Accelerating Universe and Cosmological Perturbation in the Ghost Condensate

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Abstract

In the simplest Higgs phase of gravity called ghost condensation, an accelerating universe with a phantom era \((w < -1)\) can be realized without ghost or any other instabilities. In this paper we show how to reconstruct the potential in the Higgs sector Lagrangian from a given cosmological history \((H(t), \rho(t))\). This in principle allows us to constrain the potential by geometrical information of the universe such as supernova distance-redshift relation. We also derive the evolution equation for cosmological perturbations in the Higgs phase of gravity by employing a systematic low energy expansion. This formalism is expected to be useful to test the theory by dynamical information of large scale structure in the universe such as cosmic microwave background anisotropy, weak gravitational lensing and galaxy clustering.
1 Introduction

Acceleration of the cosmic expansion today is one of the greatest mysteries in both cosmology and fundamental physics [1, 2]. Assuming that Einstein’s general relativity is the genuine description of gravity all the way up to cosmological distance and time scales, the so called concordance cosmological model requires that about 70% of our universe should be some sort of energy with negative pressure, called dark energy. However, since the nature of gravity at cosmological scales has never been probed directly, we do not know whether the general relativity is really correct at such infrared (IR) scales. Therefore, it seems natural to consider modification of general relativity in IR as an alternative to dark energy. Dark energy, IR modification of gravity and their combination should be tested and distinguished by future observations and experiments [3, 4, 5, 6, 7].

From the theoretical point of view, however, IR modification of general relativity [8, 9, 10, 11, 12, 13, 14, 15, 16, 17] is not an easy subject. Most of the previous proposals are one way or another scalar-tensor theories of gravity, and are strongly constrained by e.g. solar system experiments [18] and the theoretical requirement that ghosts be absent [19, 20, 21]. The massive gravity theory [22] and the Dvali-Gabadadze-Porrati (DGP) brane model [23] are much more interesting IR modification of gravity, but they are known to have macroscopic UV scales [24, 25]. A UV scale of a theory is the scale at which the theory breaks down and loses its predictability. For example, the UV scale of the 4D general relativity is the Planck scale, at which quantum gravity effects are believed to become important. Since the Planck scale is microscopic, the general relativity maintains its predictability at essentially all scales we can directly probe. On the other hand, in the massive gravity theory and the DGP brane model, the UV scale is macroscopic. For example, if the scale of IR modification is the Hubble scale today or longer then the UV scale would be $\sim 1,000km$ or longer. At the UV scale an extra degree of freedom, which is coupled to matter, becomes strongly coupled and its quantum effects cannot be ignored. This itself does not immediately exclude those theories, but means that we need UV completion in order to predict what we think we know about gravity within $\sim 1,000km$. Since this issue is originated from the IR modification and the extra degree of freedom cannot be decoupled from matter, it is not clear whether the physics in IR is insensitive to unknown properties of the UV completion. In particular, there is no guarantee that properties of the IR modification of gravity will persist even qualitatively when the theories are UV completed in a way that they give correct predictions about gravity at scales between $\sim 1,000km$ and $\sim 0.1mm$. 

Ghost condensation is an analogue of the Higgs mechanism in general relativity and modifies gravity in IR in a way that avoids the macroscopic UV scale \[^{1}\] \[^{2}\]. In ghost condensation the theory is expanded around a background without ghost and the low energy effective theory has a universal structure determined solely by the symmetry breaking pattern. While the Higgs mechanism in a gauge theory spontaneously breaks gauge symmetry, the ghost condensation spontaneously breaks a part of Lorentz symmetry \[^{2}\] since this is the symmetry relevant to gravity. In a gauge theory the Higgs mechanism makes it possible to give a mass term to the gauge boson and to modify the force law in a theoretically controllable way. Similarly, the ghost condensation gives a “mass term” to the scalar sector of gravity and modifies gravitational force in the linearized level even in Minkowski and de Sitter spacetimes. The Higgs phase of gravity provided by the ghost condensation is simplest in the sense that the number of Nambu-Goldstone bosons associated with spontaneous Lorentz breaking is just one and that only the scalar sector is essentially modified.

In ghost condensation the linearized gravitational potential is modified at the length scale \(r_c\) in the time scale \(t_c\), where \(r_c\) and \(t_c\) are related to the scale of spontaneous Lorentz breaking \(M\) as

\[
r_c \simeq \frac{M_{\text{Pl}}}{M^2}, \quad t_c \simeq \frac{M^2_{\text{Pl}}}{M^3}.
\]

Note that \(r_c\) and \(t_c\) are much longer than \(1/M\). The way gravity is modified is peculiar. At the time when a gravitational source is turned on, the potential is exactly the same as that in general relativity. After that, however, the standard form of the potential is modulated with oscillation in space and with exponential growth in time. This is an analogue of Jeans instability, but unlike the usual Jeans instability, it persists in the linearized level even in Minkowski background. The length scale \(r_c\) and the time scale \(t_c\) above are for the oscillation and the exponential growth, respectively. At the time \(\sim t_c\), the modification part of the linear potential will have an appreciable peak only at the distance \(\sim r_c\). At larger distances, it will take more time for excitations of the Nambu-Goldstone boson to propagate from the source and to modify the gravitational potential. At shorter distances, the modification is smaller than at the peak position because of the spatial oscillation with the boundary condition at the origin. The behavior explained here applies to Minkowski background, but in

\[^{1}\]See e.g. [27, 28, 29, 30, 31, 32, 33] for other related proposals.

\[^{2}\]Lorentz violation in particle physics has been an active field of research. See e.g. [34, 35, 36, 37, 38] and references therein. Gravity in the presence of a vector field with fixed norm is also extensively studied [39, 40, 41, 42, 43]. For relation between this theory and the gauged ghost condensation, see Appendix of ref. [33].
ref. [26] the modification of gravity in de Sitter spacetime was also analyzed. It was shown that the growing mode of the linear gravitational potential disappears when the Hubble expansion rate exceeds a critical value \( H_c \sim 1/t_c \). Thus, the onset of the IR modification starts at the time when the Hubble expansion rate becomes as low as \( H_c \).

If we take the \( M/M_{Pl} \to 0 \) limit then the Higgs sector is completely decoupled from the gravity and the matter sectors and, thus, the general relativity is safely recovered. Therefore, cosmological and astrophysical considerations in general do not set a lower bound on the scale \( M \) of spontaneous Lorentz breaking, but provide upper bounds on \( M \). If we trusted the linear approximation for all gravitational sources for all times then the requirement \( H_c \lesssim H_0 \) would give the bound \( M \lesssim (M_{Pl}^2 H_0)^{1/3} \approx 10 \text{MeV} \), where \( H_0 \) is the Hubble parameter today [26]. However, for virtually all interesting gravitational sources the nonlinear dynamics dominates in time scales shorter than the age of the universe. As a result the nonlinear dynamics cuts off the Jeans instability of the linear theory, and allows \( M \lesssim 100 \text{GeV} \) [44].

Many other aspects of ghost condensation have been explored. They include a new spin-dependent force [45], a qualitatively different picture of inflationary de Sitter phase [46, 47], effects of moving sources [48, 49], nonlinear dynamics [50, 44], properties of black holes [51, 52, 53], implications to galaxy rotation curves [54, 55, 56], dark energy models [57, 58, 59], other classical dynamics [60, 61], attempts towards UV completion [62, 63], and so on.

In the simplest setup of the ghost condensation an exact shift symmetry is assumed and there is no potential term in the Lagrangian of the Higgs sector. As a result the Higgs sector behaves like a cosmological constant plus cold dark matter for homogeneous, isotropic background evolution [44]. If the shift symmetry is not exact but is softly broken then a shallow potential is allowed in the Higgs sector Lagrangian. Recently, Creminelli, et. al [64] showed that the ghost condensation with softly broken shift symmetry can violate the null energy condition without any instabilities \(^3\). This opens up interesting possibilities of non-standard cosmology, including an accelerating universe with \( w < -1 \).

A point is that the coefficient of the time kinetic term \( \dot{\pi}^2 \) in the low energy effective action for the scalar excitation \( \pi \) is positive and of order unity. This means that there is no ghost in the ghost condensation. This also implies that in ghost condensation there is no problem analogous to the strong coupling issues which the massive gravity theory and the DGP brane model are facing with. On the other hand, the coefficient

\(^3\)Another model which violates the null energy condition without UV instabilities is proposed in [65]. It is interesting to notice that this model also breaks Lorentz symmetry.
of the space kinetic term \((\nabla \pi)^2\) in the action becomes positive when \(w < -1\). (A usual non-ghost scalar field, with \(w > -1\), has a positive coefficient for the time kinetic term and a negative coefficient for the space kinetic term.) This does not necessarily introduce instabilities since there is also a higher-derivative space kinetic term \((\nabla^2 \pi^2)/M^2\) with a negative coefficient. Actually, as far as the positive coefficient of \((\nabla \pi)^2\) is small enough, the higher-derivative space kinetic term pushes the would-be unstable modes outside the cosmological horizon so that any instabilities do not show up. This situation is realized if the violation of the null energy condition is not too large since the positive coefficient of \((\nabla \pi)^2\) is proportional to the amount of violation of the null energy condition \[64\].

In the present paper we investigate the classical dynamics of cosmology in ghost condensation with softly broken shift symmetry in more detail. Throughout this paper we shall adopt a 4D covariant action explained in Sec. 2. In Sec. 3 we show that it is always possible to find a form of the potential in the Higgs sector Lagrangian which realizes an arbitrary FRW cosmological history \((H(t), \rho(t))\). A similar result is known to hold also for a conventional scalar field with a potential, but in this case it is impossible to violate the null energy condition without a ghost. This is the origin of the folklore that the phantom \((w < -1)\) cosmology requires a ghost, which is correct for ordinary scalars. On the other hand, for the ghost condensate, the null energy condition can be violated without introducing ghosts or any other instabilities as far as the violation is weak enough \[64\]. Thus, the folklore is not correct in ghost condensation. It is probably worth stressing here again that the low energy effective field theory of ghost condensation is completely determined by the symmetry breaking pattern and does not include any ghosts.

After showing the reconstruction method in Sec. 3 we investigate cosmological perturbation around the FRW background in Sec. 4. The resulting evolution equation is summarized in subsection 4.4 and can be used to test the theory by observational data of e.g. cosmic microwave background anisotropy, weak gravitational lensing and galaxy clustering and so on.

2 4D covariant action

Since the structure of the low energy effective field theory of ghost condensation is determined by the symmetry breaking pattern, it is not compulsory to consider a 4D covariant description for the ghost condensation as far as physics in the Higgs phase is concerned. Nonetheless, it is instructive and sometimes convenient to have such a description. In this paper we shall adopt the 4D covariant description and extend the
low energy field theory developed in ref. [26] to a general FRW background driven by not only the ghost condensate itself but also other cosmological fluids ⁴. In this language, we start with a 4D covariant action principle for a scalar field, and the ghost condensation is realized as a background with a non-vanishing derivative of the scalar field which does not vanish even in Minkowski or de Sitter spacetime. Then the low energy effective field theory is obtained by expanding the covariant action around this background. Needless to say, the two approaches, one based on the symmetry breaking pattern and the other based on the 4D covariant action, completely agree when applied to physics within the regime of validity of the effective field theory.

The leading 4D covariant action for ghost condensation is given by

\[ I = \int d^4x \sqrt{-g} \left[ \frac{M^4}{8} \Sigma^2 - M^4 V(\phi) - \frac{\alpha_1}{2M^2} (\Box \phi)^2 - \frac{\alpha_2}{2M^2} (\nabla^\mu \nabla^\nu \phi)(\nabla_\mu \nabla_\nu \phi) \right], \tag{2.1} \]

where

\[ \Sigma \equiv -\frac{\partial^\mu \phi \partial^\nu \phi}{M^4}, \tag{2.2} \]

\[ M \] is the symmetry breaking scale giving the cutoff scale of the low energy effective theory, and the sign convention for the metric is \((-+++)\). Note that the potential \(M^4 V(\phi)\) is included to take into account the soft breaking of the shift symmetry [64]: \(V(\phi)\) is assumed to depend on \(\phi\) very weakly. When \(V(\phi)\) is a constant, the shift symmetry is exact. The corresponding stress-energy tensor is easily calculated as

\[ T_{\mu \nu}^{(\phi)} = \frac{1}{2} \Sigma \partial_\mu \phi \partial_\nu \phi - \frac{\alpha_1 + \alpha_2}{M^2} \left[ \partial_\mu (\Box \phi) \partial_\nu \phi + \partial_\nu \phi \partial_\nu (\Box \phi) \right] + \frac{\alpha_2}{M^2} \left\{ \nabla^\rho \left[ (\nabla_\mu \nabla_\nu \phi) \partial_\rho \phi \right] - (R^\rho_\rho \partial_\rho \phi + R^\rho_\rho \partial_\mu \phi \partial_\rho \phi) \right\} + \left[ \frac{M^4}{8} \Sigma^2 - M^4 V(\phi) + \frac{\alpha_1}{2M^2} (\Box \phi)^2 + \frac{\alpha_1}{M^2} \partial^\rho (\Box \phi) \partial_\rho \phi \right] - \frac{\alpha_2}{2M^2} (\nabla^\rho \nabla^\sigma \phi)(\nabla_\rho \nabla_\sigma \phi) \right] g_{\mu \nu}, \tag{2.3} \]

and the equation of motion for \(\phi\) is

\[ \frac{1}{2} \nabla^\mu (\Sigma \partial_\mu \phi) - M^4 V'(\phi) - \frac{\alpha_1 + \alpha_2}{M^2} \Box^2 \phi - \frac{\alpha_2}{M^2} \left( R^{\mu \nu} \nabla_\mu \nabla_\nu \phi + \frac{1}{2} \partial^\mu R \partial_\mu \phi \right) = 0. \tag{2.4} \]

The Einstein equation is

\[ M_{\text{Pl}}^2 G_{\mu \nu} = T_{\mu \nu}^{(\phi)} + T_{\mu \nu}, \tag{2.5} \]

⁴See Sec. 3.3 of Ref. [64] for the low energy effective field theory in a FRW background driven solely by the ghost condensate itself. In the present paper we include general gravitational sources as well as the ghost condensate since the inclusion of those sources is essential for the test of the theory by e.g. cosmic microwave background anisotropy, weak gravitational lensing, galaxy clustering and so on.
where \( T_{\mu}^{\nu} \) is the stress energy tensor of the other gravitational sources satisfying the conservation equation \( \nabla_\nu T^{\nu}_\mu = 0 \). As a consistency check, it is easy to confirm that

\[
\nabla_\nu T^{(\phi)\nu}_\mu = E^{(\phi)} \partial_\mu \phi,
\]

where \( E^{(\phi)} \) is the left hand side of (2.4). Thus, the stress-energy tensor of \( \phi \) satisfies the conservation equation \( \nabla_\nu T^{(\phi)\nu}_\mu = 0 \), provided that the equation of motion \( E^{(\phi)} = 0 \) is satisfied. On the other hand, if the Einstein equation is satisfied and if \( \partial_\mu \phi \) is non-vanishing then the equation of motion \( E^{(\phi)} = 0 \) follows. Throughout this paper we shall use these expressions for the stress-energy tensor and the equation of motion.

If we set \( \alpha_1 = \alpha_2 = 0 \) then the model is reduced to a kind of k-inflation [66] or k-essence [67, 68, 69]. However, in this case there is no modification of gravity in Minkowski or de Sitter spacetime. Moreover, with \( \alpha_1 = \alpha_2 = 0 \), the attractor \( \Sigma = 0 \) is unstable against inhomogeneous perturbations. In the presence of the terms proportional to \( \alpha_1 \) and \( \alpha_2 \) (\( \alpha_1 + \alpha_2 > 0 \)), the attractor is stable against small perturbations and gravity is modified in the linearized level even in Minkowski and de Sitter backgrounds. The relation between the k-inflation and the ghost condensation is in some sense similar to that between the usual potentially-dominated inflation and the Higgs mechanism [70].

3 FRW background

Ghost condensation provides the simplest Higgs phase of gravity in which there is only one Nambu-Goldstone boson associated with spontaneous Lorentz breaking. Note again that the dynamics of the Higgs phase of gravity in ghost condensation has nothing to do with ghosts. Indeed, there is no ghost within the regime of validity of the effective field theory. Moreover, the Higgs phase of gravity has universal low energy description determined solely by the symmetry breaking pattern.

In this section we consider a flat FRW ansatz

\[
g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2 \delta_{ij}dx^i dx^j,\]

\[
\phi = \phi(t),
\]

where \( i = 1, 2, 3 \), and analyze the dynamics of the homogeneous, isotropic universe. Linear perturbation around this background will be considered in the next section. With this ansatz, the stress energy tensor for the field \( \phi \) is

\[
T^{(\phi)\nu}_\mu = \begin{pmatrix}
-\rho_{\phi} & 0 & 0 & 0 \\
0 & \rho_{\phi} & 0 & 0 \\
0 & 0 & \rho_{\phi} & 0 \\
0 & 0 & 0 & \rho_{\phi}
\end{pmatrix},
\]

6
where

\[ \rho_\phi = \frac{1}{8} M^4 (4 + 3\Sigma) \Sigma + M^4 V \]
\[ + \alpha M^2 \left[ \frac{3}{2} (1 + \Sigma)(2\partial_t H - 3H^2) + \frac{1}{2} \partial_t^2 \Sigma - \frac{3}{8} \frac{1}{1+\Sigma} \right] \]
\[ + \beta M^2 \left[ -3(1 + \Sigma)\partial_t H + \frac{3}{2} H\partial_t \Sigma \right], \]

and

\[ \rho_\phi + p_\phi = \frac{M^4}{2} (1 + \Sigma) \Sigma \]
\[ + \alpha M^2 \left[ 6(1 + \Sigma)\partial_t H + (\partial_t^2 \Sigma + 3H\partial_t \Sigma) - \frac{1}{2} \frac{(\partial_t \Sigma)^2}{1+\Sigma} \right] \]
\[ - \beta M^2 \left[ (1 + \Sigma)(3H^2 + 5\partial_t H) + \frac{1}{2} (\partial_t^2 \Sigma + H\partial_t \Sigma) \right], \quad (3.3) \]

Here, we have expressed \( \partial_t \phi, \partial_t^2 \phi \) and \( \partial_t^3 \phi \) in terms of \( \Sigma = (\partial_t \phi)^2 / M^4 - 1 \) and its derivatives. The Einstein equation is

\[ 3 M^2_{Pl} H^2 = \rho_\phi + \rho, \]
\[ 2 M^2_{Pl} \partial_t H = -(\rho_\phi + p_\phi) - (\rho + p), \quad (3.5) \]

where \( H = \partial_t a / a \), and \( \rho \) and \( p \) are the energy density and the pressure of the other gravitational sources. Following the usual convention we call the first equation the Friedmann equation and the second the dynamical equation. Note that, because of the identity (2.6) and the Bianchi identity, the equation of motion for \( \phi \) automatically follows from (3.5), provided that the conservation equation for \( \rho \) and \( p \) holds:

\[ \partial_t \rho + 3H(\rho + p) = 0. \quad (3.6) \]

3.1 Low energy expansion

At this point, one might think that the expressions in (3.3) are too complicated to extract the physical picture of cosmology in the Higgs phase of gravity. Even if one could somehow manage to do so with brute force for the FRW background, it might be too optimistic to expect that the same approach works for linear perturbation around the FRW background. So let us go back and reconsider the meaning of the covariant action (2.1). This is not a full action including a UV completion but just a leading action suitable to describe physics sufficiently below the cutoff scale \( M \).
around backgrounds with $\Sigma \simeq 0$. If the Hubble expansion rate and/or the stress-energy tensor of the field $\phi$ become close to or above unity in the unit of $M$ then the low energy description is invalidated and we need a UV completion. Needless to say, the same criterion applies to the other approach based on the symmetry breaking pattern, and the two approaches agree in the regime of validity of the low energy effective theory. Therefore, all we can and should trust is what is obtained below the cutoff scale. In other words, we assume the existence of a good UV completion but never use its properties. For this reason, we can and should ignore terms irrelevant at low energies compared with the cutoff scale $M$.

To be systematic, we adopt a low energy expansion by introducing small dimensionless parameters $\epsilon_i \ (i = 0, 1, 2, 3)$ as

$$\frac{M^2}{M_{Pl}^2} = \epsilon_0, \quad \frac{H}{M} = \epsilon_1, \quad \frac{\rho}{M_{Pl}^2 M^2} = \epsilon_2^2, \quad \frac{p}{M_{Pl}^2 M^2} = O(\epsilon_2^2),$$

and

$$\text{Max}(|\Sigma|, |V|) = \epsilon_3, \quad (\Sigma = O(\epsilon_3), \ V = O(\epsilon_3)).$$

Since the time scale for the change of $H$, $\Sigma$, $V$, $\rho$ and $p$ is expected to be the cosmological time scale $1/H$, it also follows that

$$\frac{\partial^n H}{M^{n+1}} = O(\epsilon_1^{n+1}), \quad \frac{\partial^n \Sigma}{M^n} = O(\epsilon_3^{n_1}), \quad \frac{\partial^n V}{M^n} = O(\epsilon_3^{n_1}),$$

and that

$$\frac{\partial^n \rho}{M_{Pl}^2 M^{n+2}} = O(\epsilon_2^{n_1}), \quad \frac{\partial^n p}{M_{Pl}^2 M^{n+2}} = O(\epsilon_2^{n_1}),$$

for $n = 1, 2, \cdots$. The latter condition (3.10) is not necessary for the present purpose but will be used when we derive evolution equations for linear perturbation around the FRW background in the next section. Unless fine-tuned, consistency of this assignment with the Einstein equation (3.5) requires that

$$\epsilon_1^2 \simeq \text{Max}(\epsilon_0 \epsilon_3, \epsilon_2^2).$$

It is in principle possible but not practical to perform the low energy expansion with respect to all $\epsilon$’s, considering each $\epsilon_i$ as independent small parameters. Of course, if one performs the low energy expansion up to sufficiently high order with respect to all $\epsilon$’s then one can always reach the point where all relevant terms (and probably many other irrelevant terms) are included. However, this is not the most economical way to obtain results relevant to particular situations of physical interest. It is more economical and convenient to suppose some rough relations among these
small parameters to reduce the number of independent small parameters. This also makes it easier to extract physical picture out of complicated equations. Note that those relations among $\epsilon_i$ must reflect the situations of physical interest.

For example, in ghost inflation [46] we set $\Sigma = V = 0$ and include another field, say $\psi$, to end the inflation and to reheat the universe a la hybrid inflation. In this case $\epsilon_3$ vanishes. Moreover, since the stress energy tensor is dominated by the other field $\psi$, the consistency relation (3.11) implies that $\epsilon_1 \simeq \epsilon_2$. In this way, in Ref. [46] we considered $\epsilon_0$ and $\epsilon_1 (\simeq \epsilon_2)$ as two independent small parameters.

On the other hand, in the present paper we would like to consider late time cosmology in which both $(\rho_\phi, p_\phi)$ and $(\rho, p)$ may contribute to the background as gravitational sources. For this reason and because of the consistency relation (3.11), we suppose that there is a small number $\epsilon$ such that

$$\epsilon^2 = O(\epsilon^2), \quad \epsilon_0 \epsilon_3 = O(\epsilon^2), \quad \epsilon_2 = O(\epsilon^2), \quad (\text{e.g. } \epsilon \equiv \text{Max}(\sqrt{\epsilon_0 \epsilon_3}, \epsilon_2)).$$  (3.12)

We expect (and will actually show) that in the dispersion relation for the excitation of ghost condensation, $M^2/M_{Pl}^2 (= \epsilon_0)$ and $\Sigma (= \epsilon_3)$ additively contribute to the coefficient of the momentum squared. (See the expression for $C_0$ in (4.16) below.)

We would like to consider situations in which they can independently become relevant since they may become different sources of possible instabilities and interesting physical effects [64]. Thus, we refine the second relation in (3.12) to

$$\epsilon_0 = O(\epsilon), \quad \epsilon_3 = O(\epsilon), \quad (\text{e.g. } \epsilon \equiv \text{Max}(\epsilon_0, \epsilon_2, \epsilon_3)).$$  (3.13)

In summary we can introduce just one small parameter $\epsilon$ and suppose that

$$\frac{M^2}{M_{Pl}^2} = O(\epsilon), \quad \frac{\partial_t^n H}{M^{n+1}} = O(\epsilon^{n+1}), \quad \frac{\partial_t^n \Sigma}{M^n} = O(\epsilon^{n+1}), \quad \frac{\partial_t^n V}{M^n} = O(\epsilon^{n+1}),$$  (3.14)

and

$$\frac{\partial_t^n \rho}{M^{n+4}} = O(\epsilon^{n+1}), \quad \frac{\partial_t^n p}{M^{n+4}} = O(\epsilon^{n+1}),$$  (3.15)

where $n = 0, 1, 2, \cdots$. Again, the latter condition (3.15) is not necessary for the present purpose but will be used when we derive evolution equations for linear perturbation around the FRW background in the next section.

The low energy expansion not only enables us to compute various quantities in a systematic way but also helps us avoid picking up spurious modes associated with the higher derivative terms. In the 4D covariant action (2.1) the terms proportional to $\alpha_1$ and $\alpha_2$ include the square of the second time derivative of the field $\phi$. Therefore, if we take its face value then the equation of motion for $\phi$ includes up to the fourth
order time derivatives and there is in principle freedom to specify \( \phi, \frac{\partial \phi}{\partial t}, \frac{\partial^2 \phi}{\partial t^2} \) and \( \frac{\partial^3 \phi}{\partial t^3} \) as an initial condition. However, as the scaling analysis in ref. [26] shows, the time derivatives higher than the second order in the equation of motion are irrelevant at energies sufficiently below the cutoff \( M \), at least in Minkowski background. In other words, extra modes associated with those higher time derivatives have frequencies above the cutoff scale \( M \) and is outside the regime of validity of the effective field theory. For this reason, those extra modes are spurious and must be dropped out from physical spectrum of the low energy theory. In the expanding background it is not a priori completely clear how to drop the spurious modes while maintaining all physical modes. By adopting the low energy expansion this can be done in a systematic way.

### 3.2 Reconstructing the potential from \( H(t) \)

As shown in Ref. [64], many non-standard cosmology, including the phantom (\( w < -1 \)) cosmology, can be realized in the framework of ghost condensation without introducing ghosts or any other instabilities. The purpose of this subsection is to show that, given an arbitrary history of the Hubble expansion rate \( H(t) \) and gravitational sources \( \rho(t) \) and \( p(t) \), it is indeed possible to find a form of the potential \( V(\phi) \) which realizes \( H(t) \) as a solution to the Einstein equation (3.5), provided that the conservation equation \( \partial_t \rho + 3H(\rho + p) = 0 \) is satisfied. As is well known, a similar result holds also for a conventional scalar field with a potential: one can almost always find a form of the potential for a given history of the Hubble expansion rate. However, for a conventional scalar field, it is impossible to violate the null energy condition without a ghost. This is the origin of the folklore that the phantom (\( w < -1 \)) cosmology requires a ghost, which is correct for ordinary scalars. On the other hand, for the ghost condensate, the null energy condition can be violated without introducing ghosts or any other instabilities as far as the violation is weak enough. Thus, the folklore is not correct in ghost condensation. It is probably worth stressing here again that the low energy effective field theory of ghost condensation is completely determined by the symmetry breaking pattern and does not include any ghosts.

For the reconstruction of the potential \( V(\phi) \) from the expansion history \( H(t) \) and the energy density \( \rho(t) \) and pressure \( p(t) \) of known gravitational sources, we take advantage of the low energy expansion introduced in the previous subsection. We expand \( V \) and \( \Sigma \) with respect to \( \epsilon \):

\[
V = V_0 + V_1 + \cdots, \quad V_n = O(\epsilon^{n+1}),
\]

\[
\Sigma = \Sigma_0 + \Sigma_1 + \cdots, \quad \Sigma_n = O(\epsilon^{n+1}). \tag{3.16}
\]
Note that $\phi$ is expressed in terms of $\Sigma_n$ as

$$\phi = M^2 \int \sqrt{1 + \Sigma} dt = M^2 \int \left[ 1 + \frac{1}{2} \Sigma_0 + O(\epsilon^2) \right] dt. \quad (3.17)$$

The Friedmann and dynamical equations (3.5) in the lowest order in $\epsilon$ are easily solved with respect to $\Sigma_0$ and $V_0$ as

$$\Sigma_0 = -\frac{2}{M^4} \left[ 2M_{Pl}^2 \partial_t H + (\rho + p) \right],$$
$$V_0 = \frac{1}{M^4} \left[ M_{Pl}^2 (2\partial_t H + 3H^2) + p \right],$$

(3.18)

The relation (3.17) in the lowest order in $\epsilon$ is

$$\phi = M^2 (t - t_0) + M \cdot O(\epsilon), \quad (3.19)$$

where $t_0$ is a constant. Thus, for a given history $(H(t), \rho(t), p(t))$ satisfying $\partial_t \rho + 3H(\rho + p) = 0$, the potential $V$ is expressed in terms of $\phi$ as

$$V(\phi) = \frac{1}{M^4} \left[ M_{Pl}^2 (2\partial_t H + 3H^2) + p \right]_{t=t_0+\phi/M^2} + O(\epsilon^2).$$

(3.20)

It is also easy to obtain the correction to the lowest order potential. In the next-to-the-leading order the Friedmann and dynamical equations (3.5) are solved with respect to $\Sigma_1$ and $V_1$ as

$$\Sigma_1 = -\Sigma_0^2 - \frac{12\alpha}{M^2} \partial_t H + \frac{2\beta}{M^2} (5\partial_t H + 3H^2),$$
$$V_1 = \frac{1}{8} \Sigma_0^2 + \frac{3\alpha - 2\beta}{2M^2} (2\partial_t H + 3H^2).$$

(3.21)

The relation (3.17) in this order is

$$\phi = M^2 (t - t_0) + \frac{M^2}{2} \int_{t_0}^{t} \Sigma_0(t') dt' + M \cdot O(\epsilon^2),$$

(3.22)

where $t_0$ is again a constant. Thus, for a given history $(H(t), \rho(t), p(t))$ satisfying $\partial_t \rho + 3H(\rho + p) = 0$, the potential $V$ is expressed in terms of $\phi$ as

$$V(\phi) = [V_0 + \Delta t \cdot \partial_t V_0 + V_1]_{t=t_0+\phi/M^2} + O(\epsilon^3),$$

(3.23)

where

$$\Delta t = -\frac{1}{2} \int_{t_0}^{t} \Sigma_0(t') dt'$$

(3.24)

and $\Sigma_0$, $V_0$ and $V_1$ are given in (3.18) and (3.21).
4 Cosmological perturbation

In this section we derive evolution equations for cosmological perturbation around the FRW background.

We consider a scalar-type cosmological perturbation in the longitudinal gauge,

\[ ds^2 = -(1 + 2\Psi)dt^2 + a^2(1 + 2\Phi)dx^2, \quad (4.1) \]

and a general stress-energy tensor of the form

\[ T^t_t = -\rho - \left[ \rho\Delta - 3H(\rho + p)Va/\sqrt{k^2} \right] Y, \]
\[ T^t_i = (\rho + p)VaY_i, \quad T^i_t = -\frac{(\rho + p)YY^i}{a}, \]
\[ T^j_j = p + \left\{ p\Gamma + c_s^2[\rho\Delta - 3H(\rho + p)Va/\sqrt{k^2}] \right\} Y_\delta^j_j + p\Pi_\delta^j_j, \quad (4.2) \]

where \( Y \)'s are harmonics on the 3-space defined as

\[ Y = e^{ikx}, \]
\[ Y_i = -\frac{1}{\sqrt{k^2}}\partial_i Y, \quad Y_i = \delta^i_j Y_j, \]
\[ Y_{ij} = \frac{1}{k^2}\partial_i\partial_j Y + \frac{1}{3}Y_\delta^i_j, \quad Y_i^j = \delta^i_k Y_k^j. \quad (4.3) \]

Here, \( \rho \) and \( p \) are the unperturbed energy density and pressure, and \((\Delta, V, \Gamma, \Pi)\) represent the gauge-invariant perturbation of the stress-energy tensor. Physical meaning of each component is as follows [71]: \( \Delta \) is the density contrast in the slicing that the fluid velocity is orthogonal to constant time hypersurfaces, \( V \) is the fluid velocity relative to the observers normal to constant time hypersurfaces, \( \Gamma \) is the entropy perturbation and \( \Pi \) is anisotropic stress. They satisfy the perturbed conservation equation

\[ \partial_t(\rho\Delta) + 3H\rho\Delta + \left( \frac{k^2}{a^2} - 3\partial_t H \right) \frac{a}{\sqrt{k^2}}(\rho + p)V + 2Hp\Pi \]
\[ + 3(\rho + p)(\partial_t\Phi - H\Psi) = 0, \]
\[ \partial_t[(\rho + p)V] + (4 + 3c_s^2)H(\rho + p)V - \frac{\sqrt{k^2}}{a} \left[ p\Gamma + c_s^2\rho\Delta + (\rho + p)\Psi - \frac{2}{3}p\Pi \right] = 0. \quad \] (4.4)

When \( \rho \) and \( \rho + p \) are non-vanishing, these equations are rewritten as

\[ \partial_t\Delta - 3wH\Delta + (1 + w) \left[ \left( \frac{k^2}{a^2} - 3\partial_t H \right) \frac{a}{\sqrt{k^2}} V + 3(\partial_t\Phi - H\Psi) \right] + 2wH\Pi = 0, \]
\[ \partial_t V + HV - \frac{\sqrt{k^2}}{a} \left[ \frac{c^2_s}{1 + w} \Delta + \frac{w}{1 + w} \left( \Gamma - \frac{2}{3} \Pi \right) + \Psi \right] = 0, \]

(4.5)

where \( w = \frac{p}{\rho} \) and \( c^2_s = \frac{\partial p}{\partial t \rho} \).

As for the metric perturbation, for convenience, we decompose \((\Phi, \Psi)\) into the standard, general relativity (GR) part \((\Phi_{GR}, \Psi_{GR})\) and the modification part \((\Phi_{mod}, \Psi_{mod})\) as follows.

\[
\begin{align*}
\Phi &= \Phi_{GR} + \Phi_{mod}, \\
\Psi &= \Psi_{GR} + \Psi_{mod},
\end{align*}
\]

(4.6)

where the GR part is given by

\[
\begin{align*}
\frac{k^2}{a^2} \Phi_{GR} &= \frac{\rho \Delta}{2M_{Pl}^2}, \\
\frac{k^2}{a^2} (\Psi_{GR} + \Phi_{GR}) &= -\frac{p \Pi}{M_{Pl}^2}.
\end{align*}
\]

(4.7)

In the following we shall derive evolution equation of the modification part of the metric perturbation \((\Phi_{mod}, \Psi_{mod})\). For readers who are interested in application of the formalism, subsection 4.4 summarizes resulting evolution equations of the cosmological perturbation in the Higgs phase of gravity. In subsection 4.1 we introduce a systematic low energy expansion applicable to linear perturbation by extending the low energy expansion for the background given in subsection 3.1. Subsection 4.2 includes the hardest part of calculations in this paper: we derive a single master equation governing the modification of gravity without ignoring higher time derivatives. The master equation is a fourth-order ordinary differential equation for each comoving momentum. However, in subsection 4.3 we see that the fourth and third order derivative terms are irrelevant and reduce the master equation to a set of two first-order ordinary differential equations for each comoving momentum. At the same time, we remove an apparent singularity in the master equation. In subsection 4.4 we summarize the results of this section in a way which is directly applicable to actual problems. In Appendix A.1 as a simple application of the formula summarized in subsection 4.4, we consider Minkowski and de Sitter backgrounds and derive the results in ref. 26 for modification of gravity in those backgrounds. Readers who are interested in application of the formalism may go directly to subsections 4.4.
4.1 Low energy expansion

We now extend the low energy expansion developed in subsection 3.1 to linear perturbation around the FRW background. The assignment for the FRW background is summarized in (3.14) and (3.15).

For perturbation, besides the smallness of the perturbation itself controlling the validity of the linear approximation, there is one additional small parameter $\epsilon_4$ given by

$$\frac{1}{M} \frac{\sqrt{k^2}}{a} = \epsilon_4. \quad (4.8)$$

The low energy effective theory is valid only if $\epsilon_4$ is small enough. Supposing that $\Delta$, $V$, $\Gamma$ and $\Pi$ may become comparable, the perturbed conservation equation (4.5) implies that

$$\frac{\partial}{\partial t} M (\Delta, V, \Gamma, \Pi) \sim O(\epsilon_1, \epsilon_4), \quad (4.9)$$

unless the modification part $(\Phi_{\text{mod}}, \Psi_{\text{mod}})$ dominates over the GR part $(\Phi_{\text{GR}}, \Psi_{\text{GR}})$. As shown already in ref. [26], the dispersion relation for excitation of ghost condensation in Minkowski background is $\omega^2 \simeq \alpha k^4/M^2 - \alpha k^2 M^2/2M_{\text{Pl}}^2$. The first term in the right hand side is $\sim \epsilon_4^4 M^2$ and the second term is $\sim \epsilon_4^2 M^2$. In this paper we would like to generalize this dispersion relation in Minkowski spacetime to an evolution equation in the FRW background, taking both terms into account. For this purpose we suppose that these two terms may become comparable. In other words, we suppose that there is a small parameter $\tilde{\epsilon}$ such that

$$\epsilon = O(\tilde{\epsilon}^2), \quad \epsilon_4 = O(\tilde{\epsilon}), \quad \left(\text{e.g. } \tilde{\epsilon} \equiv \text{Max}(\sqrt{|\epsilon|}, \epsilon_4)\right). \quad (4.10)$$

In summary it is supposed that there is a small parameter $\tilde{\epsilon}$ such that

$$\frac{M^2}{M_{\text{Pl}}^2} = O(\tilde{\epsilon}^2),$$

$$\frac{\partial^n H}{M^{n+1}} = O(\tilde{\epsilon}^{2(n+1)}), \quad \frac{\partial^n \Sigma}{M^n} = O(\tilde{\epsilon}^{2(n+1)}), \quad \frac{\partial^n V}{M^n} = O(\tilde{\epsilon}^{2(n+1)}),$$

$$\frac{\partial^n \rho}{M^{n+4}} = O(\tilde{\epsilon}^{2(n+1)}), \quad \frac{\partial^n p}{M^{n+4}} = O(\tilde{\epsilon}^{2(n+1)}), \quad (4.11)$$

and

$$\frac{1}{M} \frac{\sqrt{k^2}}{a} = O(\tilde{\epsilon}),$$

$$\left(\frac{\partial}{\partial \frac{1}{M}}\right)^n (\Delta, V, \Gamma, \Pi) \sim O(\tilde{\epsilon}^n) \cdot (\Delta, V, \Gamma, \Pi), \quad (4.12)$$

where $n = 0, 1, 2, \cdots$. 

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4.2 Master equation

In the previous subsection we developed a low energy expansion for the FRW background and linear perturbation around it. Armed with this, we can roughly estimate how small or large a term in equations should be, as far as \((\Phi_{GR}, \Psi_{GR}), (\Delta, V, \Gamma, \Pi)\) and their time derivatives are concerned. On the other hand, we do not know a priori how small or large the time derivatives of the modification part \((\Phi_{mod}, \Psi_{mod})\) are. This can be seen only after we derive evolution equation of these quantities. In this section we derive a single fourth order master equation for the modification part, using the low energy expansion but not assuming the smallness of the time derivatives of \((\Phi_{mod}, \Psi_{mod})\).

Let us introduce two variables \(\tilde{\Phi}_{1,2}\) by

\[
\begin{align*}
\Phi_{mod} &= \tilde{\Phi}_1 + c\tilde{\Phi}_2, \\
\Psi_{mod} &= -\tilde{\Phi}_1 - (1 + c)\tilde{\Phi}_2,
\end{align*}
\]

where \(c\) is a constant. We shall see below that the source terms for equations governing \(\Phi_{1,2}\) scale like \(\propto M^2/M_{Pl}^2\) in the \(M/M_{Pl} \to 0\) limit. Thus the standard GR result is recovered in the \(M/M_{Pl} \to 0\) limit. The variable \(\tilde{\Phi}_2\) \((\propto \Phi_{mod} + \Psi_{mod})\) was defined so that it vanishes when \(\beta = 0\). The definition of the variable \(\tilde{\Phi}_1\) is not yet fixed because of the unfixed constant \(c\). The value of the constant \(c\) will later be determined so that the master equation for \(\tilde{\Phi}_1\) including matter source terms is simplified.

As for the field \(\phi\) responsible for ghost condensation, we expand it up to the linear order as

\[
\phi = M^2 \left[ \int \sqrt{1 + \Sigma} dt + \pi Y \right].
\]

We now have three unknown variables \(\tilde{\Phi}_1, \tilde{\Phi}_2\) and \(\pi\) other than matter variables \((\Delta, V, \Gamma, \Pi)\). Our task now is to obtain the evolution equation for those three variables sourced by these matter variables.

In cosmology with an ordinary scalar field the standard strategy to analyze linear perturbation in the longitudinal gauge is as follows: (i) to eliminate one of the two metric variables \((\Phi, \Psi)\), say \(\Psi\), by using the traceless part of \((ij)\)-components of the linearized Einstein equation; (ii) to eliminate the linear perturbation of the scalar field by using the \((0i)\)-components of the linearized Einstein equation; and (iii) to obtain a single master equation for the remaining metric variable, say \(\Phi\), from the \((00)\)-component of the linearized Einstein equation. (The linearized equation of motion of the scalar field and the trace part of \((ij)\)-components of the linearized Einstein equation are automatically satisfied because of the Bianchi identity.) The steps (i) and (ii) involve simple algebraic equations for variables being eliminated.
As a result the master equation is a second-order ordinary differential equation for each comoving momentum. This is consistent with the fact that, in the absence of other gravitational sources, the scalar sector includes only one propagating degree of freedom, i.e. excitation of the scalar field. If there are other gravitational sources then the master equation includes source terms due to those gravitational sources.

We can use the same strategy to analyze the linear perturbation with ghost condensation although the result will be drastically different. There are again three main steps. (To be precise, for the reason explained below, steps (ii) and (iii) are not separated but actually mixed in the present case.) (i) First we eliminate $\tilde{\Phi}_2$ by using the traceless part of $(ij)$-components of the linearized Einstein equation. (ii) Next we eliminate $\pi$ by using the $(0i)$-components of the linearized Einstein equation. (iii) Finally we obtain a single master equation for the remaining metric variable $\tilde{\Phi}_1$ from the $(00)$-component of the linearized Einstein equation. (The equation of motion of $\pi$ and the trace part of $(ij)$-components of the linearized Einstein equation are automatically satisfied because of the Bianchi identity.) The step (i) involves just an algebraic equation for $\tilde{\Phi}_2$ as in the standard case. However, the step (ii) involves a second-order differential equation for $\pi$ for each comoving momentum, contrary to the standard case. For this reason the steps (ii) and (iii) are not separated but actually mixed. After eliminating $\pi$ and its derivatives, the resulting master equation is a fourth-order ordinary differential equation for $\tilde{\Phi}_1$ for each comoving momentum. This is indeed consistent with the fact that the action for $\phi$ includes the square of the second derivative of $\phi$ and that the equation of motion for $\pi$ involves up to forth-order derivatives at least formally. In the master equation the matter variables ($\Delta, V, \Gamma, \Pi$) appear as source terms.

We adopt the low energy expansion introduced in the previous subsection to follow each step (i)-(iii). This in particular means that terms with sufficiently higher time derivatives acted on the background or matter variables can be ignored since they are considered as higher order in the expansion with respect to $\tilde{\epsilon}$. On the other hand, the time derivative acted on $\tilde{\Phi}_1$, $\tilde{\Phi}_2$ and $\pi$ is not supposed to raise the order of the low energy expansion. This is because we do not a priori know the time scale of the dynamics of these three variables in the FRW background until the evolution equation of a single master variable is obtained.

Following the steps (i)-(iii), we obtain the fourth-order master equation for $\tilde{\Phi}_1$:

$$C_4 \partial_t^4 \tilde{\Phi}_1 + C_3 \partial_t^3 \tilde{\Phi}_1 + \partial_t^2 \tilde{\Phi}_1 + C_1 \partial_t \tilde{\Phi}_1 + C_0 \tilde{\Phi}_1 = \tilde{S}_1,$$

(4.15)

where the coefficient of the second-order term is normalized to unity, and other coef-
Coefficients and the source term are given by

\[
C_4 = \frac{\alpha}{M^2} + \frac{O(\dot{\epsilon}^2)}{M^2},
\]

\[
C_3 = \frac{\alpha}{M^2} \left[ 6H - \frac{M^2(\partial_t \Sigma + 2H\Sigma)}{M^2\Sigma + 2\alpha k^2/a^2} \right] + \frac{O(\dot{\epsilon}^4)}{M},
\]

\[
C_1 = 3H - \frac{M^2(\partial_t \Sigma + 2H\Sigma)}{M^2\Sigma + 2\alpha k^2/a^2} + M \cdot O(\dot{\epsilon}^4),
\]

\[
C_0 = \frac{\alpha k^4}{M^2 a^4} + \frac{1}{2} \left( \Sigma - \frac{\alpha M^2}{M^2_{Pl}} \right) \frac{k^2}{a^2} + 2H^2 + \partial_t H - \frac{\Sigma M^4}{4 M^2_{Pl}} - \frac{M^2 H(\partial_t \Sigma + 2H\Sigma)}{M^2\Sigma + 2\alpha k^2/a^2} + M^2 \cdot O(\dot{\epsilon}^6),
\]

(4.16)

and

\[
M^2_{Pl} \frac{k^2}{a^2} S_1 = \frac{M^2}{M^2_{Pl}} \left\{ \frac{1}{8} \left( M^2\Sigma + 2\alpha k^2/a^2 \right) \left[ (1 + 3\epsilon_s^2)\rho\Delta + 3p\Gamma \right] \right. \\
+ (c - 1)\beta \left[ \frac{1}{2} \frac{k^2}{a^2} (\epsilon_s^2\rho\Delta + p\Gamma) - \frac{1}{3} \frac{k^2}{a^2} p\Pi \right] + c\beta \partial_t^2 (p\Pi) \right\} \\
+ M^6 \cdot O(\dot{\epsilon}^6).
\]

(4.17)

On the other hand, \(\tilde{\Phi}_2\) is expressed in terms of \(\tilde{\Phi}_1\) and its derivatives as

\[
\tilde{\Phi}_2 = \beta \left( D_3 \partial_t^3 \tilde{\Phi}_1 + D_2 \partial_t^2 \tilde{\Phi}_1 + D_1 \partial_t \tilde{\Phi}_1 + D_0 \tilde{\Phi}_1 + S_2 \right),
\]

(4.18)

where

\[
D_3 = \frac{4\alpha H}{M^2 \left[ M^2\Sigma + 2\alpha k^2/a^2 \right]} + \frac{O(\dot{\epsilon}^2)}{M^3},
\]

\[
D_2 = - \frac{2\alpha}{M^2[\alpha - (2c + 1)\beta]} + \frac{O(\dot{\epsilon}^2)}{M^2},
\]

\[
D_1 = \frac{4H}{M^2\Sigma + 2\alpha k^2/a^2} + \frac{O(\dot{\epsilon}^2)}{M},
\]

\[
D_0 = - \frac{2}{M^2 a^2} + \frac{M^2}{M^2_{Pl}} + \frac{4H^2}{M^2\Sigma + 2\alpha k^2/a^2} + O(\dot{\epsilon}^4),
\]

\[
M^2_{Pl} \frac{k^2}{a^2} S_2 = \frac{M^2}{2 M^2_{Pl}} \left( \rho\Delta + 2p\Pi \right) + M^4 \cdot O(\dot{\epsilon}^3).
\]

(4.19)

To obtain these expressions, we kept terms up to \(O(\dot{\epsilon}^4)\) in intermediate steps.

Having obtained the master equation, it is clear that

\[
c = 1
\]

(4.20)
gives the best choice for the definition of $\Phi_1$. This indeed simplifies the source term $S_1$ of the master equation. Moreover, the term $(M^2/M_{Pl}^2)\beta \partial_\tau^2(p) \Pi$ in $S_1$ can be absorbed by redefinition of the master variable: it does not appear if we use $\Phi_1$ and $\Phi_2$ defined by

$$
\Phi_1 \equiv \bar{\Phi}_1 + \beta \frac{a^2}{k^2} \frac{p\Pi}{M_{Pl}^2}, \\
\Phi_2 \equiv \bar{\Phi}_2,
$$

(4.21)

instead of $\bar{\Phi}_1$ and $\bar{\Phi}_2$. The master equation is now written as

$$
C_4 \partial_\tau^4 \Phi_1 + C_3 \partial_\tau^3 \Phi_1 + \partial_\tau^2 \Phi_1 + C_1 \partial_\tau \Phi_1 + C_0 \Phi_1 = S_1,
$$

(4.22)

where $C_{4,3,1,0}$ are given by (4.16) and the new source term is

$$
M_{Pl}^2 \frac{k^2}{a^2} S_1 = \frac{M^2}{8M_{Pl}^2} \left( M^2 \Sigma + 2\alpha k^2/a^2 \right) \left[ (1 + 3\epsilon^2) \rho \Delta + 3p \Gamma \right] + M^6 \cdot O(\bar{\epsilon}^5).
$$

(4.23)

The variable $\Phi_2$ is expressed in terms of $\Phi_1$ and matter variables as

$$
\Phi_2 = \beta \left( D_3 \partial_\tau^3 \Phi_1 + D_2 \partial_\tau^2 \Phi_1 + D_1 \partial_\tau \Phi_1 + D_0 \Phi_1 + S_2 \right),
$$

(4.24)

where $D_{3,2,1,0}$ and $S_2$ are given by (4.19) with $c = 1$.

Note that coefficients of the master equation (4.22) and the relation (4.24) were obtained just up to the leading order in the $\bar{\epsilon}$ expansion. It is of course possible to seek corrections to the leading terms, but we shall not do so in this paper for simplicity. Note also that we had to keep up to the fourth order in $\bar{\epsilon}$ in the intermediate steps to obtain the leading master equation. The reason for this is manifest: otherwise, we would not have been able to obtain a non-vanishing coefficient of $\Phi_1$ in (4.22) since $C_0 = O(\bar{\epsilon}^4)$ while the coefficient of $\partial_\tau^2 \Phi_1$ is unity.

### 4.3 Reduction to a set of two first order equations

Although the master equation (4.22) obtained in the previous subsection is formally fourth order, the fourth and third order derivative terms are actually irrelevant. To see this is easy: they are suppressed by the cutoff scale $M$ compared with the coefficients of the second and first derivative terms, respectively. This means that ignoring the fourth and third order derivative terms just drop out modes with frequency of order $M$ or higher, which are outside the regime of validity of the effective field theory. Properties of these modes are dependent of properties of unknown UV completion and, thus, we not only may but also must discard these modes. In other words,
we assume the existence of a good UV completion but will never use its properties. Hence, we obtain the second-order master equation

\[ \partial_t^2 \Phi_1 + C_1 \partial_t \Phi_1 + C_0 \Phi_1 = S_1, \tag{4.25} \]

where \( C_{1,0} \) and \( S_1 \) are given by (4.16) and (4.23). For the same reason, we must discard the third and second order derivative terms in the relation (4.24):

\[ \Phi_2 = \beta (D_1 \partial_t \Phi_1 + D_0 \Phi_1 + S_2), \tag{4.26} \]

where \( D_{1,0} \) and \( S_2 \) are given by (4.19) with \( c = 1 \).

The master equation (4.25) and the relation (4.26) appear to be singular when \( M^2 \Sigma + 2 \alpha k^2/a^2 \) vanishes. This does not mean that the system exits the regime of validity of the low energy effective theory but just implies that the dynamics is not described by a single master equation. Indeed, at the would-be singularity \( M^2 \Sigma + 2 \alpha k^2/a^2 = 0 \), the second-order master equation is reduced to \( \partial_t \Phi_1 + H \Phi_1 = 0 \). This suggest that the second-order master equation should be decomposed into a set of two first-order equations. To make this more explicit, let us define a new variable \( \chi \) by

\[ \chi \equiv \frac{4 (\partial_t \Phi_1 + H \Phi_1)}{M^2 \Sigma + 2 \alpha k^2/a^2}, \tag{4.27} \]

With this variable and \( \Phi_1 \) the master equation (4.25) is reduced to the following set of two first-order equations.

\[ \begin{align*}
\partial_t \Phi_1 + H \Phi_1 &= \frac{1}{4} \left( M^2 \Sigma + 2 \alpha \frac{k^2}{a^2} \right) \chi, \\
\partial_t \chi &= \left( \frac{M^2}{M_{\text{Pl}}^2} - \frac{2 k^2}{M^2 a^2} \right) \Phi_1 + S_\chi, \tag{4.28} \end{align*} \]

where the source term \( S_\chi \) is given by

\[ M_{\text{Pl}}^2 \frac{k^2}{a^2} S_\chi = \frac{M^2}{2 M_{\text{Pl}}^2} \left[ \left( 1 + 3 c^2 \right) \rho \Delta + 3 p \Gamma \right]. \tag{4.29} \]

On the other hand, the variable \( \Phi_2 \) is written in terms of \( \Phi_1 \) and \( \chi \) as

\[ \Phi_2 = \beta \left[ H \chi + \left( \frac{M^2}{M_{\text{Pl}}^2} - \frac{2 k^2}{M^2 a^2} \right) \Phi_1 + S_2 \right], \tag{4.30} \]

where

\[ M_{\text{Pl}}^2 \frac{k^2}{a^2} S_2 = \frac{M^2}{2 M_{\text{Pl}}^2} \left( \rho \Delta + 2 p \Pi \right). \tag{4.31} \]
The apparent singularity at $M^2 \Sigma + 2\alpha k^2/a^2 = 0$ does not exist in the new equations (4.28) and (4.30). As easily seen from the structure of the set of first order equations (4.28), the apparent singularity in (4.25) and (4.26) was just the reflection of the fact that the evolution of the variable $\Phi_1$ is decoupled from that of $\chi$ when $M^2 \Sigma + 2\alpha k^2/a^2 = 0$.

4.4 Results summarized

As shown in subsection 3.2 in ghost condensation it is always possible to find a form of a potential for the Higgs sector which realizes a given cosmological history $(H(t), \rho(t), p(t))$ of the FRW background, provided that the conservation $\partial_t \rho + 3H(\rho + p) = 0$ is satisfied. An important point is that the null energy condition can be violated in the Higgs sector of gravity without introducing ghosts or any other instabilities as far as the violation is weak enough [64]. Note again that the structure of the low energy effective field theory of ghost condensation is completely determined by the symmetry breaking pattern and does not include any ghosts. Thus, for example we can realize the phantom ($w < -1$) cosmology without a ghost. This is an explicit counter-example against the folklore that the phantom ($w < -1$) cosmology would require a ghost.

In this subsection we summarize the result of this section, where we have derived a set of two first order ordinary equations governing cosmological perturbations of the Higgs sector of gravity. The low energy effective theory of ghost condensation includes two dimensionless parameters of order unity $\alpha (\equiv \alpha_1 + \alpha_2 > 0)$ and $\beta (\equiv \alpha_2)$ and the cutoff scale $M$. Provided that $(\alpha, \beta, M)$ are fixed and a background cosmological history $(H(t), \rho(t), p(t))$ (satisfying the conservation equation but possibly including deviation from general relativity) is given, the evolution equations for cosmological perturbations are completely determined.

Now let us start summarizing evolution equations for cosmological perturbation. For notation the reader should refer to the second and the third paragraphs of this section.

The metric perturbation is decomposed into the standard, general relativity (GR) part $(\Phi_{GR}, \Psi_{GR})$ and the modification part $(\Phi_{mod}, \Psi_{mod})$ as (4.6), and the modification part is written as

$$
\Phi_{mod} = \Phi_1 + \Phi_2 - \beta \frac{a^2}{k^2} \frac{p\Pi}{M_{Pl}^2},
$$

$$
\Psi_{mod} = -\Phi_1 - 2\Phi_2 + \beta \frac{a^2}{k^2} \frac{p\Pi}{M_{Pl}^2}.
$$

(4.32)
The variable $\Phi_1$ is given by solving

$$\partial_t \Phi_1 + H \Phi_1 = \frac{1}{4} \left( M^2 \Sigma_0 + 2 \alpha \frac{k^2}{a^2} \right) \chi,$$

$$\partial_t \chi = \left( \frac{M^2}{M^2_{Pl}} - \frac{2}{M^2 a^2} \right) \Phi_1 + S_\chi,$$  \hspace{1cm} (4.33)

where $\chi$ is an auxiliary variable, $\Sigma_0$ is given by

$$\Sigma_0 = -\frac{2}{M^4} \left[ 2M^2_{Pl} \partial_t H + \left( \rho + p \right) \right]$$  \hspace{1cm} (4.34)

and represents deviation of the background evolution from general relativity, and the source term $S_\chi$ is given by

$$M^2_{Pl} \frac{k^2}{a^2} S_\chi = \frac{M^2}{2M^2_{Pl}} \left[ \left( 1 + 3c_s^2 \right) \rho \Delta + 3p \Gamma \right].$$  \hspace{1cm} (4.35)

On the other hand, the variable $\Phi_2$ is written in terms of $\Phi_1$ and $\chi$ as

$$\Phi_2 = \beta \left[ H \chi + \left( \frac{M^2}{M^2_{Pl}} - \frac{2}{M^2 a^2} \right) \Phi_1 + S_2 \right],$$  \hspace{1cm} (4.36)

where

$$M^2_{Pl} \frac{k^2}{a^2} S_2 = \frac{M^2}{2M^2_{Pl}} (\rho \Delta + 2p \Pi).$$  \hspace{1cm} (4.37)

Note that $\Phi_2$ vanishes if $\beta = 0$.

The source terms $S_\chi$ and $S_2$ vanish in the $M/M_{Pl} \rightarrow 0$ limit. This means that the modification part ($\Phi_{\text{mod}}, \Psi_{\text{mod}}$) is not induced by matter sources in this limit. In other words, in this limit the Higgs sector of gravity is decoupled from the gravity and the matter sectors, and the general relativity is safely recovered.

One must be aware that these equations are valid only if the physical momentum $\sqrt{k^2/a}$ is sufficiently lower than the scale of the spontaneous Lorentz breaking $M$, which plays the role of the cutoff scale of the low energy effective theory.

The coefficient $\Sigma_0$ includes information about deviation of the background evolution from general relativity. In usual approach, the deviation is considered as indication of dark energy and/or dark matter. Instead, we here replace those dark components with the Higgs mechanism of gravity, i.e. ghost condensation. The cases with $\Sigma_0 > 0$ and $\Sigma_0 < 0$ correspond to dark components with $w > -1$ and $w < -1$, respectively. In the usual approach, a dark component with $w < -1$ is called phantom and is thought to be associated with ghosts, i.e. excitations with wrong-sign kinetic energy. On the other hand, as explained in Sec. 1, there is no ghost in the ghost condensation. The set of equations summarized above can be applied to both $\Sigma_0 > 0 (w > -1)$ and $\Sigma_0 < 0 (w < -1)$ cases, including transition between these two regimes.
5 Concluding remark

In the simplest setup of the ghost condensation the shift symmetry is exact and it behaves like a cosmological constant plus cold dark matter for homogeneous, isotropic background evolution. With soft breaking of the shift symmetry, a shallow potential is allowed in the Higgs sector Lagrangian. We have investigated the classical dynamics of cosmology in ghost condensation with softly broken shift symmetry. As we have shown explicitly, it is always possible to find a form of the potential in the Higgs sector Lagrangian which realizes an arbitrary cosmological history \( (H(t), \rho(t)) \) of the visible sector of the FRW background. After showing the reconstruction method, we have derived the evolution equation for cosmological perturbations in the Higgs phase of gravity.

The strongest evidence for the accelerating expansion of the universe today comes from the supernova distance-redshift relation. Using this kind of geometrical information of the universe, it is in principle possible to reconstruct the potential for the Higgs sector by the method developed in subsection 3.2 of this paper. Once this is done, the theory acquires predictive power. By using the formalism of the cosmological perturbations summarized in subsection 4.4, this theory can be tested by dynamical information of large scale structure in the universe such as cosmic microwave background anisotropy, weak gravitational lensing and galaxy clustering.

Results in this paper have been obtained within the regime of validity of the effective field theory, whose structure can be determined solely by the symmetry breaking pattern. Therefore, while we have used a covariant 4D action of a scalar field as a tool for calculation, the final results summarized in subsection 4.4 should be universal and independent of the way the Higgs phase of gravity is realized. The same equations should hold as far as the symmetry breaking pattern is the same, with or without a scalar field.

Acknowledgements

The author would like to thank B. Feng, T. Hiramatsu, M. A. Luty, S. Saito, A. Shirata, Y. Suto, A. Taruya, K. Yahata and J. Yokoyama for useful discussions and/or comments. This work was in part supported by MEXT through a Grant-in-Aid for Young Scientists (B) No. 17740134.
Appendix

A.1 Simple example: Minkowski and de Sitter backgrounds

In this appendix, as a simple application of the formula summarized in subsection 4.4, we consider modification of gravity in Minkowski and de Sitter backgrounds. This is the situation considered in ref. [26].

For simplicity we set $\beta = 0$. In this case $\Phi_2 = 0$ and the modification part of the metric perturbation is

$$\Phi_{mod} = -\Psi_{mod} = \Phi_1.$$  \hfill (A.1)

By setting $\Sigma_0 = 0, \quad H = H_0 \quad (= \text{const.})$ \hfill (A.2)

and

$$\rho \Delta = \delta \rho, \quad c_s^2 = \Gamma = \Pi = 0,$$  \hfill (A.3)

the set of first order equations (4.33) is reduced to

$$\partial_t \Phi_{mod} + H_0 \Phi_{mod} = \frac{\alpha}{2} \frac{k^2}{a^2} \chi,$$

$$\partial_t \chi = \left( \frac{M^2}{M_{Pl}^2} - 2 \frac{k^2}{M^2 a^2} \right) \Phi_{mod} + S_{\chi},$$  \hfill (A.4)

where the source term is now given by

$$S_{\chi} = \frac{M^2}{2M_{Pl}^2} \frac{a^2}{k^2} \delta \rho = \frac{M^2}{M_{Pl}^2} \Phi_{GR}.$$  \hfill (A.5)

By eliminating $\chi$ from these equations, we obtain

$$\partial_t^2 \Phi_{mod} + 3H_0 \partial_t \Phi_{mod} + \left( \frac{\alpha}{M^2 a^4} - \frac{\alpha M^2 k^2}{2M_{Pl}^2 a^2} + 2H_0^2 \right) \Phi_{mod} = \frac{\alpha M^2 k^2}{2M_{Pl}^2 a^2} \Phi_{GR}.$$  \hfill (A.6)

This is eq. (8.27) of ref. [26] and explicitly shows that in the $M/M_{Pl} \to 0$ limit, the GR part $\Phi_{GR}$ ceases to act as the source of the modification part $\Phi_{mod}$ and the general relativity is safely recovered. By introducing the length and time scales $r_c$ and $t_c$ as

$$r_c = \frac{\sqrt{2} M_{Pl}}{M^2}, \quad t_c = \frac{2M_{Pl}}{\sqrt{\alpha M^3}},$$  \hfill (A.7)

Note that $\alpha^2$ in [26] corresponds to $\alpha$ in the present paper.
This equation is rewritten as
\[ \partial^2_t \Phi_{\text{mod}} + 3H_0 \partial_t \Phi_{\text{mod}} + \left( \frac{r^4}{t_c^4} \frac{k^4}{a^4} - \frac{r^2}{t_c^2} \frac{k^2}{a^2} + 2H_0^2 \right) \Phi_{\text{mod}} = \frac{r^2}{t_c^2} \frac{k^2}{a^2} \Phi_{\text{GR}}. \] (A.8)

In the Minkowski spacetime, by setting \( H_0 = 0 \) and \( a = 1 \) in (A.8), it is easily seen that modes with the length scale \( \sim r_c \) are unstable and that the time scale of the instability is \( \sim t_c \). This is the analogue of the Jeans instability found in ref. [26] and explained in Sec. 1. What is different from the usual Jeans instability is that this behavior in the linearized level persists even in Minkowski spacetime and modifies the linearized gravitational potential.

It is also easy to see from (A.8) that the Jeans instability disappears when the Hubble expansion rate \( H_0 \) is larger than a critical value \( H_c \sim 1/t_c \). Thus, the onset of the IR modification starts at the time when the Hubble expansion rate becomes as low as \( H_c \).

References


\(^6\)The dispersion relation for a usual fluid with the background energy density \( \rho_0 \) is \( \omega^2 = c_s^2 k^2 - \omega_J^2 \), where \( c_s \) is the sound speed and \( \omega_J^2 = 4\pi G N \rho_0 \). Long-scale modes with \( c_s^2 k^2 < \omega_J^2 \) have instability (Jeans instability) and contribute to the structure formation. In the Minkowski background, where \( \rho_0 = 0 \), \( \omega_J^2 \) vanishes and the Jeans instability for the usual fluid disappears.


[54] V. V. Kiselev, “Ghost condensate model of flat rotation curves,” arXiv:gr-qc/0406086

[55] V. V. Kiselev, “Ghost condensate model of flat rotation curves,” arXiv:gr-qc/0507126

[56] V. V. Kiselev and D. I. Yudin, “Gravitational lensing due to dark matter modelled by vector field,” arXiv:gr-qc/0603128


