Unified Split Octonion Formulation of Dyons

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ABSTRACT

Demonstrating the split octonion formalism for unified fields of dyons (electromagnetic fields) and gravito-dyons (gravito-Heavisidan fields of linear gravity), relevant field equations are derived in compact, simpler and manifestly covariant forms. It has been shown that this unified model reproduces the dynamics of structure of fields associated with individual charges (masses) in the absence of others.
Recently, the question of existence of monopole [1,2] has become a challenging new frontier and the object of more interest in connection with quark confinement problem of quantum chromo dynamics. The eight decades of this century witnessed a rapid development of the group theory and gauge field theory to establish the theoretical existence of monopoles and to explain their group properties and symmetries. Keeping in mind t’Hooft’s solutions [3] and the fact that despite the potential importance of monopoles, the formalism necessary to describe them has been clumsy and not manifestly covariant, Rajput et.al. [4,5] recently developed the self-consistent quantum field theory of generalized electromagnetic fields associated with dyons (particles carrying electric and magnetic charges). The analogy between linear gravitational and electromagnetic fields leads the asymmetry in Einstein’s linear equation of gravity and suggests the existence of gravitational analogue of magnetic monopole [6,7]. Like magnetic field, Cantani [8] introduced a new field (i.e. Heavisididian field) depending upon the velocities of gravitational charges (masses) and derived the covariant equations (Maxwell’s equations) of linear gravity. Avoiding the use of arbitrary string variables [1], we have formulated manifestly covariant theory of gravito-dyons [9-10] in terms of two four-potentials and maintained the structural symmetry between generalised electromagnetic fields of dyons [11-12] and generalised gravito-Heavisidian fields of gravito-dyons.

Octonions, or Cayley numbers, have been little used in physics to date [13-14]. Since octonions share with complex numbers and quaternion many attractive mathematical properties, one might except that they would be equally as useful as useful. However, as often happens in theoretical physics, they may occur as alternate mathematical structures which are as useful and more transparent. In our previous work [15-16] we represented electrodynamics and dyonic fields equation in terms of split octonion and its Zorn’s vector matrix realizations along with the corresponding field equations (Maxwell’s equations) and equation of motion in unique and consistent manner.

In this paper, we propose the idea of split octonions for unified theory of linear gravity and electromagnetism with the simultaneous existence of electric, magnetic, gravitational and Heavisididian charges (masses). Relevant field equations and corresponding quantization parameters have been derived in consistent, compact, simpler and manifestly covariant forms. It has been shown that this unified theory reproduces the dynamics of individual charges (masses) in the absence of others. 

An octonion is defined as,

$$X = X_0e_0 + X_ie_i$$  \hspace{1cm} (1)
where \( i = 1, 2, \ldots, 7 \), \( e_i \) are octonion unit elements satisfying following multiplication rules\[17\],

\[
e_i e_k = -\delta_{jk} e_0 + f_{jkl} e_l
\]

(2)

with

\[
f_{jkl} = +1 \text{ for } jkl = 123, 516, 624, 435, 471, 673, 672.
\]

(3)

Usually

\[
e_j (e_k e_l) \neq (e_j e_k) e_l.
\]

(4)

The commutation rules for octonion basis elements are given by,

\[
[e_j, e_k] = 2f_{jkl} e_l
\]

\[
\{e_j, e_k\} = -2\delta_{jk} e_0.
\]

(5)

Octonion conjugate is defined as,

\[
\overline{X} = X_0 e_0 - X_i e_i, \quad (i = 1, 2, \ldots, 7)
\]

(6)

and

\[
\overline{X} = X^*; \quad \overline{X} Y = \overline{Y} \overline{X}
\]

(7)

The norm \( N \) of the octonion is defined as,

\[
N(X) = X \overline{X} = X^2 = (X_0^2 + X_i^2) e_0
\]

(8)

and the inverse is defined as,

\[
X^{-1} = \frac{X}{N(X)}; \quad X^{-1} X = XX^{-1} = 1 e_0.
\]

(9)

For the split octonion algebra the following new basis is considered on the complex field (instead of real field) i.e. \[15\]

\[
u_1 = \frac{1}{2} (e_1 + ie_4), \quad u_1^* = \frac{1}{2} (e_1 - ie_4),
\]

\[
u_2 = \frac{1}{2} (e_2 + ie_5), \quad u_2^* = \frac{1}{2} (e_2 - ie_5),
\]

\[
u_3 = \frac{1}{2} (e_3 + ie_6), \quad u_3^* = \frac{1}{2} (e_3 - ie_6),
\]

\[
u_0 = \frac{1}{2} (1 + ie_7), \quad u_0^* = \frac{1}{2} (1 - ie_7),
\]

(10)

where \( i = \sqrt{-1} \) and is assumed to commute with \( e_A (A = 1, 2, \ldots, 7) \) octonion units,
\[ u_i u_j = \varepsilon_{ijk} u_k \]
\[ u_i^* u_j^* = -\varepsilon_{ijk} u_k \quad (i,j,k = 1,2,3) \]
\[ u_i u_j^* = -\delta_{ij} u_0; u_i u_o = 0; u_i^* u_j = 0 \]
\[ u_i u_j^* = -\delta_{ij} u_0; u_i u_j^* = 0; u_i^* u_j^* = 0 \]
\[ u_i^* u_i = u_i^*; u_i u_i^* = 0; u_i u_i^* = 0 \]
\[ u_i u_i^* = u_i^* \]
\[ u_0 u_0^* = u_0^* \]
\[ u_0 u_0^* = u_0^* = 0. \]

Let us introduce a convenient realization for the basis elements \((u_0, u_i, u_i^*, u_j^*)\) through the use of Pauli matrices. Identifying can do this

\[
\begin{align*}
  u_0 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, u_0^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
  u_i &= \begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}, u_i^* = \begin{pmatrix} 0 & -e_i \\ 0 & 0 \end{pmatrix} \quad (i = 1,2,3)
\end{align*}
\]

where \(1, e_1, e_2, e_3\) are quaternion units satisfying the multiplication rule \(e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k\).

As such for an arbitrary split octonion \(A\) we have [15],

\[
A = au_0^* + bu_0 + xu_i^* + yu_i
\]

\[= \left( \begin{array}{c} a \\ -\bar{x} \\ y \\ b \end{array} \right) \] (13)

which is a realization via the 2X2 Zorn’s vector matrices,

\[
\begin{pmatrix} a & \bar{x} \\ y & b \end{pmatrix}
\]

where \(a\) and \(b\) are scalars and \(\bar{x}\) and \(\bar{y}\) are three vectors with product,

\[
\begin{pmatrix} a & \bar{x} \\ \bar{y} & b \end{pmatrix} \begin{pmatrix} c & \bar{u} \\ v & d \end{pmatrix} = \begin{pmatrix} ac + \bar{x}\bar{v} & a\bar{u} + d\bar{x} - \bar{y}\times\bar{v} \\ cv + b\bar{v} + \bar{x}\times\bar{u} & \bar{y}\bar{u} + bd \end{pmatrix}
\]

and \(X\) denotes the usual vector product, \(e_i (i = 1,2,3)\) with \(e_i \times e_j = \varepsilon_{ijk} e_k\) and \(e_i e_j = \delta_{ij}\) then we can relate the split octonions to the vector matrices given by Eq.(12).

Octonion conjugation of Eq.(13) is defined as,

\[
\overline{A} = bu_0^* + au_0 - xu_i^* - yu_i = \begin{pmatrix} b & \bar{x} \\ -\bar{y} & a \end{pmatrix}
\]

The norm of \(A\) is then defined as,

\[
\overline{A} A = A \overline{A} = (ab + \bar{x}\bar{y}).1
\]

where 1 is the identity element of the algebra given by \(1 = lu_0^* + lu_o\).
The space–time four-differential operators can then be written as,

\[
\left( \frac{\partial}{\partial_4} - \nabla, \nabla - \frac{\partial}{\partial_4} \right).
\]

(17)

In order to formulate unified theory of generalised electromagnetic fields (associated with dyons) and generalised gravito-Heavisidian fields (associated with g-dyons) of linear gravity, we describe split octonion algebra and use of natural units \((c=\hbar=1)\), and gravitational constant is taken unity. Like equation two dyonic charges can be combined together to the following unified split octonion charge,

\[
Q = (e, g, m, h)
\]

\[
= e(u_0 + u_0^*) + g(u_1 + u_1^*) + m(u_2 + u_2^*) + h(u_3 + u_3^*)
\]

\[
= \left( \begin{array}{c}
e \\
e, g + e, m + e, h
\end{array} \right)
\]

where \(e, g, m, h\) are respectively electric, magnetic, gravitational and Heavisidian charges (masses). Structure \((e, g)\) represents the generalised charge of dyons in electromagnetic fields while \((m, h)\) is the generalised charge of gravito-dyons. The norm of unified split octonion charge is defined as,

\[
N(Q) = \bar{Q}Q = \bar{Q}Q
\]

\[
= \left( \begin{array}{cc}
e^2 + g^2 + m^2 + h^2 & 0 \\
0 & e^2 + g^2 + m^2 + h^2
\end{array} \right)
\]

\[
= (e^2 + g^2 + m^2 + h^2).I
\]

(19)

where

\[
\bar{Q} = e(u_0 + u_0^*) - g(u_1 + u_1^*) - m(u_2 + u_2^*) - h(u_3 + u_3^*)
\]

\[
= \left( \begin{array}{c}
e \\
e, g + e, m + e, h
\end{array} \right)
\]

(20)

The interaction of \(a^{th}\) split octonionic charge \(Q_a\) in the field of \(b^{th}\) split octonionic charge \(Q_b\) depends on the quantity,

\[
Q_a \bar{Q}_b = u_0 (e_a e_b + m_a m_b + g_a g_b + h_a h_b)
\]

\[
+ u_1 (-e_a g_b + g_a e_b + m_a h_b - h_a m_b)
\]

\[
+ u_2 (-e_a m_b + m_a e_b + g_a g_b - g_a h_b)
\]

\[
+ u_3 (-e_a h_b + h_a e_b + g_a m_b - m_a g_b)
\]

\[
+ u_0^* (e_a e_b + m_a m_b + g_a g_b + h_a h_b)
\]

\[
+ u_1^* (h_a m_b - m_a h_b - e_a g_b + g_a e_b)
\]

\[
+ u_2^* (g_a h_b - h_a g_b - e_a m_b + m_a e_b)
\]

\[
+ u_3^* (m_a g_b - g_a m_b - e_a h_b + h_a e_b)
\]
Unified split octonion valued four-potential may be introduced as,

\[
\tilde{Q}_a \tilde{Q}_b = u_0(e_a e_b + m_a m_b + g_a g_b + h_a h_b) \\
+ u_1(e_a g_b - g_a e_b + m_a h_b - h_a m_b) \\
+ u_2(e_a m_b - m_a e_b + h_a g_b - g_a h_b) \\
+ u_3(e_a h_b - h_a e_b + g_a m_b - m_a g_b) \\
+ u'_0(e_a e_b + m_a m_b + g_a g_b + h_a h_b) \\
+ u'_1(h_a m_b - m_a h_b + e_a g_b - g_a e_b) \\
+ u'_2(g_a h_b - h_a g_b + e_a m_b - m_a e_b) \\
+ u'_3(m_a g_b - g_a m_b + e_a h_b - h_a e_b)
\]  

(21.a)

Unified split octonion valued four-potential may be introduced as,

\[
V = (A_0 + B_0 + C_0 + D_0)(u_0 + u'_0) + (A_i + B_i + C_i + D_i)(u_i + u'_i)
\]

\[
= \left( \begin{array}{c}
A_0 + B_0 + C_0 + D_0 \\
(A_0 + B_0 + C_0 + D_0) e_0
\end{array} \right) + (A_i e_i + B_i e_i + C_i e_i + D_i e_i)
\]

(22)

where \(A, B, C\) and \(D\) are respectively associated with electric, magnetic, gravitational (g-electric) and Heavisidian (g-magnetic) charges. Unified split octonion valued potential (22) can also be written as,

\[
V = (A, B, C, D) = \left( \begin{array}{c}
A_0 + B_0 + C_0 + D_0 \\
(A_0 + B_0 + C_0 + D_0) e_0
\end{array} \right)
\]

(23)

\[
A = (A_0, A_1, A_2, A_3) = \left( \begin{array}{c}
A_0 \\
(A_0 e_1 + A_2 e_2 + A_3 e_3)
\end{array} \right)
\]

(24.a)

\[
B = (B_0, B_1, B_2, B_3) = \left( \begin{array}{c}
B_0 \\
(B_0 e_1 + B_2 e_2 + B_3 e_3)
\end{array} \right)
\]

(24.b)

\[
C = (C_0, C_1, C_2, C_3) = \left( \begin{array}{c}
C_0 \\
(C_0 e_1 + C_2 e_2 + C_3 e_3)
\end{array} \right)
\]

(24.c)

\[
D = (D_0, D_1, D_2, D_3) = \left( \begin{array}{c}
D_0 \\
(D_0 e_1 + D_2 e_2 + D_3 e_3)
\end{array} \right)
\]

(24.d)

are the split octonion valued potential for four-charges. Split octonion valued unified vector field associated with the quaternion charge may then be defined as

\[
\psi = (E_i + M_i + G_i + H_i)(u_i + u'_i)
\]

\[
= \left( \begin{array}{c}
0 \\
-(E_i + M_i + G_i + H_i)e_i
\end{array} \right)
\]

(25)

where \(\vec{E}, \vec{M}, \vec{G}\) and \(\vec{H}\) are electric, magnetic, gravitational and Heavisidian fields respectively and hence the following definitions,
\[ \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} A_0 + \vec{\nabla} \times \vec{B} \quad (26.a) \]

\[ \vec{M} = -\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} B_0 + \vec{\nabla} \times \vec{A} \quad (26.b) \]

\[ \vec{G} = -\frac{\partial \vec{C}}{\partial t} - \vec{\nabla} C_0 + \vec{\nabla} \times \vec{D} \quad (26.c) \]

\[ \vec{H} = -\frac{\partial \vec{D}}{\partial t} - \vec{\nabla} D_0 + \vec{\nabla} \times \vec{C} \quad (26.d) \]

where \( \vec{\nabla} = e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3 \). These generalised electric, magnetic, gravitational and Heavisidian fields satisfy the following pair of Maxwell’s equations of dyons [9-10]:

\[ \vec{\nabla} \cdot \vec{E} = j^e_0, \quad \vec{\nabla} \cdot \vec{G} = j^m_0 \\
\vec{\nabla} \cdot \vec{M} = j^g_0, \quad \vec{\nabla} \cdot \vec{H} = j^h_0 \\
\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{M}}{\partial t} - j^g \quad ; \quad \vec{\nabla} \times \vec{G} = -\frac{\partial \vec{H}}{\partial t} - j^h \\
\vec{\nabla} \times \vec{M} = -\frac{\partial \vec{E}}{\partial t} + j^e \quad ; \quad \vec{\nabla} \times \vec{H} = -\frac{\partial \vec{G}}{\partial t} + j^m \]

(27)

where \( (j^e_0, j^m_0, j^g_0, j^h_0) \) are respectively charge source densities for electric, magnetic, gravitational and Heavisidian charges while \( (\vec{j}^e, \vec{j}^m, \vec{j}^g, \vec{j}^h) \) are corresponding current source densities of these charges. Split octonion valued unified potential \(\vec{V}\) given by equation (23) and unified field \(\vec{\psi}\) are then related as,

\[ \vec{\psi} = -\frac{\partial \vec{V}}{\partial t} - \vec{\nabla} V_0 + \vec{\nabla} \times \vec{V} \quad (28) \]

Operating \( [ ] \) given by equation (17) to equation (23) we get the following unified split octonion form of potential equation,

\[
\begin{bmatrix}
\partial_0 \\ \partial_i e_i \\
\end{bmatrix}
\begin{pmatrix}
A_0 + B_0 + C_0 + D_0 & -(A_i e_i + B_i e_i + C_i e_i + D_i e_i) \\
(A_i e_i + B_i e_i + C_i e_i + D_i e_i) & A_0 + B_0 + C_0 + D_0 \\
\end{pmatrix}

= \begin{bmatrix}
\partial_0 (A_0 + B_0 + C_0 + D_0) \\
\partial_i (A_i + B_i + C_i + D_i) \\
\end{bmatrix}
\begin{pmatrix}
-(A_i e_i + B_i e_i + C_i e_i + D_i e_i) \\
(A_i e_i + B_i e_i + C_i e_i + D_i e_i) \\
\end{pmatrix}

+ \begin{bmatrix}
\partial_i e_i \\
\partial_0 (A_0 + B_0 + C_0 + D_0) \\
\partial_i (A_i + B_i + C_i + D_i) \\
\partial_0 (A_i e_i + B_i e_i + C_i e_i + D_i e_i) \\
\partial_i (A_i e_i + B_i e_i + C_i e_i + D_i e_i) \\
\end{bmatrix}
\begin{pmatrix}
(A_0 + B_0 + C_0 + D_0) \\
A_0 + B_0 + C_0 + D_0 \\
\end{pmatrix}

\begin{bmatrix}
\partial_i e_i \\
\partial_0 (A_0 + B_0 + C_0 + D_0) \\
\partial_i (A_i + B_i + C_i + D_i) \\
\partial_0 (A_i e_i + B_i e_i + C_i e_i + D_i e_i) \\
\partial_i (A_i e_i + B_i e_i + C_i e_i + D_i e_i) \\
\end{bmatrix}
\begin{pmatrix}
(A_i e_i + B_i e_i + C_i e_i + D_i e_i) \\
(A_i e_i + B_i e_i + C_i e_i + D_i e_i) \\
\end{pmatrix}

(29)
which is the unified potential equation of gravitational and electromagnetic generalised charges of dyons. Eq. (29) is invariant under split octonion Lorentz transformations. As such, the unified potential field equation (29) is covariant, compact and sympletic. Unified split octonion valued current, the components of which are given by Eq. (27), can be written as,

\[
J = \begin{pmatrix}
  j_0^e + j_0^g + j_0^m + j_0^h \\
  e_i j_i^e + e_i j_i^g + e_i j_i^m + e_i j_i^h
\end{pmatrix}
- e_i j_i^m - e_i j_i^g - e_i j_i^m - e_i j_i^h
- e_i j_i^m - e_i j_i^g - e_i j_i^m - e_i j_i^h
+ j_0^e + j_0^g + j_0^m + j_0^h
\]

(30)

where

\[
\begin{align*}
\tilde{\mathbf{A}} &= \begin{pmatrix}
  j_0^e \\
  e_i j_i^e + e_i j_i^g + e_i j_i^m + e_i j_i^h
\end{pmatrix} = j^e \\
\tilde{\mathbf{B}} &= \begin{pmatrix}
  j_0^g \\
  e_i j_i^e + e_i j_i^g + e_i j_i^m + e_i j_i^h
\end{pmatrix} = j^g \\
\tilde{\mathbf{C}} &= \begin{pmatrix}
  j_0^m \\
  e_i j_i^e + e_i j_i^g + e_i j_i^m + e_i j_i^h
\end{pmatrix} = j^m \\
\tilde{\mathbf{D}} &= \begin{pmatrix}
  j_0^h \\
  e_i j_i^e + e_i j_i^g + e_i j_i^m + e_i j_i^h
\end{pmatrix} = j^h
\end{align*}
\]

(31.a) (31.b) (31.c) (31.d)

and

\[
\tilde{\mathbf{J}} = \left(\partial^2 + \nabla^2\right) J = \partial^2 + \partial_\mu \partial_\mu = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]

(32)

\[
\tilde{\mathbf{V}} = \begin{pmatrix}
  \partial_4 \\
  -\nabla
\end{pmatrix}
\]

(33)

is the split octonion conjugate differential operator. Split octonion field octonion charge may then be written as,

\[
\tilde{\mathbf{J}}' = \begin{pmatrix}
  j_0^e + j_0^g + j_0^m + j_0^h \\
  e_i j_i^e + e_i j_i^g + e_i j_i^m + e_i j_i^h
\end{pmatrix}
- e_i j_i^m - e_i j_i^g - e_i j_i^m - e_i j_i^h
+ j_0^e + j_0^g + j_0^m + j_0^h
\]

(34)

and continuity equation is equivalent to the inner product of two split octonions given by Eq.(17) and (34) for differential operator and current. Split octonion valued field tensor density is defined as,

\[
Q_{\mu\nu} = \begin{pmatrix}
  A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu} + D_{\mu\nu} \\
  -A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu} + D_{\mu\nu}
\end{pmatrix}
\]

(35)

where

\[
Q_{\mu\nu} = V_{\mu\nu} - V_{\nu\mu} = \begin{pmatrix}
  A_{00} + B_{00} + C_{00} + D_{00} \\
  -A_{00} + B_{00} + C_{00} + D_{00}
\end{pmatrix}
\]

(36.a)
\[ A_{\mu \nu} = A_{\mu,\nu} - A_{\nu,\mu} = \begin{pmatrix} A_{0,0} & -A_{1,1}e_1 - A_{2,2}e_2 - A_{3,3}e_3 \\ A_{1,1}e_1 + A_{2,2}e_2 + A_{3,3}e_3 & A_{0,0} \end{pmatrix} \]  \hspace{1cm} (36.b)

\[ B_{\mu \nu} = B_{\mu,\nu} - B_{\nu,\mu} = \begin{pmatrix} B_{0,0} & -(B_{1,1}e_1 + B_{2,2}e_2 + B_{3,3}e_3) \\ B_{1,1}e_1 + B_{2,2}e_2 + B_{3,3}e_3 & B_{0,0} \end{pmatrix} \]  \hspace{1cm} (36.c)

\[ C_{\mu \nu} = C_{\mu,\nu} - C_{\nu,\mu} = \begin{pmatrix} C_{0,0} & -(C_{1,1}e_1 + C_{2,2}e_2 + C_{3,3}e_3) \\ C_{1,1}e_1 + C_{2,2}e_2 + C_{3,3}e_3 & C_{0,0} \end{pmatrix} \]  \hspace{1cm} (36.d)

\[ D_{\mu \nu} = D_{\mu,\nu} - D_{\nu,\mu} = \begin{pmatrix} D_{0,0} & -(D_{1,1}e_1 + D_{2,2}e_2 + D_{3,3}e_3) \\ D_{1,1}e_1 + D_{2,2}e_2 + D_{3,3}e_3 & D_{0,0} \end{pmatrix} \]  \hspace{1cm} (36.e)

are respectively field tensor of electric, magnetic, gravitational and Heavisidian charges and comma(,) denotes partial differentiation. \( A_{\mu} \) and \( B_{\mu} \) are dual invariant for generalised electromagnetic fields of dyons while \( C_{\mu} \) and \( D_{\mu} \) are dual invariant under duality transformations for generalised fields of gravito-dyons. Unified split octonion valued field tensor \( Q_{\mu \nu} \) is itself self-dual and also invariant under octonion transformations. The components of split octonion field tensor are the components of split octonion vector field \( \vec{\psi} \) given by Eq. (28). Unified split octonion valued current and split octonion tensor \( Q_{\mu \nu} \) is then related in the following manner;

\[ Q_{\mu,\nu} = \begin{pmatrix} j^e_0 + j^g_0 + j^m_0 + j^h_0 \\ e_\mu j^e_\mu + e_\mu j^g_\mu + e_\mu j^m_\mu + e_\mu j^h_\mu \end{pmatrix} = J_{\mu} \]  \hspace{1cm} (37)

which is equivalent to the following form;

\[ \vec{\psi} = J. \]  \hspace{1cm} (38)

The Eq. (38) is the split octonion unified field equation of linear gravitational fields of gravito-dyons and electromagnetic fields of dyons. In other words, this equation is split octonion form of Maxwell’s –Dirac equations of dyons and gravito-dyons.

The above analysis states that the dynamics of simultaneous existence of electric, magnetic, Gravitational and Heavisidian charges (masses). Though the existence of magnetic and Heavisidian charges is not confirmed, but sound theoretical investigations are in favour of their existence and lead to the deeper understanding of fundamental interactions and constituents of matter. From the above analysis it may also be concluded that besides the potential importance of monopoles as intrinsic part of grand unified theories, monopoles and dyons may provide even more ambitious model to purport the unification of gravitation with strong and electro weak forces.
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