Momentum Broadening in an Anisotropic Plasma

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The rates governing momentum broadening in a quark-gluon plasma with a momentum anisotropy are calculated to leading-log order for a heavy quark using kinetic theory. It is shown how the problematic singularity for these rates at leading-order is lifted by next-to-leading order gluon self-energy corrections to give a finite contribution to the leading-log result. The resulting rates are shown to lead to larger momentum broadening along the beam axis than in the transverse plane, which is consistent with recent STAR results. This might indicate that the quark-gluon-plasma at RHIC is not in equilibrium.

I. INTRODUCTION

One of the hottest debated questions in the context of ultrarelativistic heavy ion collisions is whether a “thermal” quark-gluon plasma has been created at the highest energy runs of the Relativistic Heavy-Ion Collider (RHIC). On the one hand, the success of ideal hydrodynamic fits to experimental data have been interpreted [1] to imply rapid thermalization of the bulk matter at RHIC. On the other hand, perturbative estimates of the thermalization time [2] result in much larger values, seemingly excluding the possibility of rapid thermalization (see however [3]). Plasma instabilities may help speeding up the equilibration process by rapidly isotropizing the system [4, 5, 6, 7, 8], probably by some cascade to the UV [9, 10, 11, 12, 13], although they might still be too slow to overcome the initially strong longitudinal expansion rate [14]. Thus, it is currently unclear how (and if) the quark gluon plasma created at RHIC stays locally isotropic (let alone thermal) during most of its time evolution.

Nevertheless, many theoretical calculations in the context of heavy-ion collisions assume a homogeneous and locally isotropic system. However, the physics of anisotropic plasmas (even if they are close to isotropy) can be quite different from that of isotropic plasmas, c.f. the presence of plasma instabilities in the former. Due to these potentially large differences, it might be interesting to reanalyze calculations of experimentally accessible observables in the context of anisotropic plasmas, wherever possible.

In what follows, I will thus study momentum broadening in a homogeneous but locally anisotropic system. Since I will be interested in qualitative effects only, I limit myself to considering momentum broadening for a heavy quark induced by collisions to leading logarithm (LL) accuracy. In section II I will discuss the calculational setup for anisotropic plasmas. Section III will contain the calculation of the rates governing momentum broadening, which can be done analytically in the small anisotropy limit. In section IV finally I will try to apply these results to preliminary STAR data on jet shapes and discuss possible consequences.

II. SETUP

The setup of my calculation will be based on the work by Moore and Teaney [15]. They considered a heavy quark with momentum $p$ moving in a static (thermal) medium with a
temperature $T$ and I will quickly repeat some equations from their work which I want to use in the following.

Due to collisions, the quark will lose momentum and the variance of its associated momentum distribution will broaden according to [15]

\[
\frac{d}{dt} \langle p \rangle = -p \eta_D(p) \\
\frac{d}{dt} \langle (\Delta p_\parallel)^2 \rangle = \kappa_\parallel(p) \\
\frac{d}{dt} \langle (\Delta p_\perp)^2 \rangle = \kappa_\perp(p) \\
\frac{d}{dt} \langle (\Delta p_z)^2 \rangle = \kappa_z(p),
\]
where I choose my coordinate system such that $\langle (\Delta p_\parallel)^2 \rangle = (p_\parallel - \langle p_\parallel \rangle)^2$ is the variance of the momentum distribution in the direction parallel to the direction of the quark and $\langle (\Delta p_\perp)^2 \rangle, \langle (\Delta p_z)^2 \rangle$ are the variances transverse to the direction of the quark (see also Fig.1).

The functions $\eta_D, \kappa_\parallel, \kappa_\perp, \kappa_z$ which encode average momentum loss as well as transverse and longitudinal fluctuations are calculated using kinetic theory. Schematically, they are given as [15]

\[
\frac{d}{dt} \langle p \rangle = \frac{1}{2v} \int |\mathcal{M}|^2 q^0 \left[ (1 \pm f(k-q^0)) f(k) - f(k-q^0)(1 \pm f(k)) \right] \\
\frac{d}{dt} \langle (\Delta p_\parallel)^2 \rangle = \int |\mathcal{M}|^2 q_\parallel^2 f(k) \left[ 1 \pm f(k-q^0) \right] \\
\frac{d}{dt} \langle (\Delta p_\perp)^2 \rangle = \int |\mathcal{M}|^2 q_\perp^2 f(k) \left[ 1 \pm f(k-q^0) \right] \\
\frac{d}{dt} \langle (\Delta p_z)^2 \rangle = \int |\mathcal{M}|^2 q_z^2 f(k) \left[ 1 \pm f(k-q^0) \right],
\]
where $v = p/p_0$, $f(k)$ is the gluon (quark) distribution function, $\mathcal{M}$ is the scattering matrix and $\int_k = \int \frac{d^4k}{(2\pi)^4}$ (similarly for $q$). If the transferred energy $q^0 \simeq v \cdot q$ is small, one can approximate

\[
f(k) \left[ 1 \pm f(k-q^0) \right] f(k) \left[ 1 \pm f(k-q^0) \right] \simeq f(k) \left[ 1 \pm f(k-q^0) \right] \simeq f(k) \left[ 1 \pm f(k) \right].
\]

Thus, if the quark is non-relativistic or if $q^0 \ll T$, the coefficient $\kappa_\parallel$ can be related to the energy loss rate $-\frac{dp^0}{dt}$ by $\kappa_\parallel = -\frac{q^0}{T} \frac{dp^0}{dt}$. This implies that also the other coefficient functions can be calculated by the appropriate modifications in the integrand of the energy loss rate.

A. Collisional Energy Loss

Studies of collisional energy loss of a heavy quark in isotropic systems have a long history [15, 16, 17, 18, 19, 20, 21, 22]. Assuming a perturbative expansion in powers (and logarithms) of the strong coupling $\alpha_s = \frac{g^2}{4\pi}$, the leading-order logarithmic contribution to
the collisional energy loss can be obtained by calculating the matrix elements in Eqns. (2) with HTL propagators and restricting to soft momentum transfer $q^0 < T$ where Eqns. (3) can be used \[15\].

There is, however, an alternative way by Thoma and Gyulassy \[18\] to calculate the collisional energy loss which is equivalent to calculating the HTL matrix elements and doing the integrals in Eqns. (1). It is based on the expression for the energy loss of a quark by interaction with its induced chromoelectric field,

$$\frac{dp^0}{dt} = \text{Re} \int d^4x \mathbf{J}_{\text{ext}}(t,\mathbf{x}) \cdot \mathbf{E}_{\text{ind}}^i(t,\mathbf{x}).$$

Here $\mathbf{J}_{\text{ext}}^i = q^a \mathbf{v} \delta^3(\mathbf{x} - \mathbf{v}t)$ is the current of the quark with color charge $q^a$ and velocity $\mathbf{v}$ and $\mathbf{E}_{\text{ind}}^i$ is the induced electric field. In Fourier space $Q = (q, \omega)$, $\mathbf{E}_{\text{ind}}^i$ can be written as

$$E_{\text{ind}}^i(\omega, \mathbf{q}) = i\omega (G^{ij} - G_0^{ij}) J_{\text{ext}}^j(Q),$$

where $G^{ij}$ and $G_0^{ij}$ are the full and free retarded gluon propagator, respectively.

The collisional energy loss for a theory with $N_c$ colors thus becomes

$$\frac{dp^0}{dt} = -g^2 \frac{(N_c^2 - 1)}{2N_c} \text{Im} \int \frac{d^4Q}{(2\pi)^4} \omega v^i \left( G^{ij} - G_0^{ij} \right) v^j \delta(2\pi) \delta(\omega - \mathbf{v} \cdot \mathbf{q}).$$

A convenient feature of these Equations is that all the information about the medium resides in the gluon propagator $G$. Thus, in this form Eqns. (7) are valid both for isotropic as well as for anisotropic systems.

### B. Anisotropic Plasmas

In the rest frame of a thermal system, particles move in all directions with equal probability. However, in the rest frame of a non-equilibrium system, particles might e.g. move predominantly in a plane, having an anisotropic probability distribution. Thus, dropping the assumption of isotropy, there will be at least one preferred direction in the system, which in the following I will take to be the $z$-direction. In the context of heavy-ion collisions, one can identify this direction with the beam-axis along which the system expands initially. As a model for an anisotropic system, I will assume that the anisotropic quark and gluon distribution functions $f(k)$ are related to the usual (isotropic) Fermi and Bose distributions $f_{\text{iso}}(|k|)$ as \[26\]

$$f(k) = N(\xi) f_{\text{iso}} \left( \sqrt{k^2 + \xi(\mathbf{k} \cdot \mathbf{e}_z)^2} \right),$$

where $N(\xi)$ is the Fermi-Dirac distribution and $\mathbf{e}_z$ is the unit vector in the $z$-direction.
where \( e_z \) denotes the unit-vector in the z-direction, \( \xi \) is a parameter controlling the strength of the anisotropy and \( N(\xi) \) is a normalization factor which in the following I will set to one. Note that \( \xi = 0 \) corresponds to the case of an isotropic system.

An anisotropic system by definition does not have one temperature. However, it may have a dimensionful scale that is related to the mean particle momentum and which takes over the role of a "hard" scale in much the same way the temperature does in a thermal system. For a quark gluon plasma (even out of equilibrium), the natural choice seems to be the saturation scale \( Q_s \). Since kinetic theory calculations remain valid even out of equilibrium as long as there is a separation of scales between hard and soft momentum modes, one could replace \( T \) by \( Q_s \) times some coefficient when translating formulae from an isotropic to an anisotropic system. For simplicity, however, I will refrain from doing that and keep \( T \) as a placeholder of the correct hard scale even for anisotropic systems.

As outlined before, the main difference between calculating the fluctuation coefficients Eqns. (7) in isotropic and anisotropic systems, respectively, is the form of the gluon propagator \( G \). While for isotropic systems the propagator in the HTL approximation is given by

\[
G_{\text{iso}}^{ij}(Q) = \frac{\delta^{ij} - q^i q^j/q^2}{q^2 - \omega^2 + \Pi_T(\omega, q) - \Pi_L(\omega, q)} + \frac{q^i q^j}{\omega^2 (\Pi_L(\omega, q) - q^2)}
\]

\[
\Pi_T(\omega, q) = \frac{m_D^2 \omega^2}{2 q^2} \left[ 1 - \frac{\omega^2 - q^2}{2 \omega |q|} \log \frac{\omega + |q|}{\omega - |q|} \right]
\]

\[
\Pi_L(\omega, q) = m_D^2 \left[ \frac{\omega}{2 |q|} \log \frac{\omega + |q|}{\omega - |q|} - 1 \right],
\]

one finds for systems with an anisotropy of the form Eq. (8) within the same approximation [26]

\[
G^{ij}(Q) = \frac{\delta^{ij} - q^i q^j/q^2 - \tilde{n}^i \tilde{n}^j/\tilde{n}^2}{q^2 - \omega^2 + \alpha(\omega, q)} + \frac{(q^2 - \omega^2 + \alpha(\omega, q) + \gamma(\omega, q)) q^i q^j/q^2}{(\beta(\omega, q) - \omega^2) - \tilde{n}^2 \delta^2(\omega, q)}
\]

\[
+ \frac{(\beta(\omega, q) - \omega^2) \tilde{n}^i \tilde{n}^j/\tilde{n}^2 - \delta(\omega, q) (q^i \tilde{n}^j + q^j \tilde{n}^i)}{(q^2 - \omega^2 + \alpha(\omega, q) + \gamma(\omega, q)) (\beta(\omega, q) - \omega^2) - \tilde{n}^2 \delta^2(\omega, q)},
\]

where here \( \tilde{n} = e_z - q q_z/q^2 \) and the structure functions \( \alpha, \beta, \gamma, \delta \) are generalizations of \( \Pi_T \) and \( \Pi_L \) to anisotropic systems. In the case of small anisotropy \( \xi \), they are given as [26]

\[
\alpha = \Pi_T(\tilde{\omega}) + \xi \left[ \frac{\tilde{\omega}^2}{6} (5 \tilde{q}_z^2 - 1)m_D^2 - \frac{1}{3} \tilde{q}_z^2 m_D^2 \right.
\]

\[
+ \frac{1}{2} \Pi_T(\tilde{\omega}) \left( (3 \tilde{q}_z^2 - 1) - \tilde{\omega}^2 (5 \tilde{q}_z^2 - 1) \right)
\]

\[
\left. - \tilde{\omega}^{-2} \beta = \Pi_L(\tilde{\omega}) + \xi \left[ \frac{1}{3} (3 \tilde{q}_z^2 - 1)m_D^2 \right. \right.
\]

\[
+ \Pi_L(\tilde{\omega}) \left( (2 \tilde{q}_z^2 - 1) - \tilde{\omega}^2 (3 \tilde{q}_z^2 - 1) \right) \right],
\]

\footnote{Here and in the following (unless stated otherwise) I use the specific choice of gauge \( A^0 = 0 \); physical observables are not affected by this choice as they are manifestly gauge-invariant.}
\[ \gamma = \frac{\xi}{3} (3 \Pi_T(\hat{\omega}) - m_D^2)(\hat{\omega}^2 - 1)(1 - \hat{q}_z^2), \]
\[ \delta = \frac{\xi}{3k} (4 \hat{\omega}^2 m_D^2 + 3 \Pi_T(\hat{\omega})(1 - 4 \hat{\omega}^2))\hat{q}_z, \]
where \( \hat{\omega} = \omega/|q| \) and \( \hat{q}_z = q_z/|q| \). For larger values of \( \xi \) one has to resort to numerical evaluations of \( \alpha, \beta, \gamma, \delta \) \[26\].

In all cases \( m_D \) is the isotropic Debye mass,
\[ m_D^2 = \frac{g^2 T^2}{\pi^2} \int_0^\infty dp \, p f_{iso}(\{|p|\}), \]
which for a thermal system with \( N_c \) colors and \( N_f \) light quark flavors becomes
\[ m_D^2 = \frac{g^2 T^2 N_c}{3} \left( 1 + \frac{N_f}{6} \right). \]

### III. MOMENTUM BROADENING

As can be quickly verified, the general form of the propagator Eq. (10) reduces to the simple form Eq. (2) in the limit of vanishing anisotropy \( \xi \to 0 \). Thus, regardless whether the system is isotropic or not, the fluctuation coefficients are found by inserting Eq. (10) into Eqns. (7).

The frequency integration is trivial, whereas the integration over total momentum exchange \( |q| \) contains a subtlety for non-vanishing anisotropy \( \xi \): since the static limit (\( \omega \to 0 \)) of e.g. \( \alpha \) is real and negative, the propagator develops a singularity for space-like momenta \( |q| \). Indeed, this singularity signals the presence of instabilities in the system \[25, 26\]. Since these instabilities correspond to soft gauge modes that grow nearly exponentially until they reach non-perturbative occupation numbers, one might dismiss any perturbative approach such as mine as futile. However, my calculation should be applicable up to the time where the soft mode occupation number has not grown non-perturbatively large yet. Even with fast growing instabilities, this can be a long physical timescale if either the initial fluctuations are tiny or there is a counter-acting effect such as the expansion of the system. In the latter case, one can even argue that unstable modes do not grow non-perturbatively large during plasma lifetimes at RHIC \[14\]. Therefore, even though for quantitative results one probably has to resort to numerical simulations, ignoring the effects of the non-perturbative soft gauge modes may be not such a bad approximation at least for some period of time.

As a consequence, here the propagator singularity is mainly a practical obstacle to calculating physical observables since this singularity is in general non-integrable, as was first recognized by Arnold, Moore and Yaffe \[25\].

However, the fact that \( \alpha \) has an imaginary part that vanishes only linearly in the static limit cures this problem for certain observables (namely those with numerators that also turn out to vanish in the static limit), which has been dubbed “Dynamical Shielding” \[27\].

In the formulation I have chosen, Eq. (7), there is only one power of the propagator and consequently the singularity is always integrable. However, the problem resurfaces in the next integration step, where for some of the fluctuation coefficients there is an uncanceled \( \omega \) in the denominator of the integral. Unlike the collisional energy loss, which is finite to full leading-order (LO), dynamical shielding breaks for at least some of the fluctuation coefficients. I will show in the following how to calculate these to LL accuracy.
The calculation is divided up into two parts: first I calculate the “regular” contribution within the LL approximation, which involves only quantities for which dynamical shielding works. Then I will deal with the contribution which would give the LO contribution (the constant under the log) in an isotropic system and show that in anisotropic systems, it gives a contribution to the LL.

A. Regular contributions

Limiting oneself to LL approximation, the \(|q|\) integration in Eq. (7) can be done as in Ref. [27], and one finds e.g.

\[
\kappa_{\perp}^{\text{reg}} = -\frac{g^2(N_c^2 - 1)}{2N_c} \int \frac{d\Omega_q}{(2\pi)^3} \frac{4T_\omega^2}{\omega} \frac{1}{1 - \omega^2} \log \frac{T}{m_D} \text{Im} \left[ \frac{v^2 - \omega^2 - (\hat{n} \cdot \mathbf{v})/\tilde{n}^2}{2(1 - \omega^2)} \right] \alpha \\
+ \frac{(1 - \omega^2)^2}{2\tilde{\omega}^2} \beta + \omega^2 (\hat{n} \cdot \mathbf{v})^2/\tilde{n}^2 (\alpha + \gamma) - 2\tilde{\omega}(1 - \omega^2)(\hat{n} \cdot \mathbf{v})\tilde{\delta},
\]

(13)

where \(\hat{q}_\perp = q_\perp/|q|, \hat{n} = \delta/|q|, \hat{\omega} = \mathbf{v} \cdot \mathbf{q}/|q|\).

I will consider the situation where the quark velocity is perpendicular to the beam axis, \(\mathbf{v} \cdot e_z = 0\). Thus I have \(\hat{\omega} = v\hat{q}_\perp, \hat{n} \cdot \mathbf{v} = \hat{\omega} z\), and \(\tilde{n}^2 = 1 - \hat{q}_\perp^2\). From the explicit form of the small \(\xi\) structure functions \(\alpha, \beta, \gamma, \delta\) in Eq. (11) it can be seen that their imaginary part always involves at least one power of \(\tilde{\omega}\). All the \(\tilde{\omega}\)’s in the denominator of the integrand in Eq. (13) are thus canceled, showing indeed that there is no singularity to LO accuracy.

For small anisotropies, the remaining integrations can be done analytically using Eqns. (9, 11). I find

\[
\frac{dp}{dt} = -\frac{g^2(N_c^2 - 1)}{8N_c} \frac{m_D^2}{8\pi} \log \frac{T}{m_D} \left[ \frac{1}{v} - \frac{1}{v^2} \text{Arctanh}(v) \right] \\
+ \frac{\xi}{6v^4} \left( 3v - 5v^3 - 3(1 - v^2)^2 \text{Arctanh}(v) \right)
\]

\[\kappa_{\perp}^{\text{reg}} = \frac{g^2(N_c^2 - 1) T m_D^2}{4\pi} \log \frac{T}{m_D} \left[ \frac{3}{2} - \frac{1}{2v^2} + \frac{(1 - v^2)^2}{2v^3} \text{Arctanh}(v) \right]
\]
\[
\kappa_{reg}^z = \frac{g^2(N_c^2 - 1)}{2N_c} \frac{T m_D^2}{4\pi} \log \frac{T}{m_D} \left[ \frac{3}{2} - \frac{1}{2v^2} + \frac{(1 - v^2)^2}{2v^5} \text{Arctanh}(v) \right]
\]

\[
+ \frac{\xi}{24v^5} \left( -3v + 8v^3 - 13v^5 + 3(1 - v^2)^3 \text{Arctanh}(v) \right). \tag{14}
\]

As expected, these expression reduce to the known results [15] in the isotropic limit \((\xi \to 0)\).

**B. Anomalous Contribution**

Let us now concentrate on the naive “constant under the log” contribution of Eqns. (14). To this end, it is illustrative pick out the first term of the propagator Eq. (10), and pluck it into Eqns. (7), finding

\[
\kappa_i \sim \int \frac{d^3q}{(2\pi)^3} \frac{\hat{q}^2}{\hat{\omega}^2 - \hat{q}_z^2\hat{\omega}^2} \text{Im} \frac{q^3}{q^2 - \omega^2 + \alpha(\omega, q)}, \tag{15}
\]

where the index \(i\) denotes \(i = \perp, ||, z\) and I have used the same notations as in the previous subsection. Denoting this part of the propagator as \(\Delta_A^{-1} = q^2 - \omega^2 + \alpha(\omega, q)\) allows me to rewrite

\[
\kappa_i \sim \int \frac{d^3q}{(2\pi)^3} \hat{q}_z^2 m_D^2 f(\hat{q}_z) \left( v^2 - \hat{\omega}^2 - \hat{q}_z^2\hat{\omega}^2 \right) q \Delta_A \Delta_A^*, \tag{16}
\]

where I introduced \(m_D^2 \hat{\omega} f(\hat{q}_z) = \text{Im} \alpha\) and the * means complex conjugation. This form now directly involves the squared matrix from Eqns. (2). As discussed before, the static limit of \(\alpha\) is real and negative to leading order in \(g\), which produces a non-integrable singularity in Eq. (16) (unless \(i = ||\) for which \(\hat{q}_z v = \hat{\omega}\) cuts off the singularity).

However, from the structure of Eq. (16), this singularity is reminiscent of so-called pinching singularities, which are usually due to incomplete resummations of the propagator. Indeed, I find it is plausible that \(\alpha\) has a non-vanishing imaginary part in the static limit at order \(O(g^3)\), which allows one to integrate

\[
\int dq q^3 |\Delta_A|^2 = \frac{1}{2(1 - \hat{\omega}^2)^2} \left[ \frac{\text{Re} \alpha}{\text{Im} \alpha} \text{Arctan} \left( \frac{\text{Re} \alpha + q^2(1 - \hat{\omega}^2)}{\text{Im} \alpha} \right) \right.
\]

\[
+ \frac{1}{2} \log \left( \frac{1}{2} \left| q^2 \right|^2 + 2\text{Re} \alpha q^2(1 - \hat{\omega}^2) + q^4(1 - \hat{\omega}^2)^2 \right) \right]. \tag{17}
\]

The second term in this equation gives a contribution to the LL for \(q = T \gg m_D\) which has been accounted for already in the previous subsection. The other term, however, now involves \(\text{Im} \alpha\) rather than the \(\hat{\omega}\) of Eq. (13) in the denominator. The contribution behaves as

\[
\lim_{\hat{\omega} \to 0^+} \text{Im} \alpha \sim m_D^2 (\hat{\omega} + cg), \tag{18}
\]

\[\text{See the appendix for details.}\]
where $c$ is just a number. Thus, the singularity at $\hat{\omega} \to 0^+$ is cut-off by this higher-order resummation in the self-energy. However, this also entails that the naive LO correction gives a contribution to the LL instead, as I will show in the following.

First note that for any nonzero $\hat{\omega}$, the first term on the r.h.s. of Eq. (17) is finite also for $c = 0$ and thus only contributes to LO. In order to extract a potential LL contribution, one may thus take $\hat{\omega} \to 0^+$ everywhere except for the $\hat{\omega}$ in the denominator, which one has to replace by $\hat{\omega} \to \hat{\omega} + c g$. For simplicity, I do this already in Eq. (15), using

\begin{equation}
\lim_{\hat{\omega} \to 0^+} \text{Im} \Delta_A \to + \pi \delta \left( q^2 + \text{Re} \alpha \right).
\end{equation}

As can be verified, for small anisotropies $\xi$ the leading contribution to $\kappa_i$ is entirely given by this part of the propagator.

The integral over $\hat{\omega}$ then gives $\log (g)$ together with some finite LO contribution that I ignore.

Thus, upon identifying $\log (g) = - \log \left( \frac{T_m}{D} \right)$, I find for the anomalous contributions to $\kappa_i$

\begin{align}
\kappa_{\perp}^{\text{anom}} &= - \frac{g^2 (N_c^2 - 1) T m_D^2}{2 N_c} \log \left( \frac{T}{m_D} \right) \frac{\xi v}{12}, \\
\kappa_{z}^{\text{anom}} &= - \frac{g^2 (N_c^2 - 1) T m_D^2}{2 N_c} \log \left( \frac{T}{m_D} \right) \frac{\xi v}{4},
\end{align}

such that together with the regular contributions Eq. (14), the full LL fluctuation coefficients in the small anisotropy limit are given by

\begin{align}
\kappa_{\perp} &= \frac{g^2 (N_c^2 - 1) T m_D^2}{2 N_c} \log \left( \frac{T}{m_D} \right) \left[ \frac{3}{2} - \frac{1}{2 v^2} + \frac{(1 - v^2)^2}{2 v^3} \text{Arctanh}(v) \\
&\quad + \frac{\xi}{24 v^5} \left( -9 v + 24 v^3 - 7 v^5 - 6 v^6 + 9 (1 - v^2)^3 \text{Arctanh}(v) \right) \right], \\
\kappa_{z} &= \frac{g^2 (N_c^2 - 1) T m_D^2}{2 N_c} \log \left( \frac{T}{m_D} \right) \left[ \frac{3}{2} - \frac{1}{2 v^2} + \frac{(1 - v^2)^2}{2 v^3} \text{Arctanh}(v) \\
&\quad + \frac{\xi}{24 v^5} \left( -9 v + 24 v^3 - 7 v^5 - 6 v^6 + 9 (1 - v^2)^3 \text{Arctanh}(v) \right) \right].
\end{align}

### C. Larger Anisotropies

The relevant integrals of the previous subsections may also be evaluated numerically for arbitrary $\xi$ using techniques from [26, 27]. The results are shown in Table II For sufficiently small $\xi$, the results turn out to coincide with the analytic results Eqns. (21).

From Table II it can be seen that the anomalous contribution makes up only a few percent at small $\xi$, while it becomes more important for larger anisotropies. Indeed, at larger $\xi$, besides Eq. (19) then also the other parts of the propagator contribute, further

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3 This $\delta$-function is not to be confounded with the structure function $\delta$!

4 This is because the structure function $\hat{\delta}^2$ in the denominator may be dropped to leading order in $O(\xi)$.

However, this result is specific to choosing the quark velocity perpendicular to $\mathbf{e}_z$, so that $\hat{n} \cdot \mathbf{v} \to 0$ in the static limit.
enhancing the anomalous contributions. For the total $\kappa_z$, where the anomalous contribution is stronger, this manifests itself by a decrease as a function of $v$ for larger $\xi$, whereas the regular contribution would have had the inverse trend.

Another trend that is apparent from Table I is that the ratio $\kappa_z/\kappa_\perp$ for a given anisotropy $\xi$ decreases for increasing velocity $v$. Put differently, the higher the quark’s momentum, the more circular its associated jet shape.

\section*{IV. DISCUSSION}

Both from the analytic results Eqns. (21) as well from the numeric evaluation given in Table I one finds that for anisotropic systems $\kappa_z/\kappa_\perp$ is always larger than one. In other words, a charm quark jet in an anisotropic quark-gluon plasma will generically experience more broadening along the longitudinal direction than in the reaction plane. Assuming an initial jet profile with $\langle (\Delta p_\perp)^2 \rangle_0 = \langle (\Delta p_z)^2 \rangle_0$, the ratio $\kappa_z/\kappa_\perp$ can roughly be associated with the ratio of jet correlation widths in azimuth $\langle \Delta \phi \rangle$ and rapidity $\langle \Delta \eta \rangle$ as

$$\kappa_z/\kappa_\perp \simeq \frac{\langle \Delta \eta \rangle}{\langle \Delta \phi \rangle}. \quad (22)$$

Therefore, the calculation in the previous sections generically predicts this ratio to be larger than one in anisotropic systems.

Due to the approximations in my calculation (heavy quark, leading-log, only collisional broadening), many effects that will change both $\kappa_z$ and $\kappa_\perp$ for real jets have been ignored. However, inclusion of other effects should not change my result on the ratio of $\kappa_z$ and $\kappa_\perp$ qualitatively, since $\kappa_z = \kappa_\perp$ in isotropic systems. Thus, $\kappa_z/\kappa_\perp > 1$ in anisotropic systems should hold true also for real jets. Interestingly, recent measurements of the dihadron correlation functions by STAR [28] for Au+Au collisions at $\sqrt{s} = 200$ GeV indeed seem to imply much broader correlations in $\Delta \eta$ than in $\Delta \phi$. In Fig2 STAR data for these correlation

\begin{table}[h]
\centering
\begin{tabular}{|c|ccccc|ccccc|ccccc|}
\hline
$\nu$ & $\kappa_\perp^{(\kappa_{\perp}^{\text{reg}})}$ & $\kappa_z^{(\kappa_{\perp}^{\text{reg}})}$ & $\kappa_z^{(\kappa_{\perp}^{\text{reg}})}$ & $\kappa_\perp^{(\kappa_{\perp}^{\text{reg}})}$ & $\kappa_z^{(\kappa_{\perp}^{\text{reg}})}$ & $\kappa_z^{(\kappa_{\perp}^{\text{reg}})}$ & $\kappa_\perp^{(\kappa_{\perp}^{\text{reg}})}$ & $\kappa_z^{(\kappa_{\perp}^{\text{reg}})}$ & $\kappa_z^{(\kappa_{\perp}^{\text{reg}})}$ \\
\hline
0.05 & 0.50 (0.50) & 0.98 (1.00) & 0.23 (0.23) & 1.45 (1.54) & 0.08 (0.08) & 1.72 (1.84) \\
0.15 & 0.50 (0.51) & 0.97 (1.00) & 0.22 (0.23) & 1.44 (1.54) & 0.07 (0.08) & 1.69 (1.84) \\
0.25 & 0.51 (0.51) & 0.97 (1.01) & 0.22 (0.24) & 1.40 (1.54) & 0.07 (0.08) & 1.63 (1.84) \\
0.35 & 0.52 (0.53) & 0.96 (1.02) & 0.22 (0.25) & 1.35 (1.55) & 0.07 (0.08) & 1.55 (1.84) \\
0.45 & 0.53 (0.55) & 0.96 (1.03) & 0.22 (0.26) & 1.31 (1.55) & 0.07 (0.09) & 1.47 (1.85) \\
0.55 & 0.54 (0.57) & 0.96 (1.05) & 0.23 (0.27) & 1.26 (1.55) & 0.07 (0.09) & 1.40 (1.85) \\
0.65 & 0.56 (0.60) & 0.97 (1.07) & 0.24 (0.29) & 1.22 (1.56) & 0.07 (0.10) & 1.32 (1.85) \\
0.75 & 0.58 (0.64) & 0.98 (1.10) & 0.25 (0.31) & 1.17 (1.57) & 0.07 (0.11) & 1.24 (1.85) \\
0.85 & 0.60 (0.69) & 1.00 (1.14) & 0.27 (0.34) & 1.13 (1.58) & 0.08 (1.12) & 1.16 (1.85) \\
0.95 & 0.64 (0.75) & 1.02 (1.19) & 0.30 (0.38) & 1.09 (1.59) & 0.09 (0.14) & 1.08 (1.85) \\
\hline
\end{tabular}
\caption{The coefficients multiplying $\frac{\sigma^2(\xi^2-1)}{2N_c} \frac{T m_{\pi}^2}{4\pi} \log \frac{T m_{\pi}}{m_D}$ in $\kappa_\perp, \kappa_z$ for larger values of the system anisotropy $\xi$, evaluated numerically. For convenience, also the regular contributions are given explicitly.}
\end{table}
FIG. 2: Dihadron correlation functions in azimuth $\Delta \phi$ and space-time rapidity $\Delta \eta$, for d+Au and central Au+Au at $\sqrt{s} = 200$ GeV (figure courtesy J. Putschke (STAR), Proceedings of Hard Probes 2006). For comparison, on the right plot I show an ellipse with eccentricity $e \simeq \sqrt{8/3}$ which corresponds to a ratio $\kappa_z/\kappa_\perp \sim 3$.

is compared to a general ellipsoidal shape with eccentricity $e \simeq \sqrt{8/3}$, which according to Eq. (22) I would associate with a ratio $\kappa_z/\kappa_\perp \sim 3$.

In other words, a charm quark jet with a “mean” momentum $p \simeq 4.6$ GeV (or $v \simeq 0.95$) and a “mean” system anisotropy $\xi \simeq 10$ should experience broadening roughly consistent with the ellipsoidal shape in Fig. 2 where “mean” here is referring to a mean over the whole system evolution. To get more quantitative, one would have to calculate the momentum broadening dynamically according to Eqns. (1), which can e.g. be done within a viscous hydrodynamics simulation [29, 30, 31]. However, note that as a general feature of anisotropic plasmas, one seemingly obtains jet shapes that can be much broader in rapidity than in azimuth. Other effects distorting the shape of jets in an isotropic plasma, namely flow effects [32], probably cannot account for such dramatic asymmetries unless invoking extreme flows.

Interestingly, momentum-binned data of the dihadron correlations from STAR seem to show a trend towards more circular jet shapes for higher trigger momentum, which I also found in section III.C. Precise data could thus provide further tests for calculations, maybe helping to constrain the system anisotropy.

In view of this, a finite system anisotropy could be a natural explanation for these broad rapidity correlations in central Au+Au collisions at $\sqrt{s} = 200$ GeV. Thus, turning the argument around, it may be that the sizable ratio $\langle \Delta \eta \rangle / \langle \Delta \phi \rangle$ is an indication that the plasma created at RHIC is after all not in equilibrium during a sizable fraction of its lifetime, calling into question the validity of the strongly advocated “perfect fluid” picture.

Clearly, in order to shed more light onto this issue, many caveats on the application of my result to experimental data have to be addressed. Among other things, a calculation of the full LO correction as well as results for light quarks, gluons and the inclusion of radiation effects in anisotropic plasmas would be on the wish-list. Nevertheless, I hope that I was able to highlight the feasibility and potential value of re-calculating experimentally interesting
To summarize, I have calculated the rates governing momentum broadening for a heavy quark to leading-log accuracy in an anisotropic quark-gluon plasma. The potential singularity, naively plaguing the constant under the log of these rates is shown to be cured after resummations of the NLO gluon self-energy, eventually contributing to the leading log. I expect this procedure to render finite also other observables that are affected by these sort of singularities, making their calculation feasible also in anisotropic plasmas. The rates found in this calculation suggest jet shapes that are elongated along the rapidity direction, which could explain recent STAR data and thus point towards the possibility of non-equilibrium phenomena at RHIC.

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APPENDIX A: NLO GLUON SELF-ENERGY

I will be interested in the static limit of the imaginary part of the gluon self-energy to $O(g^3)$. Following Braaten and Pisarski [33], this can be calculated by dressing the propagators and vertices of the contributing diagrams.

A further simplification occurs due to the static limit: following Rebhan [34], only static loop momenta have to be resummed and no fermionic loops contribute. Since the ghost self-energy vanishes at leading order, the only contribution to the gluon self-energy at $O(g^3)$ in the static limit are thus the two diagrams shown in Fig. 3.

Of these, the tadpole diagram (shown in Fig.3(a)) with bare vertices contributes [34]

$$\delta \Pi^{\mu \nu} = -g^2 N_c \int \frac{d^3k}{(2\pi)^3} \lim_{\omega \to 0} \left[ g^{\mu \nu} (G^{\alpha}_\alpha (\omega, k) - G^{\alpha}_0 (\omega, k)) - (G^{\mu \nu}_\alpha (\omega, k) - G^{\mu \nu}_0 (\omega, k)) \right],$$

(A1)

where $G$ and $G_0$ denote the dressed and free propagator, respectively. The dressed propa-
gator in covariant gauges can be found by inverting the relation
\[ g_{\mu\nu}K^2 + \left( \frac{1}{\rho} - 1 \right) K_{\mu}K_{\nu} + \Pi_{\mu\nu} = G_{\mu\nu}^{-1}, \] (A2)
where \( \rho \) is the gauge parameter and \( \Pi_{\mu\nu} \) is the gluon self-energy to \( O(g^2) \) which for isotropic as well as anisotropic systems can be expressed \[26\] by the functions \( \alpha, \beta, \gamma, \delta \) introduced in the main text.

For isotropic systems, inversion of Eq. (A2) is trivial and one indeed recovers the known result \[35\] for the tadpole contribution to \( \delta \Pi \).

For anisotropic systems, the inversion of Eq. (A2) is a little more involved but it is readily performed by e.g. a symbolic manipulation package on a computer. Taking the imaginary part of \( \delta \Pi_{\mu\nu} \), the only non-vanishing contribution after performing the static limit comes from the singularities of the propagator. These, however, give just the delta-functions considered in the main text, c.f. Eq. (19).

In the small-\( \xi \) limit, the integrals may then be done analytically, and one finds
\[ \lim_{\omega \to 0^+} \text{Im} \delta \alpha(\omega, k) = -\frac{g^2 N_c m_D T}{16\pi} \left( 2 - \hat{k}_z^2 \right) \sqrt{\frac{\xi}{3}} \left( 2 - \frac{1}{4} \log \left( 3 + 2\sqrt{2} \right) \right) \]
\[ \lim_{\omega \to 0^+} \text{Im} \delta \gamma(\omega, k) = \frac{g^2 N_c m_D T}{8\pi} \hat{k}_z^2 \sqrt{\frac{\xi}{3}} \left( 2 - \frac{1}{4} \log \left( 3 + 2\sqrt{2} \right) \right). \] (A3)

Therefore, there is a gauge-invariant, non-vanishing \( O(g^3) \) contribution to the imaginary part of the structure functions in the static limit. Since I find it highly improbable that the corresponding contribution from the second diagram in Fig. 3(b) or anisotropic vertex corrections cancel this contribution \emph{exactly}, I will refrain from evaluating also these contributions, and instead take Eqns. (A3) as indication that the \emph{whole} \( O(g^3) \) contribution does not vanish.

However, both diagrams will have to be recalculated at non-vanishing frequency when aiming for the full LO contribution to the fluctuation coefficients \( \kappa_i \).

\textbf{Note added:} In thermal equilibrium systems (which are necessarily isotropic) the KMS condition requires the imaginary part of the self-energy to vanish in the static limit \[36\]. Out of equilibrium, the imaginary part of the self-energy has to be an odd function of frequency \[37\], but this does not seem to preclude a discontinuity at vanishing frequency. To show unambiguously whether there is a non-vanishing imaginary part of the self-energy in the static limit for non-equilibrium system, one has to calculate the above diagrams with both resummed propagators and vertices, which is interesting and doable, but beyond the scope of this work.

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