How to Catch a Photon and Measure its Mass

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We examine the physical significance of a class of "table-top" experiments proposed by Franken and Ampulski as tests for a finite photon mass μ . Our results stem from a theorem that in the massive-photon case the fields and currents of a system are changed only by order $(\mu D)^2$ from those of the massless-photon case (D is the dimension of the system). This implies that the table-top method is only weakly sensitive to small "Yukawa" deviations from Coulomb's law, and so the theorem constitutes a rebuttal to the suggestion that the method can improve present limits on the photon mass. Instead it appears that Franken and Ampulski have devised a sensitive test of Faraday's Law.

In a recent note, Franken and Ampulski¹ (FA) have described results of a preliminary experiment, which they hope to improve later, attempting to set a limit on the photon mass. The experiment is disarmingly simple: The expected resonant frequency ω_E of a parallel resonant (LC) circuit is computed from the measured capacity, inductance, and resistance. Then the resonant frequency ω_{res} is measured. The results are compared with the formula

$$\omega_{res}^2 = \omega_E^2 + \omega_0^2, \quad (1)$$

where ω_0 is the conceivably nonzero mass of the photon measured in frequency units:

$$\omega_0 = m_\gamma c^2 / \hbar \equiv \mu c. \quad (2)$$

The last equality in Eq. (2) defines mass in wave-number or inverse-length units. The best previously published limit² corresponds in frequency units to $\omega_0 \leq 4$ Hz, so that currently available techniques could improve this result if Eq. (1) were correct.

We believe that the FA experiments should be encouraged on the simple grounds that any new test of classical electrodynamics is worthwhile. However, if the concept of a photon mass is introduced in the conventional manner—replacing Maxwell's equations by Proca's—then the crucial Eq. (1) is certainly wrong. We first give a formal demonstration of this assertion and then make some heuristic remarks on the proper in-

terpretation of the result.

The crucial step is the following theorem,³ which might be titled, "A demonstration that photons of small mass (large Compton wavelength) cannot be held in small containers":

Theorem.—If, in some experiment, relevant currents and charges and observing apparatus are confined to a region of dimensions D in all directions, and if the photon mass μ (in inverse-length units) is nonzero, then the relative change in any component of the electric or magnetic field arising from nonzero μ is given by

$$\delta F_{\alpha\beta} = O((\mu D)^2) F_{\alpha\beta}^{(0)}, \quad (3)$$

where the $F_{\alpha\beta}^{(0)}$ are typical values of the electromagnetic fields in the observation region, calculated for $\mu = 0$.

Proof.—The equations of massive electrodynamics are conventionally taken in Proca's form,

$$\partial^\alpha F_{\alpha\beta} + \mu^2 A_\beta = (4\pi/c) J_\beta, \quad (4)$$

where

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha.$$

This may be rewritten more usefully for our purposes as

$$(\square + \mu^2) F_{\alpha\beta} = (4\pi/c)(\partial_\alpha J_\beta - \partial_\beta J_\alpha). \quad (5)$$

If we take the Fourier transform of this equation in space and time, we obtain the relation for the fields and currents in "momentum space,"

$$\tilde{F}_{\alpha\beta}(k) = -i(4\pi/c)(\mu^2 - k^2)^{-1} [k_\alpha \tilde{J}_\beta(k) - k_\beta \tilde{J}_\alpha(k)], \quad (6)$$

where $k^2 \equiv k_\alpha k^\alpha \equiv (\omega/c)^2 - \vec{k}^2$ is the invariant square of the four-momentum of the "virtual photons" produced by $\tilde{J}_\alpha(k)$, the Fourier components of the current. Notice that this expression for $\tilde{F}_{\alpha\beta}$ diverges

only if k^2 goes to the value μ^2 , which means that for $\tilde{F}_{\alpha\beta}$ to diverge, *both* the momentum and energy must be small, if one of them is.

We may rewrite Eq. (6) in a suggestive form, useful if μ^2 is small compared to k^2 ,

$$\tilde{F}_{\alpha\beta}(k) = -i \frac{4\pi}{c} \left(\frac{-1}{k^2} + \frac{\mu^2}{\mu^2 - k^2} \frac{1}{k^2} \right) (k_\alpha \tilde{J}_\beta - k_\beta \tilde{J}_\alpha) = \tilde{F}_{\alpha\beta}^{(0)}(k) - \frac{\mu^2}{\mu^2 - k^2} \tilde{F}_{\alpha\beta}^{(0)}(k), \quad (7)$$

where $\tilde{F}_{\alpha\beta}^{(0)}$ is the Fourier transform of the solution of Maxwell's equations, that is, the electromagnetic field for zero mass. On going back to position space, but considering a charge and current distribution of fixed angular frequency ω , we obtain the result

$$F_{\alpha\beta}(\omega, \vec{r}) e^{-i\omega t} = F_{\alpha\beta}^{(0)}(\omega, \vec{r}) e^{-i\omega t} + \frac{\mu^2}{4\pi} \int d^3r' \frac{\exp(ik|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} F_{\alpha\beta}^{(0)}(\omega, \vec{r}') e^{-i\omega t}, \quad (8)$$

with $k = [(\omega/c)^2 - \mu^2]^{1/2} = |\vec{k}|$.

Of course, the physical fields are obtained by taking the real part of this equation. The important point for us is that the conditions assumed in the statement of our theorem permit an estimate to be made for the second term in Eq. (8):

$$\delta F_{\alpha\beta}(\omega, \vec{r}) \equiv [F - F^{(0)}]_{\alpha\beta} = O((\mu D)^2) \bar{F}_{\alpha\beta}^{(0)}(\omega), \quad (9)$$

where $\bar{F}_{\alpha\beta}^{(0)}(\omega)$ is a typical value of the $\alpha\beta$ component of the field. The only difficulty in this estimate comes from the radiation-zone contribution to the integral, but even that may be estimated straightforwardly.

Equation (9) completely and fully defines our theorem. The application to the FA experiment is simple. Let the charges and currents oscillate at a frequency near the $\mu = 0$ resonant frequency. The electromagnetic fields produced suffer a relative change of order $(\mu D)^2$. Clearly this will produce a similar relative change in the natural motion of charges and currents in these fields. Hence the resonant frequency will be changed by

$$\delta\omega_{\text{res}}/\omega_{\text{res}} = O((\mu D)^2), \quad (10)$$

and *not*, as implied in Eq. (1), by $O((\mu c/\omega_{\text{res}})^2)$. This means a reduction in sensitivity of $(\omega_{\text{res}} D/c)^2 \lesssim 10^{-16}$ for the case of FA.

The reader may be uneasy because resonance is a delicate phenomenon. Is it not possible that this small shift in fields begets a small shift in currents, which reacts on the fields, etc. thus driving the resonance to a vastly different frequency? We can safely assert that such is not the case.

Suppose there exists a frequency where the Maxwell theory yields no resonance in some circuit. This means that if currents could be made to oscillate at this frequency, the induced fields would be appreciably different in either

magnitude or phase or both from the fields required to maintain the oscillation. Thus, to produce resonance, the fields must be changed by at least their own order of magnitude. If the conditions of our theorem apply, then finite μ cannot accomplish this change. Therefore, there will also be no resonance at this frequency for $\mu \neq 0$. Working this argument backwards as well as forwards enables us to deduce that the existence and frequency of a table-top circuit resonance is imperceptibly dependent on a small μ .

Quantitatively, the situation is even worse for FA than the above estimates indicate. If, on the right-hand side of Eq. (8), quantities of order $kD(\mu D)^2$ are neglected, the effect of this remainder term is simply to modify very slightly the values of L , C , and R for the circuit. This modification will apply both to the measurement of these quantities and to the determination of the resonant frequency. As a consequence, the *measured* resonant frequency ω_{res} will be equal to the value calculated from the *measured* L , C , and R to order $kD(\mu D)^2 \lesssim 10^{-24}$! Therefore, Franken and Ampulski are making a stringent test of classical electrodynamics, but one which is exceedingly insensitive to a small photon mass. It appears to us more a test of Faraday's law, and hence of the validity of special relativity for electrodynamics.

It is instructive to apply the theorem to another case. An ideal coaxial cable is well known to have a traveling, TEM mode which propagates at the speed of light down the cable. We thought that a photon mass μ would produce a dispersion $\delta\mu \approx \mu^2/2k^2$ in the phase velocity of a wave with wave number k . Then by making the cable long enough and looking at a wavelength equal to twice the cable length, one could have obtained electric field changes of order $(\mu l)^2$, where l is the length of the cable. Equation (8) shows this is not the case. An elementary estimate gives

$\delta\vec{E}/\vec{E}_{av}$ of order $(\mu a)^2 \ln(l/a)$, where a is the transverse dimension of the cable. What went wrong?

Let us remind ourselves of the standard derivation of the form of these TEM waves. If the cable axis is in the z direction, then at zero frequency there can be a potential $\varphi(x, y)$ between the inner and outer conductors. Of course, φ assumes a constant value φ_1 on the inner conductor and a different value φ_2 on the outer conductor, and the potential φ obeys the two-dimensional Laplace equation

$$\nabla^2 \varphi(x, y) = 0. \quad (11)$$

The traveling-wave solution for nonzero frequency is

$$\vec{E} = -e^{i(\omega t - kz)} \nabla \varphi(x, y), \quad (12)$$

with $\square \vec{E} = 0$ implying

$$\omega = ck. \quad (13)$$

If finite photon mass is introduced, Eq. (11) becomes

$$(\nabla^2 - \mu^2) \varphi(x, y) = 0, \quad (14)$$

but Eqs. (12) and (13) are unchanged! The wave still propagates with phase velocity c . Only the dependence on x and y is altered, giving $\delta\vec{E}/\vec{E}_{av}$ of order $(\mu a)^2$, more than satisfying the bound deduced from our theorem.

This result is quite important to the discussion of FA: If it were possible to obtain a resonance effect sensitive to photon mass in a very long cable, then by coiling the cable one might even hope to make such a resonance measurement in a small volume—the “table-top experiment” of FA. To our own surprise, this is not even possible for a straight cable. The reason is that the TEM wave in the cable is not really a portion of a photon of fixed wave number parallel to the cable axis. Rather, it is a collection of virtual photons with high transverse momenta, produced by the Coulomb and Ampere fields of the charges and currents in the cable walls. The wave moves with velocity c because this parameter appears in the laws of Ampere and Faraday. A small photon mass only produces a tiny shift in the transverse momenta of these virtual photons.⁴

One may draw quite similar conclusions for a system that is highly extended in two dimensions, for example, a rectangular conducting box with a large square base of side s and a small height h . Again, the lowest modes have phase velocity $\bar{c}$ parallel to the base and have electric fields

parallel to $\vec{h}$ which are, however, slightly non-uniform because the photon mass produces effects of order $(\mu h)^2$.

In short, our theorem suggests a simple moral, already suspected by FA. In order to catch and keep a photon, and measure its mass, you need “one giant of a box.”⁵

We wish to thank Professor Peter Franken for bringing this subject to our attention prior to publication. The effort to find a flaw in the argument was quite educational for us. We also wish to mention that E. R. Cohen independently raised the issue of Rosa-Dorsey experiments as useful tests for finite μ . A.S.G. acknowledges the hospitality of Professor B. Zumino at CERN.

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¹P. A. Franken and G. W. Ampulski, Phys. Rev. Lett. **26**, 115 (1971).

²A. S. Goldhaber and M. M. Nieto, Phys. Rev. Lett. **21**, 567 (1968). See also a review article, A. S. Goldhaber and M. M. Nieto, “Terrestrial and Extraterrestrial Limits on the Photon Mass” (to be published).

³We came upon this theorem, which clarifies a number of arguments about photon mass experiments, in the course of preparing the review article in Ref. 2.

⁴This discussion of the coaxial cable is not completely trivial. In addition to the correct solution mentioned in the text, there exists another solution of the Proca equations in the space between the conductors—a wave with $\omega = c(\vec{k}^2 + \mu^2)^{1/2}$ obeying the two-dimensional Laplace equation in the transverse x and y variables. However, for this “solution” the scalar potential vanishes in the Lorentz gauge, implying that there are no charge-density oscillations in phase with the wave. Since the electric field points from the inner to the outer conductor, there must be charge inside the inner conductor, and this wave is not part of a consistent solution in all space.

For the related problem of a wave traveling within and parallel to the plates of a capacitor with the field perpendicular to the plates, there is a charge-free solution with $\omega = c(\vec{k}^2 + \mu^2)^{1/2}$. However, the solution is continuous across the plates—just the ordinary free-field solution—and so it is excluded by the condition of vanishing outside the capacitor. It is amusing that for $\mu > 0$ the free field has a smaller velocity than the lowest Coulombic mode in either a coaxial cable or in a capacitor.

⁵This suggests what could be an alternate title for our article, “How to Detect a Small Photon Mass: The Need for Large Apparatus or High Precision.”