Initial States: IR and Collinear Divergences

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The standard approach to the infra-red problem is to use the Bloch-Nordsieck trick to handle soft divergences and the Lee-Nauenberg (LN) theorem for collinear singularities. We show that this is inconsistent in the presence of massless initial particles. Furthermore, we show that using the LN theorem with such initial states introduces a non-convergent infinite series of diagrams at any fixed order in perturbation theory.

1. Introduction

The physical origin of the infra-red (IR) problem has long been understood: in theories with massless particles the interactions do not fall off quickly enough. Ignoring this problem, e.g., by talking about adiabatically switching off interactions, generates IR (soft and collinear) divergences in S-matrix elements.

The primary theoretical responses are twofold. For soft divergences it is widely argued that one should use the Bloch-Nordsieck (BN) trick: calculate semi-inclusive cross-sections where one sums over all emitted soft photons with energy less than some experimental energy resolution $\Delta$. If some charged fields are massless (QCD or QED if the electron mass is taken to vanish) there are also collinear divergences: then generally BN does not work and, following the work of Kinoshita\cite{Kinoshita1962} and the quantum mechanical theorem of Lee and Nauenberg\cite{LeeNauenberg1964}, it is argued that semi-inclusive cross-sections with a sum over both initial and final particles within some experimental angular resolution $\delta$ will be finite.

In practice it is most common to solely calculate infra-red safe quantities, the most famous being $g - 2$. If necessary the BN trick is used while KLN with both initial and final states is very rarely considered.

We will show that the division of just including virtual plus emission for soft divergences but virtual, emission and absorption for collinear is inconsistent. Furthermore, using the KLN theorem for initial and final soft particles leads to an infinite series of diagrams at any fixed order in perturbation theory. To be concrete we consider Coulomb scattering.

1.1. Soft and Collinear Divergences

At one loop virtual photons produce the following soft and collinear singularities in the S-matrix:

\[ \frac{1}{\epsilon}, \frac{1}{\epsilon} \ln(m), \ln^2(m), \ln(m). \]

(1)

Here dimensional regularisation ($\epsilon$) regulates the soft divergences and a small electron mass ($m$) regulates the collinear singularities. (Full results are given in\cite{Lavelle2006}, here we want to show the structure of all cancellations and non-cancellations: hence we will only give the divergent structures and often drop overall factors.)

The BN trick is to integrate over the emission of soft photons with energies less than some experimental resolution $\Delta$. These soft photons are emitted in all directions. Generally one works in the eikonal approximation – i.e., dropping higher powers of the photon momentum in the numerator. The addition of this cross-section to the one-loop cross-section which follows from results in the cancellation of all soft divergences $1/\epsilon$, $\ln(m)/\epsilon$ and introduces a dependence on $\Delta$ into the effective cross-section. The BN trick also cancels the leading collinear logs, $\ln^2(m)$. However, the sub-leading collinear logs are not can-
celled and we are left, up to overall factors, with
\[- \ln(m) \times \left[ \frac{3}{4} - \ln \left( \frac{E}{\Delta} \right) \right]. \tag{2}\]

Thus we have to go beyond the BN trick.

The standard argument now is that when the electron is almost massless it cannot be distinguished from an electron accompanied by non-soft photons emitted almost parallel to it (within an experimental angular resolution \(\delta\)). Hence one should also add semi-hard photon emission where photons parallel to the outgoing fermion share the total energy, \(E\). The diagram below thus produces a collinear divergence

\[\sim \quad \sim\]

This contributes
\[+ \frac{1}{2} \ln(m) \times \left[ \frac{3}{4} - \ln \left( \frac{E}{\Delta} \right) \right]. \tag{3}\]

The KLN idea is to include absorption of semi-hard collinear photons, so doubling the result of (3) and cancelling (2).

This seems unnatural (including soft and semi-hard collinear emission but only semi-hard absorption, i.e., no soft absorption), however, there are other terms which must be taken into account. A careful calculation yields for the contribution from semi-hard collinear photon emission:
\[- \frac{1}{2} \ln(m) \times \left[ \frac{3}{4} - \ln \left( \frac{E}{\Delta} \right) - \frac{\Delta}{E} + \frac{1}{4} \frac{\Delta^2}{E^2} \right], \tag{4}\]

where \(\Delta\) is the experimental energy resolution. What can cancel these additional collinear divergences? (We stress that it is not allowed to set \(\Delta\) to zero as the usual claim would then be that the cross-section vanishes.)

It turns out that they are artifacts of the divide between soft and semi-hard divergences. If in soft emission we go beyond the eikonal approximation one generates additional terms which, by power counting, must be soft finite but produce exactly these additional collinear logs. Hence the collinear divergent contribution to the emission cross-section is (4) rather than (3) but only upon inclusion of emission of those soft photon terms which do not produce soft divergences – the non-eikonal \(k\) terms in the intermediate fermion numerator, \(p + k + m\) – plus semi-hard emission.

This is, though, still half the result needed to cancel (2) and so a more precise statement of BN/KLN cancellation for Coulomb scattering is: add all soft emission (cancels the soft divergences and some collinear), add semi-hard emission and semi-hard absorption (to cancel the collinear logs in eq. 2) and also add soft absorption diagrams but only retaining those terms which do not generate soft divergences and are needed to cancel the extra collinear logs in eq. 2. This is clearly unphysical and shows that such a divide between BN and KLN is unacceptable.

It is thus natural to look for an approach to soft and collinear singularities in the spirit of KLN where one includes all initial and final degeneracies, i.e., including all soft photon absorption. As this will introduce soft divergences, we have to ask what can remove them. For the rest of this paper we will purely consider soft divergences and study them in the spirit of KLN.

1.2. Emission and Absorption

Since we already have all order \(e^4\) contributions to the cross-section from emission and from absorption, we now include all possible diagrams with emission and absorption. Two of them are

The apparent problem is that such diagrams are already at order \(e^3\). However, as noted by Lee and Nauenberg (see Appendix D in [2]), there is, at the level of the cross-section a connected interference contribution with the following diagram which contains a disconnected line at the level of the S-matrix:
Such contributions appear rather unfamiliar but are essential for the cancellation in Lee and Nauenberg and have been considered by a variety of authors since their paper [4–8].

These diagrams indeed produce a soft infra-red divergence. However, it does not cancel the other singularities. Rather, and closely following [2], one adds either emission plus a disconnected photon or alternatively absorption plus a disconnected photon, i.e., diagrams like

When such diagrams are squared up, one obtains both connected interference contributions

and disconnected contributions

If following LN one drops the disconnected terms and retains the connected ones, the soft divergences cancel. Since this is so important let us be explicit about how the cancellation (up to a common factor) takes place by listing the various soft contributions which sum to zero:

<table>
<thead>
<tr>
<th>virtual</th>
<th>emit</th>
<th>absorb</th>
</tr>
</thead>
<tbody>
<tr>
<td>$- \frac{1}{\epsilon}$</td>
<td>$+ \frac{1}{\epsilon}$</td>
<td>$+ \frac{1}{\epsilon}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>emit &amp; abs.</th>
<th>abs. plus disconn.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$- \frac{2}{\epsilon}$</td>
<td>$+ \frac{1}{\epsilon}$</td>
</tr>
</tbody>
</table>

This is essentially the method used by Lee and Nauenberg and more or less followed in [4, 6, 7]. However, is this cancellation physically meaningful? After all one could have added either absorption plus a disconnected line or emission plus a disconnected line. Indeed one could have included more than one disconnected line at the same order in perturbation theory! It is thus important to study such diagrams.

2. Many Disconnected Lines

There are, already at this order of perturbation theory, infinitely many connected interference contributions from the addition of disconnected lines. Indeed only the virtual loop diagrams, with no real emitted or absorbed photons, do not have such connected interference contributions. Hence such interference terms need to be taken into account in a consistent manner. To the best of our knowledge this has been only seriously considered by Ito [5] and by Akhoury, Sotiropoulos and Zakharov [8] (whose proposal is essentially identical to that of Ito).

Their idea is to include all possible disconnected photons and combine them into connected and disconnected contributions to the cross-section. Their claim is that the disconnected terms factor out and that the connected terms combine in a way such that the soft divergences cancel. To summarise their idea consider the sum of probabilities $P_{mn}$

$$
\sum_{mn} (e + m \text{ soft photons} \rightarrow e + n \text{ soft photons}). (5)
$$

At order $e^4$ one has to include the following probabilities $P_{00}$ (virtual loop); $P_{01}$ (emission of a soft photon); $P_{10}$ (absorption of a soft photon); $P_{11}$ (both emission and absorption of a soft photon). Their assertion is that if one sums over all

$$\text{Total prob.} = \sum_{mna} D(m - a, n - a)! \frac{1}{(m - a)! (n - a)!} \times \left[ P_{00} + P_{01} + P_{10} + (P_{11} - P_{00}) \right]. (6)$$

The terms in the square bracket are the sum of connected diagrams, $D(m - a, n - a)$ are the factorised disconnected terms and the factorials are combinatorial factors.

It is then argued that soft divergences cancel in the square bracket and the disconnected terms will cancel by normalisation. However, it seems very strange that the virtual terms $P_{00}$ cancel in (6) since for the virtual loop diagrams any disconnected photons will automatically be factorised as disconnected loops in the cross-section (there is nothing for them to connect to!)

$^1$Disconnected at the level of cross-sections as in the last diagram drawn above.
Closer inspection shows that this is a result of writing the virtual terms as a difference of two infinite series. These series can be shown not to converge. Consider the diagram formed from interference with disconnected photons (the last diagram but one above). The initially disconnected lines are essentially just delta functions and they may be ‘unravelled’, i.e., their contribution is exactly equal to that of the diagram

without the disconnected line. In fact the combinatorial factors all cancel in these connected terms. Hence in the apparent factorial suppression of higher terms is indeed only apparent.

To restate this result: for a diagram that contributes a soft divergence of $1/\epsilon$, the same diagram with one disconnected photon will, in the connected cross-section, also contribute $1/\epsilon$; further every additional disconnected soft photon will add another $1/\epsilon$. Hence we have to combine infinite, non-converging series and the result is not well defined.

We conclude that this line of argument is not safe. Indeed, as shown in [3], it can be used to argue that tree level scattering vanishes. (At order $\epsilon^2$ there is only the probability $P_{00}$ and in a similar fashion to these terms can be argued to cancel.)

3. Conclusions

We have seen that in the presence of initial and final state degeneracies it is not allowed to use the Bloch-Nordsieck trick to handle soft divergences and the Lee-Nauenberg theorem to treat collinear ones. This led us to consider the use of disconnected diagrams which generate connected contributions at the level of the cross-section. Although it is possible to produce an apparent cancellation by only including sufficient degeneracies to do it, such a truncation is not in any way justified. There are in fact infinitely many contributions from disconnected photons at any fixed order in perturbation theory. Previous attempts to sum all possible diagrams and factor out the disconnected terms were shown to be unsafe: the infinite series do not converge and are not well defined.

We note that diagrams with emission and absorption on the same leg produce IR finite double pole terms on that leg. It would be interesting to see if this can be understood as a (finite) mass shift.

It would seem attractive to consider different initial and final states (based upon coherent states perhaps). Such work would need to render the series convergent and cancel the soft divergences. It is not immediately clear how this works and this is under investigation. A completely different approach will be the subject of D. McMullan’s talk at this conference.

We conclude that there does not exist a good understanding of the cancellation of soft divergences (or forward collinear divergences [3]) in perturbation theory and that more work in this area is urgently required.

Acknowledgments

It is a pleasure to thank E. Bagan for discussions on this topic. ML also thanks the Université de Montpellier II for hospitality before the meeting and P. Grangé for related discussions.

REFERENCES