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GEOMETRY OF THE N POINT P SPACE FUNCTION
OF QUANTUM FIELD THEORY *)

H. Epstein
I.H.E.S. - Bures sur Yvette

V. Glaser
CERN - Geneva

and

R. Stora
CERN - Geneva
and
C.N.R.S. - Marseille

A B S T R A C T

The algebra generated by time-ordered product operators is investigated from a new point of view which is applied to the study of the generalized retarded products.

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INTRODUCTION

The geometry of the n point p space function of quantum field theory was first simultaneously developed by Araki ¹⁾ and Ruelle ²⁾, following previous but less extensive work by Polkinghorne ³⁾ and Steinman ⁴⁾. The construction of these authors which can be found in original papers and reviews is intuitively based on a preliminary study of the one-dimensional space-time situation - conventional non-relativistic quantum mechanics - which leads to the relativistic situation studied by these authors.

A more recent approach ⁵⁾ is suggested by perturbation theory studies ⁶⁾ as well as experience gained in the study of local analyticity properties of the scattering amplitudes in quantum field theory ⁷⁾. One natural way to supplement the Wightman axioms ⁸⁾ - or, with minor modifications which do not affect analyticity statements, to exploit the Haag-Araki ⁹⁾ axioms - is to assume the existence of operator valued distributions depending on n space-time points $(x_1 \dots x_n) = X$, conventionally called time-ordered products of n field operators, whose products leave stable a dense domain in Hilbert space, which contains the vacuum vector Ω . Time-ordered products are assumed to fulfil the causal factorization property ⁶⁾ :

$$T(X) = T(I) T(I')$$

if $I \gtrsim I'$ where $I' = X \setminus I$ and $I \gtrsim I' = (x \mid x_i - x_j \in \bar{V}^- \forall i \in I, j \in I')$. It is the latter condition that undergoes a minor modification in the Haag-Araki formulation of Q.F.T. One can thus naturally introduce the partially retarded operator ⁷⁾

$$R_I(X) = T(X) - T(I') T(I)$$

with support $(x \mid x_i - x_j \in \bar{V}^+ \text{ some } i \in I, j \in I')$. Let now

$$(\Omega, R_I(X) \Omega) = r_I(X); (\Omega, T(X) \Omega) = t(X)$$

Applying spectrum properties to Fourier transforms, one obtains

$$\tilde{r}_I(p) = \tilde{t}(p) \quad \text{if } p_I \notin S_I^-$$

where $S_I^- = \{0\} \cup \bar{V}_I^-, \bar{V}_I^-$ being the closed convex hull of the lower sheet of a hyperboloid of mass M_I depending on selection rules and spectrum.

These properties can be extended to more general expressions which we shall now construct.

1. - THE ALGEBRA GENERATED BY TIME-ORDERED PRODUCTS

One considers the vector space $\mathcal{C}$ spanned by ordered monomials

$$\prod_{k=1}^{k=\nu} T(I_k) \equiv T(I_1) \dots T(I_\nu), \quad I_1 \cup \dots \cup I_\nu = X, \quad I_i \cap I_j = \emptyset, \\ i, j = 1 \dots \nu,$$

whose generic element will be denoted $\hat{I}$. If $I \subseteq X$, one defines $\hat{I}$ by linear extension of

$$\hat{I} \prod_{k=1}^{k=\nu} T(I_k) = \prod_{k=1}^{k=\nu} T(I \cap I_k) T(I' \cap I_k)$$

with the convention $\hat{X} = \hat{\emptyset} = 1, T(\emptyset) = 1$. One has the following properties :

$$(i) \quad \hat{I}^2 = \hat{I}$$

$$(ii) \quad \hat{I}_1 \hat{I}_2 = \hat{I}_2 \hat{I}_1 \iff I_1 \subseteq I_2 \text{ or } I_2 \subseteq I_1$$

$$\left(I_1' \cap I_2 = I_1 \cap I_2' \rightarrow I_1' \cap I_2 = I_1 \cap I_2' \cap I_2 = \emptyset \right) \\ (iii) \quad (\mathbb{1} - \widehat{I_1 \cup I_2}) \prod_{r=1}^{r=\nu} \hat{K}_r \hat{I}_1 \hat{I}_2 = \prod_{r=1}^{r=\nu} \hat{K}_r \hat{I}_1 \hat{I}_2 (\mathbb{1} - \widehat{I_1 \cup I_2}) \\ = \prod_{r=1}^{r=\nu} \hat{K}_r \hat{I}_1 (\mathbb{1} - \widehat{I_1 \cup I_2}) \hat{I}_2 = 0$$

$$(iv) \quad \text{same as (iii) with } \cup \rightarrow \cap$$

$$(v) \quad \begin{aligned} \hat{I}' \hat{I} &= \hat{I} \\ \hat{I} \hat{I}' &= \hat{I}' \end{aligned}$$

(vi) $T(X)$ is cyclic :

$$T(K_1) \dots T(K_v) = \hat{I}_1 \dots \hat{I}_v T(X)$$

$$I_x = \bigcup_{s=1}^{s=x} K_s$$

This allows to define a partial order on $\mathcal{C}$:

$$T(K_1) \dots T(K_v) > T(L_1) \dots T(L_\mu)$$

if the partition $(L_1 \dots L_\mu)$ refines the partition $(K_1 \dots K_v)$.

$$(vii) \quad \text{If } I_1 \cap I_2 = \phi, \quad \hat{I}_1 \hat{I}_2 = \widehat{I_1 \cup I_2} \quad \hat{I}_2 = \hat{I}_2 \widehat{I_1 \cup I_2}$$

$$(viii) \quad \text{If } I_1 \cup I_2 = X, \quad \hat{I}_1 \hat{I}_2 = \widehat{I_1 \cap I_2} \quad \hat{I}_2 = \hat{I}_2 \widehat{I_1 \cap I_2}$$

(ix) Eigenstates of $\hat{I}$:

$$\hat{I} \Theta = \Theta \xLeftrightarrow{\text{Def.}} \Theta \in \mathcal{N}(I).$$

vectors of the form

$$\Theta = \prod_{k=1}^{k=v} T(I_k)$$

with $I_k \subset I$ or $I_k \subset I'$ belong to $\mathcal{N}(I)$.

Conversely, all elements of $\mathcal{N}(I)$ are of this form :

$$\text{If } \hat{I} \prod_{k=1}^{k=v} T(I_k) \neq \prod_{k=1}^{k=v} T(I_k) \in \mathcal{D}(I),$$

$$(\mathbb{1} - \hat{I}) \prod_{k=1}^{k=v} T(I_k) \neq 0, \quad \hat{I} (\mathbb{1} - \hat{I}) \prod_{k=1}^{k=v} T(I_k) = 0.$$

In other words, either a monomial is non-decomposable [in $\mathcal{X}(I)$], and is an eigenstate of $\hat{I}$ with eigenvalue 1, or it is decomposable [in $\mathcal{X}(I)$] and there corresponds to it an eigenstate of $\hat{I}$ with eigenvalue 0 - such eigenstates of $\hat{I}$ are thus put in a one-to-one correspondence with all partitions of X and thus span the whole of $\mathcal{C}$.

2. - PRECELLS 5), 10)

We now look for generalizations of the r_I 's, with support properties in x space, and with coincidence properties in p space with $\tilde{t}(P)$.

Def 5)

A precell $\mathcal{S}$ is a set of proper parts of X ($\neq \emptyset, X$) such that

- 1) If $I \in \mathcal{S}$, $I' \notin \mathcal{S}$,
Let $\mathcal{S}' = \{I' \mid I \in \mathcal{S}\}$.
If $I \notin \mathcal{S}$, then $I' \in \mathcal{S}$.
- 2) If $I_1 \in \mathcal{S}$, $I_2 \in \mathcal{S}$, $I_1 \cap I_2 = \emptyset$
then $I_1 \cup I_2 \in \mathcal{S}$.
- 3) If $I_1 \in \mathcal{S}$, $I_2 \in \mathcal{S}$, $I_1 \cup I_2 = X$
then $I_1 \cap I_2 \in \mathcal{S}$

[a consequence of 1) and 2) applied to $\mathcal{S}'$]

Def

$$\Theta_{\mathcal{S}'} = \prod_{I \in \mathcal{S}} (1 - \hat{I}) \Theta$$

$$\Theta_{\mathcal{S}'} = (\Omega, \Theta_{\mathcal{S}'}, \Omega)$$

Precell Lemma ⁵⁾ (elementary)

If $I_1 \in \mathcal{J}, I_2 \in \mathcal{J}$

Then $I_1 \cap I_2 \in \mathcal{J}$ or $I_1 \cup I_2 \in \mathcal{J}$

Def (paracell) ¹⁰⁾

A paracell is a set $\mathcal{J}$ of proper parts of X for which the precell lemma holds.

Th

If $\mathcal{J}$ is a paracell, the product

$$\prod_{I \in \mathcal{J}} (1 - \hat{I})$$

is independent of the order of the factors.

Sketch of proof

Let $I_1 \dots I_N$ be an arbitrary ordering of $\mathcal{J}$, we have to show that for any permutation π

$$(1 - \hat{I}_1) \dots (1 - \hat{I}_N) = (1 - \hat{I}_{\pi(1)}) \dots (1 - \hat{I}_{\pi(N)})$$

Suppose π is the transposition $N \rightleftharpoons N-1$. The property is true by (ii), (iii), (iv). Suppose the property has been proved for all π 's such that

$$\pi(1)=1, \pi(2)=2, \dots, \pi(N-p)=N-p,$$

The passage from p to $p+1$ follows easily from the induction hypothesis and application of (ii), (iii), (iv). In fact the proof shows the following.

Lemma

Let $A_1 \dots A_N$ be a sequence of not necessarily distinct proper subsets of X , such that for every pair i, j , there exists k such that $A_k = A_i \cup A_j$ or $A_k = A_i \cap A_j$, then for every permutation π of $1 \dots N$

$$(1 - \hat{A}_1) \dots (1 - \hat{A}_N) = (1 - \hat{A}_{\pi(1)}) \dots (1 - \hat{A}_{\pi(N)})$$

- Support properties

Lemma

$$(1 - \hat{I}) \odot_{\mathcal{I}'} = 0 \quad \text{if } I' \geq I, \quad I \in \mathcal{I}$$

hence support $\odot_{\mathcal{I}'} \subset C_I = \left\{ x / x_i - x_j \in \bar{V}_+ , \text{ some } i \in I, j \in I' \right\}$

$$= \bigcup_{\substack{i \in I \\ j \in I'}} (x / x_i - x_j \in \bar{V}^+) = \bigcup_{\substack{i \in I \\ j \in I'}} C_{ij}$$

(use causal factorization property).

- Corollary

$$\odot_{\mathcal{I}'} = \prod_{I \in \mathcal{I}} (1 - \hat{I}) \odot$$

has support contained in

$$\bigcap_{\substack{I \in \mathcal{I} \\ I' \in \mathcal{I}'}} \bigcup_{\substack{i \in I \\ j \in I'}} C_{ij}$$

This support can be analyzed as follows. A choice ^{5), 7)} h is a map $I \rightarrow X$ such that $h(I) \in I$. Let

$$C_h^{\mathcal{I}} = \bigcap_{I \in \mathcal{I}} C_{h(I)h(I')}$$

With this notation

$$\bigcap_{\substack{I \in \mathcal{I} \\ I' \in \mathcal{I}'}} \bigcup_{\substack{i \in I \\ j \in I'}} C_{ij} = \bigcup_h C_h^{\mathcal{I}}$$

Choices h which yield the support can be restricted to choices compatible ⁵⁾ with $\mathcal{J}$ as follows. Let

$$\mathcal{J}^+ = \{j \in X, \{j\} \in \mathcal{J}\}$$

$$\mathcal{J}^- = X \setminus \mathcal{J}^+ = \mathcal{J}'^+$$

One can show ⁵⁾ that for each choice h , there exists a "compatible choice" h' such that

$$h'(I) \in \mathcal{J}^+ \quad \text{if } I \in \mathcal{J}$$

$$h'(I) \in \mathcal{J}^- \quad \text{if } I \in \mathcal{J}'$$

and

$$C_h^{\mathcal{J}} \subset C_{h'}^{\mathcal{J}}$$

from which there follows

$$\text{supp. } \textcircled{H}_{\mathcal{J}} \subset \bigcup_{h \text{ compatible}} C_h^{\mathcal{J}}$$

One can furthermore show ⁵⁾ that each $C_h^{\mathcal{J}}$ is of the form

$$\bigcap_{\substack{i \in \mathcal{J}^+ \\ (ij) \in \mathcal{C}_h}} \{x \mid x_i - x_j \in \nabla^+\}$$

where $\mathcal{C}_h$ is a tree graph with vertices $i \in X$ and links (ij) , $i \in \mathcal{J}^+$, $j \in \mathcal{J}^-$. Such trees were first discovered by Bros ¹¹⁾.

- Analyticity properties

It is easy to see that the Fourier transform of a translation invariant distribution with support in C_h can be extended into a holomorphic function defined on the hyperplane

$$p_x \equiv \sum_{i=1}^n p_i = 0$$

and analytic in the tube with imaginary basis $\tilde{C}_h$ dual to C_h , given by the parametric form ^{1), 9)}

$$p = \sum_{(ij) \in \mathcal{C}_h} s_{(ij)} t_{(ij)}$$

where

$$\mu_{(ij)} \in V^+, \quad S_{(ij)} = (S_{(ij)}^1, \dots, S_{(ij)}^n),$$

$$S_{(ij)}^k = \delta_i^k - \delta_j^k$$

This parametric form is due to Araki ^{1),9)} and has the following nice property :
if a translation invariant distribution has support $C_h^{\mathcal{J}} \cap C_{h'}^{\mathcal{J}}$, its Fourier transform can be extended into the tube with imaginary basis

$$p = \sum S_\lambda \mu_\lambda$$

where $\mu_\lambda \in V^+$ and S_λ are the co-ordinates of exposed one-dimensional facets of the convex hull

$$S = t \sum_{(ij) \in C_h} S_{(ij)} \rho_{(ij)} + (1-t) \sum_{(ij) \in C_{h'}} S'_{(ij)} \rho'_{(ij)}$$

$$0 \leq t \leq 1, \quad 0 \leq \rho_{(ij)}, \quad 0 \leq \rho'_{(ij)}.$$

Alternatively, the tube corresponding to C_h can be described by the set of $n-1$ conditions of the type $\text{Im } p_{I(ij)} \in V^+, \quad I(ij) \in \mathcal{J}$, obtained by writing

$$\sum_{i=1}^{i=n} p_i x_i = \sum_{(ij) \in \mathcal{J}_h} p_{I(ij)} (x_i - x_j) \mod \sum_{i=1}^{i=n} p_i.$$

A description of the latter kind does not exist for convex hulls, in general.

3. - SPECTRUM PROPERTIES OF $T_{\mathcal{J}}$

We look at Fourier transforms of vacuum expectation values, $\tilde{t}_{\mathcal{J}}(P)$ obtained for

$$\Theta = T(X) .$$

Expanding the product

$$T_{\mathcal{J}}(X) = \prod_{I \in \mathcal{J}'} (1 - \hat{I}) T(X)$$

and looking at the last factor of each term, we see that

$$\begin{aligned} \tilde{t}_{\mathcal{J}}(P) &= \tilde{t}(P) & \text{if } p_I \notin S_I^- \\ & & \forall I \in \mathcal{J} \end{aligned}$$

4. - DISCONTINUITY FORMULAE

Def

If $\mathcal{J}$ is a precell, I is called a boundary of $\mathcal{J}$ ($I \in \partial\mathcal{J}$) if

$$(\mathcal{J} \setminus \{I\}) \cup \{I'\}$$

is a precell.

Boundary Lemma ⁵⁾

I is a boundary of $\mathcal{J}$ if

$$\begin{aligned} \text{(i)} \quad I_1 \cup I_2 &= I, \quad I_1 \cap I_2 = \emptyset, \quad I_1 \neq I, \quad I_1 \in \mathcal{J} \\ &\rightarrow I_2 \in \mathcal{J}' \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad I'_1 \cup I'_2 &= I', \quad I'_1 \cap I'_2 = \emptyset, \quad I'_1 \neq I', \quad I'_1 \in \mathcal{J}' \\ &\rightarrow I'_2 \in \mathcal{J} \end{aligned}$$

(i) and (ii) are equivalent to (i) and :

$$(iii) \quad I_1 \cap I_2 = I, \quad I_1 \cup I_2 = X, \quad I_1 \neq I, \quad I_1 \in \mathcal{I} \\ \rightarrow I_2 \in \mathcal{I}'$$

If I is a boundary of $\mathcal{I}$ we shall define

$$\mathcal{I}_I^+ = \mathcal{I}, \quad \mathcal{I}_I^- = (\mathcal{I} \setminus \{I\}) \cup \{I'\} \\ \mathcal{I}_I = (\mathcal{I} \setminus \{I\})$$

Using the commutativity of $\prod_{I \in \mathcal{I}}$ $(1 - \hat{I})$ and leaving I as the last factor, one easily gets

$$\prod_{J \in \mathcal{I}_I^+} (1 - \hat{J}) - \prod_{J \in \mathcal{I}_I^-} (1 - \hat{J}) = \prod_{J \in \mathcal{I}_I} (1 - \hat{J}) [\hat{I}' - \hat{I}]$$

Hence the commutator formula

$$T_{\mathcal{I}_I^+} - T_{\mathcal{I}_I^-} = \prod_{J \in \mathcal{I}_I} (1 - \hat{J}) [T(I'), T(I)]_-$$

Write now

$$\hat{J} = \hat{J}_{I'} \cdot \hat{J}_I = \hat{J} \cap I' \cdot \hat{J} \cap I$$

as a direct product which allows to write, if $K \subset I, K' \subset I'$

$$\hat{J} T(K) T(K') = \hat{J}_{I'} \cdot \hat{J}_I T(K) T(K') = \hat{J}_I T(K) \cdot \hat{J}_{I'} T(K')$$

where

$$\hat{J}_I T(K) = T(K \cap J \cap I) T(K \cap J' \cap I)$$

$$\hat{J}_{I'} T(K') = T(K' \cap J \cap I') T(K' \cap J' \cap I')$$

Let now

$$\Sigma_I = \{ K \mid K \subset I, K \in \mathcal{S}_I \}$$

$$\Sigma_{I'} = \{ K \mid K \subset I', K \in \mathcal{S}_I \}$$

These are precells from the precell property of $\mathcal{S}$ and the boundary property of I .

Now if $J \in \mathcal{S}_I$, either $J \cap I$ is in Σ_I or $J \cap I'$ is in $\Sigma_{I'}$ (or both) because if it were not so, both $J \cap I$ and $J \cap I'$ would be in $\mathcal{S}'$ and since they are disjoint, their union, J , would be in $\mathcal{S}'$ which is not true. Hence J is of one of the following types :

- (i) $K \cup K', K \in \Sigma_I, K' \in \Sigma_{I'}$.
- (ii) $K \cup \emptyset, K \in \Sigma_I$.
- (iii) $\emptyset \cup K', K' \in \Sigma_{I'}$.
- (iv) $K \cup I', K \in \Sigma_I$.
- (v) $I \cup K', K' \in \Sigma_{I'}$.
- (vi) $K \cup L', K \in \Sigma_I, L' \subset I', (L' \neq I', \emptyset)$.
- (vii) $L \cup K', L \subset I, (L \neq I, \emptyset), K' \in \Sigma_{I'}$.

Using again the precell property of $\mathcal{S}$, the commutativity of $\prod_{J \in \mathcal{S}_I} (1 - \hat{J})$ and the idempotency identities

$$(1 - \hat{K})^2 = (1 - \hat{K}), \quad (1 - \hat{K}')^2 = 1 - \hat{K}'$$

we can write

$$\prod_{J \in \mathcal{S}_I} (1 - \hat{J}) = \prod_{\substack{K \in \Sigma_I \\ K' \in \Sigma_{I'}}} (1 - \hat{K}' \hat{K}) (1 - \hat{K}) (1 - \hat{K}') \dots$$

$$\begin{aligned} \dots & \cdot \prod_{\substack{K \in \Sigma_{I'} \\ L \in I}} (1 - \hat{K}' \hat{L}) (1 - \hat{K}') \cdot \prod_{\substack{L' \in I' \\ K \in \Sigma_I}} (1 - \hat{L}' \hat{K}) (1 - \hat{K}) \\ & = \prod_{\substack{K \in \Sigma_I \\ K' \in \Sigma_{I'}}} (1 - \hat{K}') (1 - \hat{K}) \end{aligned}$$

where we have used

$$(1 - \hat{K}' \hat{K}) (1 - \hat{K}) (1 - \hat{K}') = (1 - \hat{K}') (1 - \hat{K})$$

$$(1 - \hat{K}' \hat{L}) (1 - \hat{K}') = (1 - \hat{K}')$$

$$(1 - \hat{L}' \hat{K}) (1 - \hat{K}) = (1 - \hat{K})$$

on account of the idempotency

$$\hat{K}^2 = \hat{K}, \quad \hat{K}'^2 = \hat{K}'$$

The commutativity of the factors $(1 - \hat{K})$, $(1 - \hat{K}')$ allows to split these factors :

$$\begin{aligned} T_{\mathcal{I}_I^+} - T_{\mathcal{I}_I^-} &= \prod_{\substack{K \in \Sigma_I \\ K' \in \Sigma_{I'}}} (1 - \hat{K}) (1 - \hat{K}') [T(I'), T(I)] \\ &= \left[\prod_{K' \in \Sigma_{I'}} (1 - \hat{K}') T(I'), \prod_{K \in \Sigma_I} (1 - \hat{K}) T(I) \right] \\ &= \left[T_{\Sigma_{I'}}(I'), T_{\Sigma_I}(I) \right] \end{aligned}$$

This is the so-called Ruelle discontinuity formula ^{2),5)}.

5. - STEINMANN'S IDENTITIES 4), 1), 2)

Let I, J such that $I \not\subset J, I \not\subset J', J \not\subset I, J \not\subset I'$, hence $I' \not\subset J, I' \not\subset J', J \not\subset I', J' \not\subset I'$. Consider four cells admitting I, J , as boundaries :

$$\mathcal{S}_{IJ}^{++} = \mathcal{S}_{IJ} \cup I \cup J = \mathcal{S}_I \cup J = \mathcal{S}_J \cup I$$

$$\mathcal{S}_{IJ}^{+-} = \mathcal{S}_{IJ} \cup I \cup J' = \mathcal{S}_I \cup J' = \mathcal{S}_{J'} \cup I$$

$$\mathcal{S}_{IJ}^{-+} = \mathcal{S}_{IJ} \cup I' \cup J = \mathcal{S}_{I'} \cup J = \mathcal{S}_J \cup I'$$

$$\mathcal{S}_{IJ}^{--} = \mathcal{S}_{IJ} \cup I' \cup J' = \mathcal{S}_{I'} \cup J' = \mathcal{S}_{J'} \cup I'$$

Then

$$T_{\mathcal{S}_{IJ}^{++}} - T_{\mathcal{S}_{IJ}^{+-}} = \left[T_{\Sigma_I^+}(I), T_{\Sigma_{I'}^+}(I') \right]$$

where

$$\begin{aligned} \Sigma_I^+ &= \left\{ K \mid K \subset I, K \in \mathcal{S}_{IJ} \right\} \\ &\text{since } J \not\subset I \\ \Sigma_{I'}^+ &= \left\{ K \mid K \subset I', K \in \mathcal{S}_{IJ} \right\} \\ &\text{since } J \not\subset I' \end{aligned}$$

Similarly

$$T_{\mathcal{S}_{IJ}^{-+}} - T_{\mathcal{S}_{IJ}^{--}} = \left[T_{\Sigma_I^-}(I), T_{\Sigma_{I'}^-}(I') \right]$$

where

$$\begin{aligned} \Sigma_I^- &= \left\{ K \mid K \subset I, K \in \mathcal{S}_{IJ} \right\} \\ &\text{since } J' \not\subset I \\ \Sigma_{I'}^- &= \left\{ K \mid K \subset I', K \in \mathcal{S}_{IJ} \right\} \\ &\text{since } J' \not\subset I' \end{aligned}$$

Hence

$$\begin{aligned}\sum_I^+ &= \sum_I^- \\ \sum_{I'}^+ &= \sum_{I'}^-\end{aligned}$$

and

$$T_{\mathcal{I}\mathcal{J}}^{++'} - T_{\mathcal{I}\mathcal{J}}^{+-'} - T_{\mathcal{I}\mathcal{J}}^{-+'} + T_{\mathcal{I}\mathcal{J}}^{--'} = 0$$

The so-called Steinmann-Ruelle relations ^{1),2)}.

6. - CELLS ^{1),2)}

The interpretation of $\tilde{t}(P)$ as a piecewise boundary value of analytic functions goes via the consideration of cells.

Def ⁵⁾

A precell $\mathcal{S}$ is a cell if, in R^n , on the hyperplane $S_X = \sum_i^n S_i = 0$, the set of conditions

$$S_I = \sum_{i \in I} S_i > 0 \quad \forall I \in \mathcal{S}$$

is non empty. This set will be called $\Gamma_{\mathcal{S}}$. Clearly, if $S \in \Gamma_{\mathcal{S}}$ $S_I < 0$ if $I \in \mathcal{S}'$.

One can easily see that Steinmann-Ruelle relations can only connect together cells whose Γ 's are within a hypercell $\Gamma_I : S_i > 0 \ i \in I, S_i < 0 \ i \in I'$, i.e., which all contain $i, i \in I$ and $[i]'$, $i \in I'$. The one dimensional boundaries of Γ_I are the vectors $S_{ij}, i \in I, j \in I'$, $S_{ij}^k = \delta_i^k - \delta_j^k$, we already met. One can prove that for a cell $\mathcal{S}$, the admissible $\tilde{C}_h$ are such that Γ_h :

$$S = \sum_{(ij) \in \tilde{C}_h} S_{(ij)} \rho_{(ij)} \quad , \quad \rho_{(ij)} > 0$$

contain $\Gamma_{\mathcal{S}}$.

A more detailed result (Bros ¹¹⁾) shows that for all $\mathcal{J}$'s such that $\Gamma_{\mathcal{J}} \subset \Gamma_I$ there is a decomposition of the form

$$\tilde{t}_{\mathcal{J}} = \sum_h \Theta_{\mathcal{J},h} \tilde{f}_h$$

where the $\tilde{f}_h$ are holomorphic in the tube with imaginary basis $\tilde{C}_h$ (the dual of C_h), where

$$\begin{aligned} \Theta_{\mathcal{J},h} &= 1 & \text{if } \Gamma_h \supset \Gamma_{\mathcal{J}} \\ \Theta_{\mathcal{J},h} &= 0 & \text{if } \Gamma_h \not\supset \Gamma_{\mathcal{J}} \end{aligned}$$

The foregoing results are best summarized within the framework of the following geometrical construction.

Consider an n dimensional space $\{S = (S_1, \dots, S_n)\}$ and the hyperplane

$$\sum_{i=1}^n S_i = 0$$

One draws all the hyperplanes

$$S_I \equiv \sum_{i \in I} S_i = 0 \quad (= S_{I'})$$

which bound geometrical cells $\Gamma_{\mathcal{J}}$ within which all S_I 's have a prescribed sign. One can easily read off this diagram the various relevant S_{ij} 's in the preceding formulae.

Examples are shown on Figs. 1, 2, 3.

7. - TRUNCATION

One has still to make a slight modification in order to make the foregoing construction useful. The regions in p space where the $\tilde{t}(P)$ coincide with $\tilde{t}(P)$ do not cover the whole of P space because of the occurrence of the vacuum contribution $\{0\}$ to the spectrum S_I^+ . One can however, define connected distributions $\tilde{t}_c(P)$, $\tilde{t}_{\mathcal{S},c}(P)$ [which turn out to be identical to $\tilde{t}_{\mathcal{S}}(P)$] such that all previously established properties hold. The regions where $\tilde{t}_c(P)$ coincide with the $\tilde{t}_{\mathcal{S}}(P)$'s cover all of P space when $\mathcal{S}$ goes over all possible cells, S_I^+ being then reduced to $\bar{V}_I^+$. Although this is quite important, we shall not give any more detail here.

The bridge can be made with the construction of Haraki and Ruelle by establishing the following identity ^{5),12)}

$$T_{\mathcal{S}} = \sum_{\nu} (-)^{\nu+1} \sum_{\substack{I_1 \cup \dots \cup I_{\nu} = X \\ I_j \cap I_k = \emptyset \\ K_{\mathcal{S}} = I_1 \cup \dots \cup I_{\nu} \in \mathcal{S} \\ \mathcal{S} = 1, \dots, \nu-1}} T(I_1) \dots T(I_{\nu})$$

from which one easily recovers the spectrum properties, the discontinuity formula and the Steinmann identities, but from which support properties are hard to get.

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FIG.1
 $n = 3$

FIG.2

$n = 4$

A hypercell $ABCA'B'C'$

Typical cells : $AA'bc$ (3)

$AbcH$ (6)

$A'B'cH'$ (6)

$abcHH'$ (1)

$HABC$ (2)

Typical Bros Trees : $AA'BC'$ (6)

$A'BBC$ (6)

FIG.3

$n=5$