Delta_{gh}) = 1R_k^{(2)}(\Delta_{gh}), \quad R_k^k(Y) = 1R_k^{(2)}(Y), \]

where \(R_k^{(n)}(z)\) is a suitable profile function chosen to suppress the propagation of field modes with momenta below \(k\). We will use the so-called optimized cutoff [18] \(R_k^{(n)}(z) = (ak^n - z)\theta(ak^n - z)\), with \(a = 1\) unless otherwise stated.

We restrict ourselves to the one-loop approximation, which in the context of the ERGE consists of taking into account only the explicit dependence of \(R_k(z)\) on \(k\), neglecting the implicit dependence due to the presence of running couplings in the cutoff function. (In the case of the Einstein–Hilbert action, where the r.h.s. of the ERGE can be computed exactly, it is known that this approximation does not change the general behaviour.)

The traces are expanded in integrals of powers of the curvature using the heat kernel expansion coefficients given in [19]. This procedure provides a logically and computationally independent derivation of the beta functions.

The beta functions of the dimensionless couplings appearing in (1) turn out to be:

\[ \beta_\Lambda = -2\hat{\Lambda} + \frac{1}{(4\pi)^2} \frac{133}{10} \lambda^2, \]
\[ \beta_\omega = - \frac{1}{(4\pi)^2} \frac{25 + 1098 \omega + 200 \omega^2}{60} \lambda, \]
\[ \beta_\theta = \frac{1}{(4\pi)^2} \frac{7(56 - 171 \theta)}{90} \lambda. \]

They agree with those calculated in dimensional regularization [4-6]. The coupling \(\lambda\) has the usual logarithmic approach to asymptotic freedom, while the other two couplings have the FP values \(\omega_* \approx (-5.467, -0.0228)\) and \(\theta_* \approx 0.327\). Of the two roots for \(\omega\), the first turns out to be UV-repulsive, so the second has to be chosen [4-6].

The beta functions of \(\hat{\Lambda}\) and \(\hat{G}\) are:

\[ \beta_\Lambda = -2\hat{\Lambda} + \frac{1}{(4\pi)^2} \frac{1 + 20\omega^2}{256\pi G \omega^2} \lambda^2 + \frac{1 + 86\omega + 40\omega^2}{12\omega} \lambda A \]

\[ \beta_\hat{G} = 2\hat{G} - \frac{1}{(4\pi)^2} \frac{3 + 26\omega - 40\omega^2}{12\omega} \lambda \hat{G} - q(\omega) \hat{G}^2. \]

where \(q(\omega) = (83 + 70\omega + 8\omega^2)/18\pi\). The first two terms in each beta function exactly reproduce the results of
[4–6], the remaining ones are new. The origin of the new terms can be easily understood: the old terms were originally derived as coefficients of 1/ε poles in dimensional regularization, which correspond to logarithmic divergences in the effective action and in the heat kernel expansion, which we use, are given by the $B_4$ coefficient. The new terms come from the $B_2$ and $B_0$ coefficients, which in a conventional calculation of the effective action would correspond to quadratic and quartic divergences. Dimensional regularization is ill-suited to compute these terms. It is important to stress that our “Wilsonian” calculation of the beta functions does not require any UV regularization. Then, the only ambiguity is in the choice of the cutoff functions, but no reasonable choice could remove the $B_2$ and $B_0$ terms.

To picture the flow of $\Lambda$ and $\bar{G}$, we set the remaining variables to their FP values $\omega = \omega_*$, $\theta = \theta_*$, and $\lambda = \lambda_*$ = 0. Then, defining $q_* = q(\omega_*) \approx 1.440$ the flow equations (8) can be solved analytically:

\[
\dot{\Lambda}(t) = \frac{(2\pi\Lambda_0 - \bar{G}_0(1 - e^{4t})e^{-2t})}{\pi(2 - q_*G_0(1 - e^{2t}))},
\]

\[
\dot{\bar{G}}(t) = \frac{2\bar{G}_0e^{2t}}{2 - q_*\bar{G}_0(1 - e^{2t})}.
\]

The resulting flow in the $(\Lambda, \bar{G})$-plane is shown in Fig.1. It has two FPs: the Gaussian FP at $\Lambda = \bar{G} = 0$ and another one at

\[
\Lambda_* = \frac{1}{\pi q_*} \approx 0.221, \quad \bar{G}_* = \frac{2}{q_*} \approx 1.389.
\]

The attractivity properties of these FPs are determined by the stability matrix

\[
M_{ij} = \frac{\partial \dot{\bar{G}}}{\partial \phi_i} = \begin{pmatrix} -2 - q_*\bar{G} & 0 \\ 2 - 2q_*\bar{G} & 2 \end{pmatrix}
\]

At the Gaussian FP the eigenvalues of $M$ are $(-2, 2)$; the attractive eigenvector points along the $\Lambda$ axis and the repulsive eigenvector has components $(1, 2\pi)$. As expected on general grounds [8], the eigenvalues are the opposite of the canonical dimensions of $\Lambda$ and $G$. At the non-Gaussian FP the eigenvalues of $M$ are $(-4, -2)$ with the same eigenvectors as before. The FP given by (10) is UV-attractive in all five couplings.

There is a “critical” trajectory joining the Gaussian to the non-Gaussian FP, which is tangent to the repulsive eigenvector in the origin and is actually given by $\dot{\bar{G}}(t) = 2\pi\Lambda(t)$ for all $-\infty < t < \infty$.

From this calculation one can derive some physical predictions. The first is the UV-limit of the cosmological constant in Planck units $\Lambda G = \Lambda\bar{G}$, which is well known to be gauge-independent and is also independent of the cutoff parameter $a$. In contrast to [2–6], we find that $\Lambda G$ tends to the finite value $2/(\pi q_*^2) \approx 0.307$. Of course this is an asymptotic value and to compare it with the value which is observed in cosmology one would have to run the RG down to extremely low values of $k$.

Another prediction is the asymptotic value $-2\omega_*/3 \approx 0.0152$ for the ratio between the coefficients of $R^2$ and $G^2$. It is interesting to observe that the flow induced by a large number $N$ of minimally coupled matter fields gives for this ratio the value $5n_S/(3n_S + 18n_D + 36n_M)$, where $n_S$, $n_D$ and $n_M$ are the numbers of scalar, Dirac and gauge fields [13]. This number is also quite small in realistic unified theories. Thus both with and without matter it seems that, in the UV limit, fluctuations of the conformal tensor will be more suppressed than fluctuations of the Ricci tensor.

The flow that we find here is almost identical to the flow obtained in the large $N$ limit [13], where the coefficients $\omega_*$, $\theta_*$ and $q_*$ are determined by $n_S$, $n_D$ and $n_M$. A remarkable feature of the large $N$ limit, in conjunction with the use of optimized cutoffs, is that all higher powers of curvature are absent at the FP. This raises the hope that asymptotically safe gravity may be describable by a finite number of terms in the action (generically, one would expect to have infinitely many terms, with relations between the coefficients such that only a finite number of parameters is left arbitrary).

Our flow is also similar to the one obtained in the Einstein–Hilbert truncation [11], where, however, the critical exponents at the non-Gaussian FP are complex, resulting in a spiralling approach to the FP. This similarity may be somewhat surprising, because in the Einstein–Hilbert truncation the higher derivative terms are absent while they dominate the dynamics. To some extent it can be understood by the following argument. In gravity at low energies the couplings do not run, and therefore the relative importance of the terms in the action can be determined simply by counting the number of derivatives of the metric. For example, at low momenta $p \ll \sqrt{Z}$ (recall that $Z$ is the square of the Planck mass), the terms in the action (1) with four derivatives are suppressed relative to the term with two derivatives by a factor $p^2/Z$. This is not the case in the FP regime: if we consider

![Figure 1: The flow in the $(\Lambda, \bar{G})$-plane](image-url)
phenomena occurring at an energy scale \( p \), then also the
couplings should be evaluated at \( k \approx p \). If there is a
nontrivial FP, \( Z \) runs exactly as \( p^2 \) and therefore both
terms are of order \( p^4 \). This is just a restatement of the
fact that in the Einstein–Hilbert truncation the graviton
has an anomalous dimension equal to two, making its
propagator behave like \( p^{-4} \) at high energy.

Partial results for the four-derivative couplings, but
going beyond one loop, have been derived using the
ERGE in \cite{16}. Using a spherical background, where
\( \int d^4x \sqrt{g} C^2 = 0 \), \( \int d^4x \sqrt{g} R^2 = 384 \pi^2 \) and \( \chi = 2 \), the
beta function of the combination \(-\frac{\chi}{3} + \frac{1}{192 \pi^2} \xi \) can be
derived. In the absence of further input it is impossible to
disentangle the beta functions of the individual couplings.
Nevertheless, this provides valuable information.
In particular, since a finite FP-value was found for a
combination of couplings, this calculation suggests that
the asymptotic freedom of \( \lambda, \lambda / \omega \) and \( \lambda / \theta \) that we find
here may be only a feature of the approximations that
we made, and that in a more accurate calculation some
or all of these coefficients will reach finite values instead
of running logarithmically. One also expects, as in \cite{16},
that the degeneracy of the stability matrix is lifted and
that all couplings are either relevant or irrelevant.

To summarize, we have found that higher derivative
gravity has a fixed point with the following properties
at one loop: \( \Lambda \) and \( G \) are nonzero and UV-relevant,
while the couplings of the terms quadratic in curvature
are asymptotically free and marginal. Experience with
the Einstein–Hilbert truncation suggests that the FP will
persist in the exact (as opposed to one loop) treatment,
up to a finite shift of the FP-values of the couplings,
and of the critical exponents. The Gaussian FP is unstable:
even an infinitesimal value for \( G \) will generate a nonvanishing \( \Lambda \) and push the system towards the other FP.

Among other things, these results solve the second of
the problems mentioned in the introduction. Concerning
the issue of unitarity, we can say, from our Wilsonian
point of view, that the presence of ghost poles at the
Planck scale has to be assessed by considering the action
\( \Gamma_k \) for \( k \approx m_{\text{Planck}} \), which is probably very different from
the FP action. Thus, tree level analyses of the FP action
are of little significance, as already pointed out in \cite{2,3,7}.

In view of these results we think that higher derivative
gravity deserves renewed attention.

Acknowledgements

RP would like to thank M. Reuter for useful conversations.
We would also like to thank G. de Berredo-Peixoto
and I. Shapiro for correspondence on their work.

References