Quantum Collective QCD String Dynamics

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Abstract. The string breaking model of particle production is extended in order to help explain the transverse momentum distribution in elementary collisions. Inspired by an idea of Bialas', we treat the string using a collective coordinate approach. This leads to a chromo-electric field strength which fluctuates, and in turn implies that quarks are produced according to a thermal distribution.

1. Introduction

We study the production process of hadronic particles in elementary collisions based on the string breaking mechanism of QCD. The non-Abelian SU(3)-color gauge group leads to confinement of the color electromagnetic field lines at scales of roughly 1 fm and above. Thus, a $q\bar{q}$ pair produced in, say, an $e^+e^-$ collision will have a chromo-electric flux tube connecting the quark and antiquark. The tube increases in length as the quarks separate, and eventually there is sufficient energy in the field to produce another $q\bar{q}$ pair. The rate of production is calculated via Schwinger's formula \cite{1}, generalized to consider transverse particle momentum \cite{2}. This is often referred to as the string breaking mechanism of particle production.

Despite its successes, one troublesome detail of the QCD-string model has not been fully explained. The predicted transverse momentum distribution of quarks is Gaussian \cite{2}, while the observed spectrum of final state particles is thermal, or very nearly so \cite{3} \cite{4}. Naive averaging over constituent quark transverse momenta cannot make an exponential distribution out of a Gaussian one. Further, it is not natural to suppose that there is sufficient rescattering for thermalization in systems as small as those in $e^+e^-$ and $p\bar{p}$ collisions. Hagedorn thus spoke before QCD was developed of 'pre-established thermal equilibrium' \cite{3}. While the string model itself has been refined several times, nothing like a thermal $p_{\perp}$ spectrum has been predicted by any of these refinements until recently.

This discrepancy invites new ideas. Bialas showed that a random fluctuation of the string tension could produce particles in a thermal spectrum \cite{5}, if the appropriate initial distribution is used. He proposed that these fluctuations originate from the stochastic nature of the QCD vacuum. There has been further work on an extensions of this idea using dynamical fluctuations of the string in time \cite{6}. This method seems like a
promising way to proceed, and the purpose of this paper is to improve the justification for these fluctuations, as well as to propose a possible origin.

We separate the transverse and longitudinal dynamics of the string, introduce collective coordinates for the transverse dynamics of the string and quantize them. The reason that we will proceed this way is the following order of magnitude consideration: due to the very high momentum in the longitudinal direction (constituent quarks may have $p_L \approx 100\,\text{GeV}$ or more, in a 1 TeV $p\bar{p}$ collision) it is reasonable to treat this part of the dynamics classically. However, the typical transverse momentum of a produced $q\bar{q}$ pair will be on the order of the temperature (which we are seeking to justify) $T \approx 160\,\text{MeV}$. The radius of the string is about 1 fm, approximately the same magnitude as the DeBroglie wavelength of the produced particles.

Our approach is inspired by prior collective coordinate approaches in many-body quantum systems. One well-known example is the excitation spectrum of vibrations and rotations of large nuclei.[7] In the simplest liquid drop version of this model, Coulomb repulsion competes with nuclear surface tension. Quantization of the Hamiltonian resulting from these interactions leads to an excitation spectrum which can be verified experimentally.

In the string model, we once again have two competing energy considerations. The field lines will tend to spread in order to minimize their energy content. However, the vacuum confines color; this effect can be effectively reproduced by implementing in transverse direction the bag pressure and energy density dynamics. One might ask where the many-body quality arises in the string picture, since there is just a single $q\bar{q}$ pair anchoring the ends of the string. The answer is in the large number of virtual gluons which constitute the string QCD fields. We consider the string to be a collective excitation of this gluon sea.

In section 2 we first look at the classical string picture, namely the energy balance and the Hamiltonian that results. Quantizing this Hamiltonian yields a harmonic oscillator-type wave equation for the transverse dimension of the string. We explore some of the properties of the solutions to this equation. Then, in section 3 we fold the resulting probability distribution for the string tension with the Schwinger formula for pair production.[1, 2]. This in fact generates an exponential $p_\perp$ spectrum.

2. Wave Equation

2.1. String Hamiltonian

In preparation for quantization, we will first write down a Hamiltonian, focusing here on the energy density per unit length. We combine the contributions to the energy density from the chromo-electric field and the vacuum pressure to obtain the classical Hamiltonian density

$$\sigma = \frac{1}{2} E_L^2 A + BA.$$ (1)
The two terms are reminiscent of the usual kinetic and potential energies. Since the chromo-electric field is longitudinal, we will denote it by $E_L$ to avoid confusion with energy.

Suppose the field lines are broken by a $q\bar{q}$ pair produced, each having color charge $g$. In the case where the field is longitudinal, we may simplify the non-Abelian Gauss’ law down to its usual form:

$$\int \sum_a \bar{E}_a \lambda_a \cdot d\vec{A} = \frac{1}{2} g.$$  

(2)

The factor $\frac{1}{2}$ arises comparing the elementary currents in QCD and QED:

$$j_\mu^a \equiv \frac{g}{2} \bar{\Psi} \gamma_\mu \lambda_a \Psi, \quad j_\mu \equiv e \bar{\psi} \gamma_\mu \psi.$$  

(3)

Note that we have taken $E_L$ to be constant here. This is easily justified: under the constraint of a fixed flux, the total energy is minimized by a uniform field. Due to the presence of the constraint equation (2), one eliminates the dependence on, say, $E_L$, and obtains:

$$\sigma = \frac{g^2}{8A} + BA.$$  

(4)

We can see the interplay between field and vacuum energy more clearly now. Minimizing $\sigma$ with respect to $A$ gives

$$A_{\text{classical}} = \frac{g}{2\sqrt{2B}},$$  

(5)

$$E_{L_{\text{classical}}} = \sqrt{2B},$$  

(6)

and thus a classical string tension of:

$$\sigma = \frac{g}{2} \sqrt{2B} = \frac{g}{2} E_{L_{\text{classical}}}. $$  

(7)

2.2. Position-momentum collective dynamics

The string arises as a quasi particle state (presumably) in the full theory of QCD. Thus, its dynamics must be treated in an appropriate quantized way. However, as we noted earlier, the longitudinal evolution of the string should be safe to approximate classically, at least in our initial crude treatment. Furthermore, we assume the string is rigid. To be precise, there is no spatial bending of the string, nor spatial dependence in the field strength.

There are two reasons that we make this assumption. First, though the energy may fluctuate between bag and field energy, we suppose that the configuration of either is such that their separate energy contents are minimized. Second, treating the complexities of a “wavy” string is well beyond the scope of this paper. The chromo-electric field now contains only 1 degree of freedom, the longitudinal strength. Therefore, its conjugate coordinate, which will be the string cross section area or something related, will retain only 1 degree of freedom. Thus, it appears that the initially 2-dimensional transverse dynamics may in fact be successfully modeled by a 1-dimensional wave equation. Despite
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the simplifications, it will still be a problem with a constraint, equation (2) (Gauss’ law), and an unusual looking Hamiltonian.

To deal with the transverse dynamics as a quantum problem in an expedient manner, we make the following further simplification. As is evident the problem arising should still be closely related to the original one, though, admittedly, there is no absolute guarantee of complete isomorphic relation. We take the constraint equation (2) and substitute it just once into the Hamiltonian, equation (1), yielding:

\[ H_\sigma = \frac{1}{4} g E_L + BA. \]  

We want to quantize this Hamiltonian in terms of Hermitian operators with the usual dimensions of distance and momentum. Since our degrees of freedom are \(A\) and \(E_L\), which dimensions are square of distance and momentum, we introduce

\[ x = \sqrt{A} \]  
\[ p = \sqrt{\frac{g}{2}} E_L. \]

Here we have defined \(p\) to be the root of the classical energy density. Since the square root may take on positive or negative values, we will allow \(x, p \in (-\infty, +\infty)\). We make \(x\) and \(p\) into canonically conjugate operators, and so they satisfy the commutation relation

\[ [x, p] = ig \]  

The justification for our choice of commutator will follow. Indeed, \(H_\sigma\) now looks like the Hamiltonian of a harmonic oscillator:

\[ H_\sigma = \frac{1}{2} p^2 + Bx^2. \]  

Working in the \(p\)-representation leads to the wave equation:

\[ \left( \frac{1}{2} p^2 - Bg^2 \frac{\partial^2}{\partial p^2} \right) \psi_n = \sigma_n \psi_n. \]  

This is a very familiar equation, and its eigenvalues are:

\[ \sigma_n = \sqrt{2B} g^2 \left( n + \frac{1}{2} \right). \]  

Note that if the dimension of the problem is in fact not 1 but \(d\), the \(\frac{1}{2}\) would have to be multiplied by \(d\), and there would be additional \(n\)’s.

As a matter of convenience, let us define

\[ T_0 \equiv \sqrt{\frac{\sigma_n}{2\pi(2n + 1)}} \]  

Then we may express the eigensolutions of the problem as

\[ \psi_n(p) = \frac{A_n}{\sqrt{T_0}} \mathcal{H}_n(\sqrt{p^2/4\pi T_0^2}) e^{-p^2/8\pi T_0^2}, \]  

where \(\mathcal{H}_n\) are Hermite polynomials, and \(A_n\) is a dimensionless normalization.
2.3. Properties of the quantized string

Let us now take stock of some of the features that have become evident in our model. First of all, we may calculate the expected cross sectional area and field strength; because

\[ \langle \frac{1}{2}p^2 \rangle = \langle Bx^2 \rangle = \frac{1}{2}\sigma_n \] (17)

we obtain

\[ \langle E_L \rangle = \frac{2\sigma}{g} = \sqrt{2B}(2n+1), \] (18)

\[ \langle A \rangle = \frac{\sigma_n}{2B} = \frac{g}{\sqrt{8B}}(2n+1). \] (19)

In both cases there is agreement between the ground state and the classical picture.

Now, we want to see if the constraint, as well, is satisfied. It is reasonable to look at the product of the expectation values and see if Gauss’ Law results.

\[ \langle E_L \rangle \langle A \rangle = \frac{g}{2}(2n+1)^2 \] (20)

By using this combination of operators we see that the ground state at least satisfies (2). This also clarifies our choice of commutator in equation (11). Whether or not this excludes excited states of the string remains to be seen. For if we consider different combinations of expectation values that still appear to be related to the constraint, we attain different results. Consider, for example,

\[ \langle xp \rangle \langle px \rangle = \frac{g}{2}. \] (21)

This should be related to the constraint, and it indicates that all excited string states may satisfy equation (2). Finding the correct combination of operators to verify if the constraint equation (2) is satisfied remains a difficult problem. However, these preliminary results strongly suggest that at least one, if not more, string states are admissible solutions.

3. Thermal \( p_\perp \) Spectrum

We are now ready to investigate the spectrum of particles produced from this string. First, let us look at the probability distribution \( P(E_L) \). Taking the probability density \( \psi_n^2 \) and changing coordinates back to the field strength, we obtain

\[ dP_n(E_L) = \frac{B_n}{T_0\sqrt{gE_L}} T_n^2(\sqrt{gE_L/8\pi T_0^2})e^{-gE_L/8\pi T_0^2}dE_L. \] (22)

\( B_n \) is again a dimensionless normalization. Note that the factor \( (gE_L)^{-1/2} \) arises from the change in measure. The singularity arising for even \( n \) is not troubling; it is an artifact of the change in measure and never worse than \( E_L^{-1/2} \).

Next, we review the traditional particle production spectrum. The probability per unit 4-volume of a quark pair of momentum \( p_\perp \) being produced is given by the Schwinger mechanism, as expounded by Casher, Neuberger and Nussinov[2]:

\[ dP(p_\perp) = \frac{gE_L}{8\pi^3}e^{-2\pi m_1^2/gE_L}d^2p_\perp. \] (23)
This is in fact only the first term in an infinite sum, but the approximation is appropriate for our current level of precision. We consider each string breaking to be, essentially, a measurement of $E_L$. For now, let us suppose that the strings are formed in the ground state, with string tension $\sigma_0$. Thus, over several string breakings, the observed spectrum of produced quarks is obtained by folding equations (22) and (23) over $E_L$.

Performing this folding results in the quark transverse production probability per unit 4-volume,

$$dP_{\text{quark}}(p_\perp) = \frac{C T_0^2}{g} (1 + m_\perp/T_0) e^{-m_\perp/T_0} d^2 p_\perp.$$  \hspace{1cm} (24)

This agrees with previous results up to a linear prefactor\cite{5, 6}. As a final step, consider the composite hadron spectrum. We will obtain this by first rewriting the distribution in terms of the transverse mass (fortunately, it is already more or less in such a form), and then folding two distributions together, corresponding to quark-antiquark or quark-diquark pairings. The final hadron spectrum thus obtained is

$$\frac{dP_{\text{hadrons}}(m_\perp)}{dm_\perp} \propto \left( 2X_3 - \frac{6}{5}X_5 + X_\perp (-3X_2 + 2X_3 + 3X_4) \right.$$  

$$- X_\perp^2 (3X_2 + 2X_3) + X_\perp^3 + X_\perp^4 + \frac{1}{5}X_\perp^5 \right) e^{-X_\perp}$$  \hspace{1cm} (25)

where

$$X_n \equiv \left( \frac{m_1}{T_0} \right)^n + \left( \frac{m_2}{T_0} \right)^n, \quad X_\perp \equiv \frac{m_\perp}{T_0}$$

and $m_1$ and $m_2$ are the constituent quark/diquark/antiquark masses. Note that the largest root of the prefactor is $m_\perp = m_1 + m_2$, and thus the probability is always positive, as expected. This can be seen trivially in the case $m_1 = m_2 = 0$. The specific relationship \cite{15} is the same as in the aforementioned works, though the pre-factor shifts slightly the fitted temperature. Making use of a fairly standard value for the string constant, $\sigma_0 \approx 0.9$ GeV/fm, gives $T_0 \approx 165$ MeV.

4. Conclusions

We have considered the string between a $q\bar{q}$ pair as a quasiparticle excitation of the gluon field. Thus, we justified introducing collective quantization of transverse dynamics. This implies that the internal chromo-electric field is not constant, but rather when the string breaks and the field is observed, it takes on values according to a quantum probability distribution. Folding this distribution with the traditional Schwinger-Casher-Neuberger-Nussinov result leads to an intrinsically thermal $p_\perp$ spectrum. The emerging phenomenology is quite satisfactory; e.g. it produces a thermal spectrum. However, a more complete theory will certainly lead to further insights including refinement of the temperature parameter.

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5. References

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