Slow rolling, inflation, and quintessence

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PACS. 98.80.-k – Cosmology.
PACS. 98.80.Jk – Mathematical and relativistic aspects of cosmology.
PACS. 98.80.Cq – Particle-theory and field-theory models of the early universe.

Abstract. – We comment on the choice of the quintessence potential, examining the slow-roll approximation in a minimally coupled theory of gravity. We make some considerations on the potential behaviors, the related $\Gamma$ parameter, and their relationships to phantom cosmology.

Introduction. – In order to discriminate models for dark energy we usually assign a scalar field potential $V = V(\varphi)$ \cite{1–3}. Within the already available theoretical framework, the behaviors of its first and second derivatives ($V' \equiv dV/d\varphi$ and $V'' \equiv d^2V/d\varphi^2$) have been studied, and some constraints from observational data have also been derived. In this context, the $\Gamma$ function was first introduced in \cite{4} to characterize acceleration properly with a tracking behavior, via a suitable theorem. Another very popular (and closely related) approach is that of using the so called scaling solutions \cite{5, 6}. As shown in \cite{7}, this first led to exclude possibly good forms of the potential like the exponential one $V(\varphi) \sim \exp(-\lambda \varphi)$, in fact shown in \cite{7} as still deserving attention, since it meets all the usual constraints posed by observational data. This was proved, in particular, with $\lambda \equiv \sqrt{3/2}$ \cite{7–12}.

Let us reconsider here some features of the potential for the quintessence $Q$ field \cite{4, 13}, also thinking of the possibility of phantom cosmology, and comment on the slow-rolling conditions introduced in early inflation. Moreover, we consider the $\Gamma$ parameter and other aspects of the potential with respect to the slow-roll approximation.

Slow-roll conditions. – In inflation there is a phase of (almost) exponential cosmic expansion. Its definition in terms of the scale factor $a$ of the universe is given by a positive acceleration, $\ddot{a} > 0$. In standard cosmology this is a requirement on the cosmic content, since if the $\Lambda$-term is zero or absorbed into the energy density $\rho$ of a suitable fluid, we have $\rho + 3p < 0$ independently of the spacetime curvature. Being $\rho$ positive, we must introduce a fluid with negative pressure, $p < 0$, for example a single scalar field $\varphi$. Assuming homogeneity and
isotropy, the energy density $\rho_\varphi$ and pressure $p_\varphi$ are written in terms of kinetic and potential contributions

$$\rho_\varphi = \frac{1}{2} \dot{\varphi}^2 + V(\varphi), \quad p_\varphi = \frac{1}{2} \dot{\varphi}^2 - V(\varphi),$$  

(1)

where an inflationary model is given once we assume a potential $V$.

The Friedman equations of a homogeneous and isotropic spatially flat universe are ($c = 1$)

$$H^2 = \frac{8\pi G}{3} \left( \rho_m + V(\varphi) + \frac{1}{2} \dot{\varphi}^2 \right),$$  

(2)

$$\dot{H} = -4\pi G \left( \rho_m + p_m + \dot{\varphi}^2 \right),$$  

(3)

and

$$\ddot{\varphi} + 3H \dot{\varphi} + \frac{dV}{d\varphi} = 0,$$  

(4)

where $H \equiv \dot{a}/a$, and $\rho_m$ and $p_m$ are the energy density and pressure of the matter component, which are null in early inflation. In this last case, the slow-roll approximation [14, 15] leads to discard the last term in eq. (2) and the first term in eq. (4), so that

$$H^2 \simeq \frac{8\pi G}{3} V, \quad \dot{H} \simeq -4\pi G \dot{\varphi}^2, \quad 3H \dot{\varphi} \simeq -V'.$$  

(5)

This is equivalent to requiring that: i) the universe is in a phase of acceleration; ii) the field is slowly varying, or slowly rolling down its potential ($\dot{\varphi}^2 \ll 2V$); iii) the acceleration of the scalar field is also small ($\ddot{\varphi} \ll 3H \dot{\varphi}$). This is sufficient to guarantee inflation, but it is not also necessary, since in principle inflation can take place even if slow-roll conditions are violated [15].

Let us introduce the two slow-roll parameters [14, 15]

$$\varepsilon(\varphi) \equiv \frac{3}{2} \left( 1 + w_\varphi \right), \quad \eta(\varphi) \equiv \frac{\dot{\varphi}}{H \dot{\varphi}},$$  

(6)

where $w_\varphi \equiv p_\varphi/\rho_\varphi$ is the equation of state of the scalar fluid, and require that

$$\varepsilon(\varphi) \ll 1, \quad |\eta(\varphi)| \ll 1.$$  

(7)

In scalar field cosmology, when $\dot{\varphi}^2 = -2\dot{H}$, $\varepsilon$ and $\eta$ are related to $H$ and its time derivatives

$$\varepsilon = -\frac{\dot{H}}{3H^2 + \dot{H}}, \quad \eta = \frac{\ddot{H}}{2H \dot{H}}.$$  

(8)

Anyway, we think it is important to discuss the slow-roll approximation also when matter is present. Thus, we go back to eq. (1) and differentiate it with respect to time, and, from eq. (3) (with $p_m = 0$), get

$$V'' \dot{\varphi} = 12\pi G \dot{\varphi}(\rho_m + \dot{\varphi}^2) - 3H^2 \dot{\varphi} - \frac{d(\dot{\varphi})}{dt}.$$  

(9)

¿From $\dot{\varphi} \neq 0, V \neq 0$ and $d(\dot{\varphi})/dt \ll \dot{\varphi}$, we thus find (for $\varepsilon \neq 3$)

$$\frac{V''}{V} = 4\pi G \left[ (3 - 2\eta) \frac{\rho_m}{V} + \frac{6(\varepsilon - \eta)}{3 - \varepsilon} \right].$$  

(10)
(The value $\varepsilon = 3$ is excluded here and means $w_\phi = 1$, typical for stiff matter.) We can also rewrite eq. (4) as $(V')^2 = H^2 \dot{\varphi}^2 (3 + \eta)^2$ and get

$$
\left( \frac{V'}{V} \right)^2 = \frac{16\pi G \varepsilon}{3} \left( \frac{3 + \eta}{3 - \varepsilon} \right)^2 \left[ 3 + (3 - \varepsilon) \frac{\rho_m}{V} \right]. \quad (11)
$$

We have now two subcases we discuss separately in the following.

**Inflation.** With $\rho_m = 0$, if $\varepsilon \ll 1$ we can Taylor expand $w_\phi$ to the first order in $\dot{\varphi}^2/(2V)$, and obtain $w_\phi \approx -1 + \dot{\varphi}^2/V$, while, due to the first of eq. (6), we find $\varepsilon \approx 3\dot{\varphi}^2/(2V)$. With $\eta \ll 1$, eq. (4) tells us that $\dot{\varphi}^2 \approx V^2/(16H^2)$, and eq. (2), with $\varepsilon \ll 1$, becomes the first of eq. (5). Assuming eq. (7), it is not difficult to prove that, to the first order, it is

$$
\varepsilon \approx \frac{1}{16\pi G} \left( \frac{V'}{V} \right)^2, \quad \varepsilon - \eta \approx \frac{1}{8\pi G} \frac{V''}{V}.
$$

(12)

Thus, we can find that, if the universe is in the slow-rolling regime, both the relative slope and curvature of the potential $V$ are small, being $V'/V \ll 1 (GeV)^{-1}$ and $V''/V \ll 1 (GeV)^{-2}$.

(With $\hbar = c = 1$, $G = 6.842 \times 10^{-39} (GeV)^{-2}$, and $[\varphi] = eV$, $[\dot{\varphi}] = [V] = [\rho_\phi] = eV^4$, $[\dot{\varphi}^2] = [V'] = eV^3$, $[V''] = eV^2$.)

The relations in eq. (12) are found assuming $\varepsilon, \eta \ll 1$, so that they cannot be used to prove that small $V'/V$ and $V''/V$ imply $\varepsilon, \eta$ to be small, too. To show this, let us rewrite eq. (4) (with $\rho_m = 0$, $V'/V \ll 1 (GeV)^{-1}$, $V''/V \ll 1 (GeV)^{-2}$, and $d(\dot{\varphi})/dt \ll \dot{\varphi}$) and the first of eq. (6)

$$
V'' \approx 12\pi G \dot{\varphi}^2 - 3H^2\eta, \quad \dot{\varphi}^2 = \frac{2\varepsilon V}{3 - \varepsilon}, \quad (13)
$$

so that from eqs. (6) and the first of (12) we get

$$
\frac{V''}{V} \approx 24\pi G(\varepsilon - \eta). \quad (14)
$$

Also, the scalar field equation can be written as $V'' = H^2 \dot{\varphi}^2 (3 + \eta)^2$, and from both the first Friedman equation and the second of eq. (13) it is

$$
\left( \frac{V'}{V} \right)^2 \approx 16\pi G \varepsilon \left( \frac{3 + \eta}{3 - \varepsilon} \right)^2 \approx 16\pi G \varepsilon. \quad (15)
$$

So, we obtain the two equations (14) and (15) for $\varepsilon$ and $\eta$. To the first order in $V'/V$ and $V''/V$, they give eq. (12) again.

Thus, in the early inflationary scenario asking for $\varepsilon$ and $\eta$ to be small is a necessary and sufficient condition for $V'/V$ and $V''/V$ to be small, too. The inflaton potentials are just selected through their relative slopes and curvatures.

**Quintessence.** Now, there is no fundamental reason why the scalar field should obey to the slow-roll approximation. There is a difficulty of relating the slow-roll parameters with the potential today, which is mainly due to the impossibility of neglecting the ordinary matter contribution. Let us in fact consider $V = V(Q)$ and, as supposed before, assume that $\dot{Q} \neq 0$ and also that $d(\dot{Q})/dt$ is negligible with respect to $Q$ and $\dot{Q}$. From eqs. (10) and (11), imposing $\varepsilon, \eta \ll 1$ does not imply that the relative slope and curvature be small again. The same is true for the viceversa. To make it possible, $\rho_m/V$ should in fact be at least of the same order of magnitude of $\varepsilon$ and $\eta$, but now the ratio $\rho_m/V$ is of order unity. Even if today $Q$ is such that
\( \dot{Q}^2 < 2V(Q) \), so giving rise to acceleration, it is also not necessarily slowed down. Thus, \( \varepsilon \) and \( \eta \) are now not so crucial as before in the cosmological description, while keeping information on the local behavior of the scalar field.

As an example, consider the dark energy model (with both \( Q \) and CDM) studied in [8], with an exponential potential \( V(Q) \sim \exp(-\sqrt{3/2}Q) (8\pi G = 1) \); the equations are generally and exactly solved, leading to [8]

\[
\begin{align*}
\varepsilon &= -1 + \frac{2(1 + 2\tau^2)^2}{1 + 7\tau^2 + 6\tau^4}, \\
\eta &= \frac{3(1 + 3\tau^2 + 2\tau^6)}{2(1 + 3\tau^2 + 4\tau^4 + 4\tau^6)}.
\end{align*}
\]

where \( t_s \) is a time scale such that \( H(t_s) \equiv t_s^{-1} \), i.e., of the order of the age of the universe, and \( \tau \equiv t/t_s = H(t_s)t \) a suitable dimensionless time taking the value \( \tau \simeq 1 \). Posing \( t_s = 1 \) for simplicity, from eq. (8)

\[
\varepsilon = -1 + \frac{2(1 + 2\tau^2)^2}{1 + 7\tau^2 + 6\tau^4}, \\
\eta = \frac{3(1 + 3\tau^2 + 2\tau^6)}{2(1 + 3\tau^2 + 4\tau^4 + 4\tau^6)}.
\]

So, choosing for instance \( \tau = 0.82 \) (as in [8]) leads to \( \varepsilon \simeq 0.31 \) and \( \eta \simeq -0.90 \), which is not consistent with slow roll (i.e., \( \varepsilon, \eta < 1 \)).

We can also see that, on the other hand, if we generally assume

\[
V(Q) = V_0 \exp\left(-\sqrt{\frac{2}{p}}Q\right), \quad a(t) = a_0 t^p,
\]

with \( p \) a generic parameter, we find

\[
Q(t) = \sqrt{2p} \ln \left( \sqrt{\frac{V_0}{p(3p - 1)}} t \right).
\]

This means that we have

\[
H(t) = \frac{p}{t}, \quad \dot{Q} = \sqrt{\frac{2p}{t}}, \quad \ddot{Q} = -\frac{\sqrt{2p}}{t^2}, \quad V(t) = p(3p - 1)t^{-2},
\]

satisfying eqs. (14) and (2). On the other hand, since in general

\[
\varepsilon = \frac{3\dot{Q}^2}{2V} = \frac{3}{3p - 1}, \quad \eta = \frac{\ddot{Q}}{H\dot{Q}} = -\frac{1}{p},
\]

\( p = 4/3 \) (and \( 2/p = 3/2 \)) in fact implies \( \varepsilon = 1 \) and \( \eta = -3/4 \), so that such parameters are not negligible with respect to 1.

Furthermore, \( \varepsilon \) needs to be greater than zero for the usual accelerated universe. As a matter of fact, when \( \varepsilon < 0 \) (and, as usual, \( V > 0 \)), we get \( \dot{Q}^2 < 0 \) from eq. (21). That is, we are in presence of phantom cosmology [16], with an equation of state \( w_Q < -1 \) and the prediction of a big rip in the future [17]. Although the kinetic energy of \( Q \) has to be negative [18], this kind of cosmology is indeed compatible with observational constraints, such that \(-1.38 < w_Q < -0.82 \) [19]. A related interesting example can be found in [20], where exact solutions for scalar-tensor theories are used to implement dark energy models with varying \( G \) and \( \Lambda \), such that phantom cosmology can be recovered without any big rip, since \( \rho_Q \) in fact results always decreasing in the future. (This appears to be peculiar, indeed, to scalar fields nonminimally coupled to gravity.)
Now, let us discuss what happens when \( V'' / V \) and \( V' / V \) are small. In order to understand whether and when eqs. (10) and (11) lead to contemporary small values for \( V'' / V \) and \( V' / V \), we here choose to present only indicative reasonings. For sake of simplicity, let us in fact assume that \( V'' / V \) and \( V' / V \) are both so small that we can take them directly null

\[
\frac{V''}{V} = 0 = 3 - 2\eta + \frac{6(\varepsilon - \eta)}{3 - \varepsilon} \quad (22)
\]

and

\[
\left( \frac{V'}{V} \right)^2 = 0 = \varepsilon \left( \frac{3 + \eta}{3 - \varepsilon} \right)^2 \left[ 3 + (3 - \varepsilon)\delta \right], \quad (23)
\]

where we posed \( \delta \equiv \rho_m / V > 0 \) (always, for \( V > 0 \)). Solving for \( \varepsilon \) and \( \eta \) gives

\[
\varepsilon = 3 \frac{3\delta + 2}{3\delta - 2}, \quad \eta = -3, \quad (24)
\]

which shows that \( \varepsilon < 0 \) (corresponding to phantom cosmology) when \( \delta < 2 / 3 \). In this approximated situation, we can examine better which are the allowed non contradictory values for \( \varepsilon \) and \( \eta \), so finding that, with the assumption \( \delta > 0 \) (including also phantom cosmology), we can only accept couples of values of \( \varepsilon, \eta \) such that either \( \varepsilon \neq 0, \eta = -3 \) or \( \varepsilon = 0, \eta \neq \pm 3 \). Anyway, we must not forget that we have taken \( V'' / V = V' / V = 0 \), that is, a more extreme case than the one with \( V'' / V \) and \( V' / V \) simply small. But we have also seen that our hypotheses cannot yield both \( \varepsilon = 0 \) and \( \eta = 0 \), since eq. (22) in fact becomes \( \delta = 0 \). This then implies that the usual slow-rolling regime only belongs to a scalar-field dominated universe, typical for early inflation but not for quintessence today.

As a final side remark, note that, last but not least, assuming \( \varepsilon = 0 \) is equivalent to consider \( w_Q = -1 \), i.e., the cosmological constant.

The \( \Gamma \) function. – The \( \Gamma \) function is defined as

\[
\Gamma \equiv \frac{V V''}{(V')^2} = \frac{V'' / V}{(V' / V)^2}, \quad (25)
\]

and was first introduced in [4]. It then revealed not so interesting as supposed [7, 21], since it is not necessary now to consider slow roll for the \( Q \)-field. Anyway, \( \Gamma \) still remains a significant indicator, since it is skillfully built from a suitable ratio with the derivatives of the potential. Even if it has been shown that \( \Gamma > 1 \) cannot be considered as obvious for a correct quintessence tracking behavior, that remains a sufficient condition [7]. (See also [21] for related illuminating comments.) For example, a single exponential (\( \Gamma = 1 \)), or a suitable combination of two of them (\( \Gamma < 1 \)), is indeed compatible with observational constraints [7, 12].

It is interesting to find all the potentials giving rise to strictly constant values of \( \Gamma \). Considering eq. (25) as a differential equation for \( V = V(Q) \), a direct investigation in fact gives (for a constant \( \Gamma \)):

i) \( \Gamma = 0 \Rightarrow V = \alpha Q + \beta \),

ii) \( \Gamma = 1 \Rightarrow V = \beta \exp(-\alpha Q) \),

iii) \( \Gamma = -1 \Rightarrow V = \sqrt{\alpha Q + \beta} \),

iv) \( \Gamma \neq 0, \pm 1 \Rightarrow V = (\alpha Q + \beta)^{1/(1-\Gamma)} \),

where \( \alpha, \beta \) are suitable integration constants. Case iii) includes the well known and most commonly used potential \( V \sim Q^{-\lambda} \) (for \( \Gamma = 1 + 1 / \lambda \)).

When \( w_Q \) is nearly constant, with \( w_m \approx 0 \) (i.e., an asymptotic dominance of the scalar field), we anyway have \( \Gamma \approx 1 \). In such a case, the potential must be very close to exponential
(see also [22]). But the condition on \( w_Q \) is strictly verified only asymptotically, so indicating that the potential must be exponential there, whatever its functional form before. (Note that an exponential potential in fact leads to some variation of the values of \( w_Q \) in the present period [7].)

Assuming both the conditions in eq. (7) and \( V = V_0 \exp(-\lambda \varphi) \) (with \( \lambda \) a generic constant), eq. (12) now transforms into

\[
\varepsilon \simeq \frac{1}{16\pi G} \lambda^2, \quad \varepsilon - \eta \simeq \frac{1}{8\pi G} \lambda^2, \tag{26}
\]

which evidently lead to \( 0 < \varepsilon \simeq -\eta \ll 1 \). If we instead assume \( V = V_0 \varphi^{-\alpha} \) (\( \alpha > 0 \)), eq. (12) gives

\[
\varepsilon \simeq \frac{1}{16\pi G} \frac{\alpha^2}{\varphi^2}, \quad \varepsilon - \eta \simeq \frac{\alpha(1 + \alpha)}{8\pi G \varphi^2}, \tag{27}
\]

so that \( 0 < \varepsilon \simeq \alpha \eta / (2 + \alpha) \).

We have used \( \varphi \) instead of \( Q \) since this is valid only in the early inflationary period; from the definition of \( \Gamma \), and using eq. (12) (as well as eqs. (14) and (15)), we in fact find

\[
\Gamma \equiv \frac{V V''}{(V')^2} \simeq \frac{\varepsilon - \eta}{2\varepsilon}, \tag{28}
\]

which of course reproduces both the results found above for \( \varepsilon \) in the early inflationary stage.

Also, eqs. (10) and (11) lead to the general expression

\[
\Gamma \equiv \frac{VV''}{(V')^2} = \frac{3}{4\varepsilon} \frac{(3 - 2\eta)(3 - \varepsilon)^2 \delta + 6(\varepsilon - \eta)(3 - \varepsilon)}{(3 + \eta)^2([3 + \delta(3 - \varepsilon)])} \simeq \frac{2V - \rho_m}{3V + \rho_m} + \frac{1}{\varepsilon} \left( \frac{\rho_m}{V + \rho_m} + \frac{2 \eta}{3} \right), \tag{29}
\]

where \( \varepsilon \neq 0 \) is assumed. When \( \delta = 0 \) (asymptotic scalar field dominance), this gives back eq. (28) for \( \varepsilon, \eta \ll 1 \); when \( \delta \neq 0 \), eq. (30) implies (also without taking \( \varepsilon, \eta \ll 1 \))

\[
\Gamma \simeq \frac{3\delta + 2(\varepsilon - \eta)}{4\varepsilon(1 + \delta)} \simeq \frac{3\delta}{4\varepsilon(1 + \delta)} \tag{30}
\]

for \( \delta \simeq 1 \). This is equivalent to \( \Gamma \gg 1 \) but is not always valid, due to the possible asymptotic behavior \( \delta \to 0 \), when the \( Q \) content completely dominates the universe. It can be characteristic, together with eq. (28), only for cosmology with \( w_Q > -1 \).

It is interesting to notice that we recover a phantom energy scenario (with \( w_Q < -1 \)) only for \( \Gamma < 0 \), as can be soon seen from eqs. (21) and (30). This sheds new light on how the parameter \( \Gamma \) works with this kind of energy.

The presence of ordinary matter today complicates the relationship between \( \varepsilon \) and \( \eta \). We have already seen that the condition \( \varepsilon \ll 1 \) is in fact tuned by the non negligible ratio \( \delta \equiv \rho_m/V(Q) \). This is well illustrated, for example, assuming again an exponential potential. In this case, eq. (24) gives

\[
\varepsilon \simeq -\frac{1}{1 + 5\delta} \left[ 3\delta + 2(1 + \delta)\eta \right], \tag{31}
\]

while eq. (30), valid for \( \delta \neq 0 \) and \( \varepsilon, \eta \ll 1 \), yields

\[
\varepsilon \simeq \frac{-2\eta + 3\delta}{3(1 + 2\delta)}. \tag{32}
\]
So, we find that \( \varepsilon < 0 \) (phantom cosmology) only when \( \eta > 3\delta/2 > 0 \), being \( \delta > 0 \) always. This then inverts the sign contraposition between \( \varepsilon \) and \( \eta \) found for the exponential potential in the early inflationary stage.

From the above considerations it appears, first of all, that the situation usually depicted in quintessence is in fact quite different from the inflationary scenario. This seems a rather trivial observation, but, as a matter of fact, some confusion or lack of clearness on this point is still present in the literature.

The main results of this paper lie in a careful analysis of the relationship between slow rolling parameters and the \( \Gamma \) function. This goes in the direction to illustrate the problems related to slow roll, also illuminating how the phantom energy may enter the game. Such considerations are actually interesting for the remarkable role this strange kind of energy is playing in cosmology recently. Only new data will give a way to discriminate in the near future among the various proposals till now posed by theory, but now it still makes sense to speculate on them.

REFERENCES