SEEKING EXACTLY SOLVABLE MODELS OF TRAVERSABLE WORMHOLES SUPPORTED BY PHANTOM ENERGY

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Abstract. The purpose of this paper is to obtain exact solutions of the Einstein field equations describing traversable wormholes supported by phantom energy. Their relationship to exact solutions in the literature is also discussed, as well as the conditions required to determine such solutions.

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1. Introduction

Wormholes may be defined as handles or tunnels in the spacetime topology linking different universes or widely separated regions of our own universe [1]. Renewed interest in the subject is due in part to the discovery that our universe is undergoing an accelerated expansion [2, 3], that is, \( \dot{a} > 0 \) in the Friedmann equation \( \dot{a}/a = -\frac{4\pi}{3}(\rho + 3p) \). (Our units are taken to be those in which \( G = c = 1 \).) The acceleration is caused by a negative pressure dark energy with equation of state \( p = -K\rho \), \( K > \frac{1}{3} \) and \( \rho > 0 \). A value of \( K > \frac{1}{3} \) is required for accelerated expansion; \( K = 1 \) corresponds to a cosmological constant [4]. Of particular interest is the case \( K > 1 \), referred to as phantom energy. For this case, \( \rho + p < 0 \), in violation of the null energy condition. (The null energy condition requires the stress-energy tensor \( T_{\alpha\beta} \) to obey \( T_{\alpha\beta}k^\alpha k^\beta \geq 0 \) for all null vectors.) It is well known that the violation of the null energy condition is a necessary condition for the existence of wormholes [1]. In this context such matter is usually called exotic. Phantom energy could therefore automatically qualify as a candidate for exotic matter, except for one problem: the notion of dark or phantom energy applies to a homogeneous distribution of matter in the universe, while wormhole spacetimes are necessarily inhomogeneous. Fortunately, the extension to spherically symmetric inhomogeneous spacetimes has been carried out. (See Ref. [5] for details.)
Returning to the definition of phantom energy, we saw that the condition $K > 1$ results in a violation of the null energy condition. So if $K < 1$, the null energy condition is met and turns out to have a direct effect on the so-called flare-out condition. We shall return to this point in Section 4.

A recurring problem in the general theory of relativity is finding exact solutions to the Einstein field equations. In the case of phantom-energy supported wormholes several solutions already exist in the literature [5, 6, 7]. The strategy in this paper is to start with a general line element, together with the above equation of state, and to determine the conditions required to obtain explicit exact solutions, both old and new. Two new solutions are discussed.

One problem is that not everyone uses the terms exact and explicit in the same sense. Thus an “exact” solution may contain an intractable integral, while an “explicit” solution may contain functions defined only implicitly. Our primarily interest is therefore centered on elementary functions, rather than arbitrary functions. For present purposes these may be defined as functions of a single variable built up by using that variable and constants together with a finite number of algebraic operations, composition, forming trigonometric functions and their inverses, and constructing exponents and logarithms.

While the derivative of an elementary function is elementary, the integral may not be. For example, $\int e^{x^2} \, dx$ is not an elementary function, or, as it is often expressed, the integral cannot be written explicitly (or in closed form or in finite terms.) General criteria for integration in finite terms can be found in Refs. [8, 9].

We have similar requirements for the solution of differential equations. We are interested in finding solutions that can be expressed explicitly in terms of elementary functions, as opposed to infinite-series or numerical solutions.

The solutions in Refs. [5, 6, 7] mentioned above are examples of exact solutions in the sense defined here.

2. The problem

Consider the general line element

$$ds^2 = -e^{2\Phi(r)} \, dt^2 + e^{2\alpha(r)} \, dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2),$$

where $\Phi$ and $\alpha$ are functions of the radial coordinate $r$. The function $\Phi$ is called the redshift function. We require that $e^{2\Phi(r)}$ never be zero to avoid an event horizon. The function $\alpha$ has a vertical asymptote at
the throat \( r = r_0 \):

\[
\lim_{r \to r_0^+} \alpha(r) = +\infty.
\]

The reason is its relationship to the shape function \( b(r) \):

\[
e^{2\alpha(r)} = \frac{1}{1 - \frac{b(r)}{r}}.
\]

It follows that

\[
b(r) = r(1 - e^{-2\alpha(r)}).
\]

The shape function determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram. To obtain a traversable wormhole, the shape function must obey the usual flare-out conditions at the throat [1]: \( b(r_0) = r_0, \) \( b'(r_0) < 1, \) and \( b(r) < r \). Another requirement is asymptotic flatness, that is, \( b(r)/r \to 0 \) as \( r \to \infty \). Because of the spherical symmetry, the nonzero components of the stress-energy tensor are \( T_{00} = \rho(r), T_{11} = p(r), \) and \( T_{22} = T_{33} = p_t(r) \), the transverse pressure. The components of the Einstein tensor in the orthonormal frame are given next [10]:

\[
(4) \quad G_{ij} = \frac{2}{r} e^{-2\alpha(r)} \alpha'(r) + \frac{1}{r^2}(1 - e^{-2\alpha(r)}),
\]

\[
(5) \quad G_{\hat{r}\hat{r}} = \frac{2}{r} e^{-2\alpha(r)} \Phi'(r) - \frac{1}{r^2}(1 - e^{-2\alpha(r)}),
\]

\[
(6) \quad G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = e^{-2\alpha(r)} \left( \Phi''(r) + (\Phi'(r))^2 + \frac{1}{r} \Phi'(r) - \frac{1}{r} \alpha'(r) \right).
\]

From the Einstein field equations \( G_{\hat{\alpha}\hat{\beta}} = 8\pi T_{\hat{\alpha}\hat{\beta}} \) and the equation of state \( p = -K\rho \), we have \( G_{i\hat{i}} = 8\pi \rho \) and \( G_{r\hat{r}} = 8\pi(-K\rho) \), giving us the following system of equations:

\[
(7) \quad G_{i\hat{i}} = 8\pi T_{i\hat{i}} = 8\pi \rho = \frac{2}{r} e^{-2\alpha(r)} \alpha'(r) + \frac{1}{r^2}(1 - e^{-2\alpha(r)}),
\]

\[
(8) \quad G_{r\hat{r}} = 8\pi T_{r\hat{r}} = 8\pi(-K\rho) = \frac{2}{r} e^{-2\alpha(r)} \Phi'(r) - \frac{1}{r^2}(1 - e^{-2\alpha(r)}).
\]

Substitution yields

\[
\frac{2}{r} e^{-2\alpha(r)} \alpha'(r) + \frac{1}{r^2}(1 - e^{-2\alpha(r)})
\]

\[
= -\frac{1}{K} \frac{2}{r} e^{-2\alpha(r)} \Phi'(r) + \frac{1}{K} \frac{1}{r^2}(1 - e^{-2\alpha(r)}).
\]
After rearranging the terms,

\[ K\alpha'(r) = -\Phi'(r) - \frac{1}{2r}(e^{2\alpha(r)} - 1)(K - 1). \]

This equation shows the close relationship between \( \Phi'(r) \) and \( \alpha'(r) \) and hence between \( \Phi(r) \) and \( \alpha(r) \). Since \( \alpha(r) \to +\infty \) as \( r \to r_0^+ \), there is a distressing tendency for \( e^{2\Phi(r)} \) to go to zero as \( r \to r_0^+ \). So, while the existence of exotic matter does help to satisfy a basic requirement, the equation of state makes it very difficult to obtain an exact solution without an event horizon.

3. The redshift function

Returning to Eq. (9), one way to solve this equation is to insert the redshift function \( \Phi'(r) \equiv 0 \), resulting in \( \Phi = \text{constant} \). This equation is readily solved and leads to

\[ e^{2\alpha(r)} = \frac{1}{1 - \left(\frac{r_0}{r}\right)^{1-1/K}}. \]

This is the solution in Ref. [6].

The only other possibility is \( \Phi(r) = \frac{1}{2}\ln(r_1/r) \), for some constant \( r_1 \), which allows the solution of Eq. (9) by separation of variables, that is, by factoring \( 1/r \). This approach yields

\[ e^{2\alpha(r)} = \frac{1}{\left(1 - \frac{1}{K}\right)\left(1 - \frac{r_0}{r}\right)}. \]

This is the solution in Ref. [7].

It follows that to obtain an exact solution, \( \Phi'(r) \) must depend directly on \( \alpha(r) \) and \( \alpha'(r) \) and therefore indirectly on the shape function. This dependence may be expressed as \( \Phi'(r) = F[\alpha(r)]\alpha'(r) \), for some elementary function \( F \).

4. The general case

Suppose we write Eq. (11) as follows:

\[ -K\alpha'(r) - \frac{1}{2r}(e^{2\alpha(r)} - 1)(K - 1) = \Phi'(r). \]

By the above discussion, \( \Phi'(r) \) must have the form

\[ \Phi'(r) = F[\alpha(r)]\alpha'(r). \]

So by Eq. (12),

\[ -\frac{K\alpha'(r)}{e^{2\alpha(r)} - 1} - \frac{1}{2r}(K - 1) = \frac{F[\alpha(r)]\alpha'(r)}{e^{2\alpha(r)} - 1}. \]
To obtain an exact solution, we must be able to solve Eq. (14) in a closed form and to express the integral of $Φ'(r)$ in finite terms. Finding $Φ(r)$ in an exact form does not, of course, guarantee the absence of an event horizon. For example, if $F ≡ −1$ in Eq. (13), then

$$Φ(r) = −α(r) + C$$

and $e^{2Φ(r)} = 0$ at the throat, which is a typical outcome!

The simplest way to meet all these requirements is by letting

$$F[α(r)] = \frac{K}{e^{2α(r)}}.$$  

Then, Eq. (14) becomes

$$−\frac{Kα'(r)}{e^{2α(r)} − 1} − \frac{1}{2r}(K − 1) = −\frac{Kα'(r)}{e^{2α(r)}(e^{2α(r)} − 1)}$$

with initial condition $α(r_0) = +∞$. The solution is

$$e^{2α(r)} = \frac{1}{\ln(\frac{r}{r_0})^{(K−1)/K}}.$$  

Integrating $−Kα'(r)e^{−2α(r)}$ and substituting the expression for $e^{2α(r)}$ yields

$$Φ(r) = \frac{1}{2} \ln C \left( \frac{r}{r_0} \right)^{K−1};$$

$C$ is the constant of integration, which needs to be determined from the junction conditions. So, the line element is

$$ds^2 = −C \left( \frac{r}{r_0} \right)^{K−1} dt^2 + \frac{1}{\ln(\frac{r}{r_0})^{(K−1)/K}} dr^2 + r^2(dθ^2 + \sin^2θ \, dφ^2).$$  

This solution has most of the required features. For example, since $b(r) = r(1 − e^{−2α(r)})$ by Eq. (3), we have

$$b(r) = r \left[ 1 − \ln \left( \frac{r}{r_0} \right)^{\frac{K−1}{K}} \right].$$

It is easily checked that $b(r_0) = r_0$, $b'(r_0) = 1/K < 1$, and $b(r) < r$. (Observe that if $K < 1$, then $b'(r_0) > 1$ and the flare-out condition is no longer satisfied.)

Unfortunately, the resulting spacetime is not asymptotically flat since $b(r)$ eventually decreases. Accordingly, the wormhole material must be cut off at some $r = a$ and joined to an external Schwarzschild spacetime. A natural choice for $r = a$ is the value for which $b(r)$ becomes a maximum. From the critical value of $b'(r) = 0$, we get $a = r_0 e^{1/(K−1)}$. At first glance this does not look like a large distance. According to Ref. [11], however, $K$ is likely to be very close to unity.
Matching our interior solution to the exterior Schwarzschild solution

\[ ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \]

at some \( r = a \) requires continuity of the metric. As noted in Ref [12], since the components \( g_{\hat{\theta} \hat{\theta}} \) and \( g_{\hat{\phi} \hat{\phi}} \) are already continuous due to the spherical symmetry, one needs to impose continuity only on the remaining components at \( r = a \):

\[ g_{\hat{t} \hat{t}}(\text{int})(a) = g_{\hat{t} \hat{t}}(\text{ext})(a) \quad \text{and} \quad g_{\hat{r} \hat{r}}(\text{int})(a) = g_{\hat{r} \hat{r}}(\text{ext})(a) \]

for the interior and exterior components, respectively. These requirements, in turn, imply that

\[ \Phi(\text{int})(a) = \Phi(\text{ext})(a) \quad \text{and} \quad b(\text{int})(a) = b(\text{ext})(a). \]

In particular,

\[ e^{2\alpha(r)} = \frac{1}{1 - \frac{b(a)}{a}} = \frac{1}{1 - \frac{2M}{a}}. \]

So we need to determine \( M = \frac{1}{2} b(a) \), the total mass of the wormhole for \( r \leq a \):

\[ M = \frac{1}{2} b(a) = \frac{1}{2} a \left[ 1 - \ln \left( \frac{a}{r_0} \right)^{\frac{K-1}{K}} \right]. \]

Choosing \( a = r_0e^{1/(K-1)} \), we get

\[ M = \frac{1}{2} r_0e^{\frac{1}{K-1}} \left( 1 - \frac{1}{K} \right). \]

Returning to \( \Phi(r) \), we now have

\[ C \left( \frac{a}{r_0} \right)^{\frac{K-1}{K}} = 1 - \frac{2M}{a} \]

or

\[ C = \frac{1 - 2M/a}{(a/r_0)^{K-1}}. \]

For \( a = r_0e^{1/(K-1)} \), \( C = 1/(Ke) \). So the line element becomes

\[ ds^2 = -\frac{1}{Ke} \left( \frac{r}{r_0} \right)^{K-1} dt^2 + \frac{1}{\ln \left( \frac{r}{r_0} \right)^{(K-1)/K}} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2). \]

While the metric is continuous on the junction surface \( r = a \), the derivatives may not be. This behavior needs to be taken into account.
when discussing the surface stresses. The following forms, proposed by Lobo \[6, 12\], are suitable for this purpose:

\[
\sigma = -\frac{1}{4\pi a} \left( \sqrt{1 - \frac{2M}{a}} - \sqrt{1 - \frac{b(a)}{a}} \right)
\]

and

\[
\mathcal{P} = \frac{1}{8\pi a} \left( \frac{1 - \frac{M}{a}}{\sqrt{1 - \frac{2M}{a}}} - [1 + a\Phi'(a)] \sqrt{1 - \frac{b(a)}{a}} \right).
\]

It is clear that the surface stress-energy \(\sigma\) is zero. The surface tangential pressure \(\mathcal{P}\) turns out to be positive. To see this, consider \(\Phi'_{+}\), the “jump” at \(r = a = r_0 e^{1/(K-1)}\); \(\lim_{r\to a^+} \mathcal{P}\) is positive since we are dealing with a Schwarzschild spacetime \[12\]. Substituting \(\Phi'_-(a) = \lim_{r\to a^-} \Phi'(r)\) shows that \(\lim_{r\to a^-} \mathcal{P} = 0\).

An important consideration affecting the traversability is the proper distance \(\ell(r)\) from the throat to a point away from the throat:

\[
\ell(r) = \int_{r_0}^{r} \frac{dr}{\sqrt{\ln\left(\frac{r}{r_0}\right)^{(K-1)/K}}},
\]

which is finite; in fact, \(\ell(r_0) = 0\). Unless \(K\) is extremely close to unity, \(\ell(r)\) is not going to be excessively large. For example, if \(K = 1.1\) and \(r = 2r_0\), then \(\ell(r) \approx 7.1r_0\).

A final consideration is the time dilation near the throat. Let \(v = d\ell/d\tau\), so that \(d\tau = d\ell/v\) (assuming that \(\gamma = \sqrt{1 - (v/c)^2} \approx 1\)). Since \(d\ell = e^{\alpha(r)} dr\) and \(d\tau = e^{\Phi(r)} dt\), we have for any coordinate interval \(\Delta t\):

\[
\Delta t = \int_{t_a}^{t_b} dt = \int_{\ell_a}^{\ell_b} e^{-\Phi(r)} \frac{d\ell}{v} = \int_{r_a}^{r_b} \frac{1}{v} e^{-\Phi(r)} e^{\alpha(r)} dr.
\]

Going from the throat to \(r\), we get

\[
\Delta t = \int_{r_0}^{r} \frac{\sqrt{Ke}}{v} \sqrt{\left(\frac{r_0}{r}\right)^{K-1}} \frac{dr}{\sqrt{\ln\left(\frac{r}{r_0}\right)^{(K-1)/K}}}
\leq \int_{r_0}^{r} \frac{\sqrt{Ke}}{v} \frac{dr}{\sqrt{\ln\left(\frac{r}{r_0}\right)^{(K-1)/K}}},
\]

which is also well behaved near the throat.
5. Other solutions

Returning to Eq. (13), the choices for $F$ appear to be severely limited. At least one other possibility is $F[\alpha(r)] = -2K/(e^{2\alpha(r)} + 1)$. So by Eq. (12),

$$-K \frac{\alpha'(r)}{e^{2\alpha(r)} - 1} - \frac{1}{2r} (K - 1) = -\frac{2K \alpha'(r)}{e^{4\alpha(r)} - 1}.$$ 

This equation can also be readily solved to yield

$$ds^2 = -C \left( \frac{r}{r_0} \right)^{2(K-1)} dt^2 + \frac{1}{(\frac{r}{r_0})^{(K-1)/K} - 1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

This solution has some of the same features as the solution in the previous section: $b(r)$ satisfies the flare-out conditions and attains a maximum value at

$$a = r_0 \left( \frac{2K}{2K - 1} \right)^{\frac{K}{K-1}}.$$ 

It is readily shown that if $r = a$ is the junction surface, then

$$M = r_0 \frac{K - 1}{2K - 1} \left( \frac{2K}{2K - 1} \right)^{\frac{K}{K-1}}$$

and

$$C = \frac{(2K - 1)^{2K - 1}}{(2K)^{2K}}.$$ 

The cases considered so far may very well exhaust the possibilities for getting exact solutions without event horizons. For example, $F[\alpha(r)] = -1/\sqrt{e^{2\alpha(r)} - 1}$ in Eq. (13) leads to

$$\Phi(r) = -\int \frac{\alpha'(r) dr}{\sqrt{e^{2\alpha(r)} - 1}} = -\tan^{-1} \sqrt{e^{2\alpha(r)} - 1} + C,$$

which is finite at $r = r_0$. So there is no event horizon. Unfortunately, the differential equation (12) leads only to an implicit solution for $e^{\alpha(r)}$:

$$-\frac{K}{2} \left[ \ln(e^{2\alpha(r)} - 1) - 2\alpha(r) \right] - \frac{K - 1}{2} \ln r$$

$$= -\frac{1}{2} \left( -2 \tan^{-1} \sqrt{e^{2\alpha(r)} - 1} - \frac{2}{\sqrt{e^{2\alpha(r)} - 1}} \right) + C.$$

This equation cannot be solved explicitly for $e^{\alpha(r)}$.

The choice $F[\alpha(r)] = 1/(e^{2\alpha(r)} - 1)^2$ is even worse. Not only do we get a strictly implicit solution, but the resulting metric has an event
horizon. Other plausible choices, such as

\[ F[\alpha(r)] = \frac{1}{(e^{2\alpha(r)} - 1)^n} \quad \text{for} \quad n > 2 \]

or

\[ F[\alpha(r)] = \frac{1}{(e^{2\alpha(r)} - 1)^{m/n}} \]

are even more complicated and do not yield explicit solutions.

Specifying \( \alpha(r) \) and solving for \( \Phi(r) \) leads to similar difficulties; judging from Eq. (12),

\[ \Phi'(r) = -K\alpha'(r) - \frac{1}{2r}(e^{2\alpha(r)} - 1)(K - 1), \]

if

\[ \int_{r_0}^{r} \frac{1}{2r}(e^{2\alpha(r)} - 1)(K - 1)dr \]

is finite, then

\[ \Phi(r) = -K\alpha(r) - \int_{r_0}^{r} \frac{1}{2r}(e^{2\alpha(r)} - 1)(K - 1)dr \]

is likely to lead to an event horizon at \( r = r_0 \). But, as we have seen, \( \Phi(r) \) can be determined in certain special cases, such as Eqs. (10) and (11).

Another special case is discussed in Ref. [5]: if \( \rho(r) = \rho_0 \), a constant for \( r_0 \leq r \leq r_1 \), then Eq. (7) becomes

\[ 8\pi \rho_0 = \frac{2}{r} e^{-2\alpha(r)}\alpha'(r) + \frac{1}{r^2}(1 - e^{-2\alpha(r)}). \]

So

\[ 8\pi \rho_0 r^2 = 2r e^{-2\alpha(r)}\alpha'(r) + (1 - e^{-2\alpha(r)}) = b'(r) \]

by Eq. (3). This equation yields \( b(r) \) and hence \( \alpha(r) \) and \( \Phi(r) \), although the details are complicated.

6. Conclusion

It is shown in this paper that for wormholes supported by phantom energy the only specific choices for the redshift function \( \Phi \) that lead to explicit exact solutions without an event horizon are \( \Phi(r) \equiv \text{constant} \) and \( \Phi(r) = \frac{1}{2}\ln(r_1/r) \). Otherwise \( \Phi(r) \) must depend on \( \alpha(r) \) in such a way that \( \Phi'(r) = F[\alpha(r)]\alpha'(r) \), for some elementary function \( F \). The choices for \( F \) are severely limited. Two new solutions are obtained.
References


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