SPIN CONSTRAINTS ON THE PARTICLE SPECTRUM
IN A DUALITY BOOTSTRAP

J.E. Mandula ^{*)}

Massachusetts Institute of Technology
Department of Applied Mathematics
Cambridge, USA

J. Weyers

CERN, Geneva, Switzerland

G. Zweig ^{*)}

California Institute of Technology,
Pasadena, California, USA

A B S T R A C T

The constraints on the particle spectrum following from the requirement that sums of resonances yield the spin structure of the leading exchange are discussed.

^{*)} Alfred P. Sloan Foundation Fellow.

1. INTRODUCTION

This paper is a discussion of the effects of external particle spin on the duality conjecture¹⁾ that scattering amplitudes shorn off their diffractive components²⁾, can be calculated either by summing their single resonance contributions or by summing their Regge exchanges.

Although the extent to which duality is a valid principle remains unclear, it is a very powerful constraint on the particle spectrum. Precisely how powerful depends on how widely duality may be imposed. Dual resonance models³⁾, for example, embody a maximal statement of duality: all reactions at all momentum transfers are dual. We here discuss a more conservative scheme in which duality is to be imposed on all reactions simultaneously but only in forward scattering, although we indicate how to enlarge the momentum transfer range. We obtain a large particle spectrum not through the analytic matching of Regge pole and resonance contributions to a scattering amplitude but rather by matching their contributions to many different spin amplitudes at a single momentum transfer. The content of the scheme is expressed as a set of sum rules on the resonance contributions to forward scattering amplitudes.

The specific technical assumptions with which we implement duality are:

- 1) Semi-local duality at $t = 0$ (we operationally define semi-local duality by requiring that only one particle on a trajectory participates in each duality sum rule).
- 2) The leading meson exchange, the vector-tensor nonets of exchange degenerate trajectories, have $M = 0$; and the leading exchange degenerate baryon trajectories have $M = \frac{1}{2}$ (the Toller quantum number M is the allowed s -channel helicity flip of an exchanged trajectory in the forward direction. Note that as a consequence of MacDowell symmetry, both natural parities are present in the leading baryon exchange at $u = 0$).
- 3) There are no exotic mesons or baryons.

From the requirement that there be no exotic resonances, the exchange degeneracies between mesonic and between baryonic trajectories were inferred. Meson trajectories are required to occur in exchange degenerate nonets, while several patterns of baryonic exchange degeneracy were discovered⁴⁾.

In addition, there are new constraints which follow from the requirement that resonances, when summed, yield amplitudes with the same spin structure as that given by the leading exchange. These constraints are implemented in a manner exactly complementary to the calculation of the exchange degeneracies. The duality constraints that are relevant for exchange degeneracy come from reactions with

exotic direct channels; that is, those in which the resonance side of the duality equation vanishes. The new constraints are those in which the Regge side of the equation vanishes, either because the crossed channel is an exotic SU(3) representation or because the spins of the external particles are such that the leading Regge exchange makes no contribution in either forward or backward scattering.

These constraints have already been solved⁵⁾ in the simplest case that exhibits all the complications of the problem. The smallest mesonic spectrum that satisfies the constraints in all reactions involving the pseudoscalar and vector nonets contains for each value of C three PC = P (-1)^J = +1 nonets, two PC = P (-1)^J = -1 nonets and one PC = -P(-)^J = +1 nonet.

In this paper we extend the previous calculation to the general case. We impose the duality constraints on all reactions with baryon number zero and one involving non-exotic mesons and baryons with any spins as the external particles. The strongest constraints come from the elastic scattering of high spin particles and from inelastic reactions in which the natural parity of the initial and final mesons are different. The solution to the duality constraints in other reactions are trivially derived from these.

The constraints on forward and backward scattering have striking simplicity -- many amplitudes vanish. By contrast, the constraints on finite angle scattering are weaker and more complex; we shall not treat them here.

The constraints on the spin structure of forward and backward scattering explicitly involve the coupling constants of resonances to external particles. Let us denote by $A_{\mu\nu}^{a,n}$ the coupling of the n-th SU(3) representation $\underline{a}$ of resonances to the (μ, ν) helicity state of the initial external particles, and by $B_{\lambda\sigma}^{a,n}$ the coupling to the (λ, σ) helicity state of the final particles. The constraints are most simply expressed in terms of vectors $\vec{A}_{\lambda\sigma}^a$ and $\vec{B}_{\lambda\sigma}^a$ whose components are $A_{\mu\nu}^{a,n}$ and $B_{\lambda\sigma}^{a,n}$.

In meson-meson scattering, the forward and backward constraints are redundant. The absence of exotic exchanges implies the constraint

$$\sum_b X^{ab} \vec{A}_{\mu\nu}^b \vec{B}_{\lambda\sigma}^b = 0 \quad (\mu - \nu = \lambda - \sigma) \quad (1.1)$$

where X^{ab} is the crossing matrix and $\underline{a}$ an exotic SU(3) representation. The M = 0 nature of the leading Regge exchange gives rise to the constraints

$$\vec{A}_{\mu\nu}^a \cdot \vec{B}_{\lambda\sigma}^a = 0 \quad (\mu - \nu = \lambda - \sigma \text{ and } |\mu - \lambda| > 0) \quad (1.2)$$

and if the initial and final mesons have opposite natural parity, the positive natural parity of the leading exchange provides the additional constraints

$$\vec{A}_{0\nu}^a \cdot \vec{B}_{0\sigma}^a = 0 \quad (\nu = \sigma) \quad (1.3)$$

The forward and backward constraints on meson-baryon scattering are independent. The forward constraints are exactly the same as those on meson-meson scattering above, with $\vec{A}_{\mu\nu}^a$ and $\vec{B}_{\mu\nu}^a$ vectors whose components are couplings of resonances to external meson-baryon channels. The backward constraints are simplest when expressed in terms of a metric $\underset{\sim}{g}$ in the resonance vector space, where

$$g_{nn'} = \delta_{nn'} \eta_{P_n} \quad (1.4)$$

and η_{P_n} is the parity of the n-th resonance. The absence of exotic baryon exchanges implies the constraints

$$\sum_b \chi^{ab} \vec{A}_{\mu\nu}^b \cdot \underset{\sim}{g} \cdot \vec{B}_{\lambda\sigma}^b = 0 \quad (\mu-\nu = \lambda-\sigma) \quad (1.5)$$

with $\underline{a}$ an exotic baryon representation while the $M = 1/2$ nature of the leading baryon exchange results in

$$\vec{A}_{\mu\nu}^a \cdot \underset{\sim}{g} \cdot \vec{B}_{\lambda\sigma}^a = 0 \quad (\mu-\nu = \lambda-\sigma \text{ and } |\mu-\sigma| > \frac{1}{2}) \quad (1.6)$$

If an amplitude is not forced to be zero because of the character of the leading Regge exchange, we require that it be non-zero. The solutions to these equations give the meson and baryon spectra in this truncated duality model.

The minimum meson spectrum satisfying these constraints in the scattering of all mesons with spins from 0 to L and both P and C contains the following nonets of mesons: $J^{PC} = (2L)^{++}, (2L-1)^{--}, (2L-2)^{++}, \dots, 1^{--}, 0^{++}, 0^{++}; (2L-1)^{+-}, (2L-2)^{-+}, (2L-3)^{+-}, \dots, 3^{+-}, 2^{-+}, 2^{-+}, L^{+-}, 1^{+-}; 2^{--}, 1^{++}$. The constraints would still be satisfied if any of these nonets of mesons have spins which exceed those indicated by an even integer. If one imagined that the solutions with $L = 1, 2, 3, \dots$ represented the particle spectrum at a given mass region in a theory where all particles lie on linearly rising Regge trajectories, the simplest pattern of Regge trajectories would contain a leading natural parity +, PC + exchange degenerate nonet of trajectories each with an infinite set of subsidiary trajectories spaced by even integers, the first of which is doubled; a doubled natural parity -, PC - exchange degenerate nonet of trajectories with an infinite set of non-degenerate subsidiaries also spaced by even integers; and a single non-degenerate natural parity -, PC + exchange degenerate nonet of trajectories. Note that although there is a proliferation of trajectories with $C(-1)^J = +1$, with both natural parities, only a single exchange degenerate nonet of trajectories with the A_1 quantum numbers $[C(-1)^J = -1, PC = +]$ is required by these constraints.

The well-known difficulty which duality encounters in baryon-antibaryon scattering⁷⁾ invites query into the precise significance of this mesonic spectrum. If, as has been suggested⁸⁾, duality is only approximately correct at moderate energies, then the physical particle spectrum will agree with, or at least contain, the low mass states in this model. In particular our model predicts additional low-lying scalar (0^{++}) and axial (1^{+-}) nonets beyond the multiplets expected from the quark model⁹⁾.

On the other hand it should be noted that in a theory in which resonance saturation is valid in all reactions (in particular in baryon-antibaryon scattering), our calculation gives the complete minimal spectrum of non-exotic resonances but a rich spectrum of high mass exotics would of course have to be added.

There is an important qualitative distinction between the minimal sets of approximately degenerate resonances that we have been describing and the multiplets associated with approximate symmetries. When a group underlies a multiplet structure, particles occur only in certain definite combinations -- complete multiplets. But any solution to these duality constraints may be augmented by adjoining any additional set of particles, and it will be possible to satisfy the constraints with the larger set as well.

The question of how closely degenerate the sets of resonances we have described should be is the same as the question of over how small an energy region is the "average of the resonances" implicit in the definition of semi-local duality. One clue to the answer is the observation that particles on exchange degenerate trajectories must be included in individual semi-local duality constraints. This certainly suggests that the deviations from degeneracy are at least one BeV^2 .

In Section 2 we derive and discuss the duality constraints and in Section 3 we explicitly solve them for the mesons. Section 4 describes the outlook for further progress.

2. DERIVATION OF THE CONSTRAINTS

Because mesonic and baryonic trajectory exchanges have very different $\text{SU}(3)$ and spin structures, we will treat the constraints associated with these exchanges separately.

A) Meson exchange

The basic assumption, from which all the mesonic exchange constraints follow, is that the dominant mesonic trajectories are the nonets of trajectories on which lie the vector and tensor mesons, and that these trajectories have $M = 0$. The value $M = 0$ is what would be required of a factorized trajectory that coupled to the crossed channel pseudoscalar meson -- pseudoscalar meson system at $t = 0$. It implies that any amplitude with helicity flip at either vertex (or both) vanishes at $t = 0$.

The imaginary part of the amplitude for the processes $I + II \rightarrow III + IV$, on the assumption that all scattering occurs through narrow resonances is

$$\text{Im } f_{\mu\nu;\lambda\sigma}^a(\theta) = \sum_n A_{\mu\nu}^{a,n} d_{\mu-\nu,\lambda-\sigma}^{J_n}(\theta) B_{\lambda\sigma}^{a,n} \delta(s - m_n^2) \quad (2.1)$$

The superscript a denotes the direct channel $SU(3)$ representation of the reaction and n labels the multiplets of resonances, $A_{\mu\nu}^{a,n}$ is the coupling of the n^{th} multiplet to particles I and II with helicities μ and ν respectively; $B_{\lambda\sigma}^{a,n}$ is its coupling to particles III and IV with helicities λ and σ ; and m_n and J_n are its mass and spin. Integrating over a range of energies R , and evaluating equation (1) at $t = 0$ gives

$$\int_R ds \text{Im } f_{\mu\nu;\lambda\sigma}^a(0) = \sum_{n \in R} A_{\mu\nu}^{a,n} d_{\mu-\nu,\lambda-\sigma}^{J_n}(0) B_{\lambda\sigma}^{a,n} \quad (2.2)$$

The requirement that the dominant mesonic exchange be purely $M = 0$ and $t = 0$ immediately gives the constraints

$$\sum_{n \in R} A_{\mu\nu}^{a,n} B_{\lambda\sigma}^{a,n} = 0 \quad (\mu - \nu = \lambda - \sigma \text{ and } |\mu - \lambda| > 0) \quad (2.3a)$$

The restriction to $\mu - \nu = \lambda - \sigma$ results from the fact that $d_{\mu-\nu,\lambda-\sigma}^J(0) = \delta_{\mu-\nu,\lambda-\sigma}$. Note that the equations have the form of orthogonality equations:

$$\vec{A}_{\mu\nu}^a \cdot \vec{B}_{\lambda\sigma}^a = \sum_n A_{\mu\nu}^{a,n} B_{\lambda\sigma}^{a,n} = 0 \quad (\mu - \nu = \lambda - \sigma \text{ and } |\mu - \lambda| > 0) \quad (2.3b)$$

where the dot product is taken in the vector space whose coordinates are labelled by the resonances so that the dimension of the space equals the number of $SU(3)$ multiplets of resonances. For elastic scattering, $A = B$.

When the natural parity changes at a mesonic vertex, there is an additional constraint that arises because the leading trajectory has positive natural parity, and so does not contribute to amplitudes which describe inelastic forward scattering $I + II \rightarrow III + IV$ in which either particles I and III or II and IV or both have opposite natural parity and zero helicity. For reactions in which particles I and III are mesons with opposite natural parity, the additional constraint at $t = 0$ is

$$\vec{A}_{0\nu}^a \cdot \vec{B}_{0\sigma}^a = 0 \quad (\nu = \sigma) \quad (2.4)$$

and if particles II and IV have opposite natural parity.

$$\vec{A}_{\mu 0}^a \cdot \vec{B}_{\lambda 0}^a = 0 \quad (\mu = \lambda) \quad (2.5)$$

In meson-meson scattering the constraints associated with spin and those associated with SU(3) exoticity factor since the exchange of a nonet ($\underline{8} \oplus \underline{1}$) of exchange degenerate trajectories with coupling ratios given by previous exchange degeneracy calculations⁹⁾, expressed in terms of direct channel SU(3) representations has exactly the same coupling ratio of octet and singlet. Technically, this specific ratio of SU(3) octet and singlet amplitudes is an eigenvector of the SU(3) crossing matrix with eigenvalue +1. Accordingly, we will drop the SU(3) index "a" in all further discussion of meson-meson scattering. Each component of $\vec{A}_{\mu\nu}$ will represent 18 states of the same natural parity, $\underline{8} \oplus \underline{1}$ of even spin and $\underline{8} \oplus \underline{1}$ of odd spin.

The most important constraints are those on elastic scattering and on reactions involving particles of different natural parity and zero helicity. These constraints are

$$\vec{A}_{\mu\nu} \cdot \vec{A}_{\lambda\sigma} = 0 \quad (\mu - \nu = \lambda - \sigma \text{ and } |\mu - \lambda| > 0) \quad (2.6)$$

If particles I and III have opposite natural parity,

$$\vec{A}_{0\nu} \cdot \vec{B}_{0\sigma} = 0 \quad (\nu = \sigma) \quad (2.7)$$

and if particles II and IV have opposite natural parity

$$\vec{A}_{\mu 0} \cdot \vec{B}_{\lambda 0} = 0 \quad (\mu = \lambda) \quad (2.8)$$

The other analogue equations involving inelastic scattering turn out to be redundant.

The problem we wish to solve is the determination of the minimum number of particles needed to satisfy these duality constraints, which is the same as the minimum dimensionality of a space in which all these orthogonality relations can hold without imposing any further orthogonality relations that do not correspond to some valid duality constraint. That is to say, we will not admit solutions with "accidental zeros".

The natural parity of each direct channel mesonic or baryonic resonance is reflected in the symmetries of its couplings to the external particles. If $\overline{\eta}_P = (-1)^J \eta_P$ denotes the natural parity of a particle, the relations are that

$$A_{\mu\nu}^{a,n} = \overline{\eta}_{P,I} \overline{\eta}_{P,II} \overline{\eta}_{P,n} A_{-\mu,-\nu}^{a,n} \quad (2.9)$$

$$B_{\lambda\sigma}^{a,n} = \overline{\eta}_{P,III} \overline{\eta}_{P,IV} \overline{\eta}_{P,n} B_{-\lambda,-\sigma}^{a,n} \quad (2.10)$$

If particles I and II (or III and IV) are identical mesons, then the PC of a direct channel meson resonance is also reflected in the symmetries of its couplings. If η_{PC} denotes the PC of a resonance and particles I and II are identical mesons,

$$A_{\mu\nu}^{a,n} = \eta_{PC,n} A_{-\nu-\mu}^{a,n} \quad (2.11)$$

and if III and IV are identical mesons

$$B_{\lambda\sigma}^{a,n} = \eta_{PC,n} B_{-\sigma-\lambda}^{a,n} \quad (2.12)$$

B) Baryon exchange

There are several complications associated with the formulation of these duality constraints on meson-baryon scattering, and especially on backward meson-baryon scattering, which is dominated by baryon trajectory exchange. So although we will not solve the constraints on meson-baryon scattering here, we will nonetheless write them down in detail.

We make only the following assumptions about the dominant baryonic exchange: that it has $M = 1/2$ and that it is non-exotic. At $u = 0$ any Regge exchange has equal amounts of both natural parities simply as a consequence of McDowell symmetry. If we require that the positive and negative natural parity exchanges separately factorize at $u = 0$, and that the leading exchange does not decouple from πN at that point, the ensuing constraints are equivalent to the demand that the leading baryonic exchanges have $M = 1/2$, and this observation is the rationale for that assumption.

The SU(3) structure of the leading baryonic exchange is expected to change at ultra-high energies, because at presently available energies the nucleon trajectory dominates backward meson-baryon scattering while at ultra-high energies the Δ and $5/2^-$ exchange degenerate trajectories should become dominant. Were we to take either as the dominant exchange, there would be further constraints coming from the partial knowledge of the SU(3) structure of their couplings obtained from exchange degeneracy calculations. Because of the ambiguity inherent in the choice of constraints, we elect to write down neither.

In order to display the backward scattering constraints we evaluate a finite energy integral of the imaginary part of a backward meson-baryon scattering amplitude:

$$\int_R ds \operatorname{Im} f_{\mu\nu,\lambda\sigma}^a(\pi) = \sum_{n \in R} A_{\mu\nu}^{a,n} d_{\mu-\nu,\lambda-\sigma}^{J_n}(\pi) B_{\lambda\sigma}^{a,n} \quad (2.13)$$

The requirement that the baryonic exchanges have $M = 1/2$ gives the constraints

$$\sum_{n \in R} A_{\mu\nu}^{a,n} d_{\mu-\nu,\lambda-\sigma}^{J_n}(\pi) B_{\lambda\sigma}^{a,n} = 0 \quad (|\mu + \sigma| > 1/2) \quad (2.14)$$

If we observe that $d_{\alpha\beta}^J(\pi) = (-1)^{J-\alpha} \delta_{\alpha-\beta}$, that $B_{-\lambda-\sigma}^{a,n} = \bar{\eta}_{P,III}, \bar{\eta}_{P,IV}, \bar{\eta}_{P,n} B_{\lambda\sigma}^{a,n}$ [Eq. (10)], and that $\bar{\eta}_{P,n} = (-1)^{Jn} \eta_{P,n}$, then these constraints are seen to be pseudo-orthogonality relations:

$$\sum_{n \in R} \eta_{P,n} A_{\mu\nu}^{a,n} B_{-\lambda-\sigma}^{a,n} = 0 \quad (\mu - \nu = \sigma - \lambda \text{ and } |\mu + \sigma| > \frac{1}{2}) \quad (2.15)$$

There are only constraints when $\mu - \nu = \sigma - \lambda$ because only then is $d_{\mu-\nu, \lambda-\sigma}^J(\pi)$ non-vanishing. If we define a metric in the particle vector space with elements

$$g_{nn'} = \delta_{nn'} \eta_{P,n} \quad (2.16)$$

these backward meson-baryon scattering constraints can be written in the compact inner product notation.

$$\vec{A}_{\mu\nu}^a \cdot g \cdot \vec{B}_{-\lambda-\sigma}^a = 0 \quad (\mu - \nu = \sigma - \lambda \text{ and } |\mu + \sigma| > \frac{1}{2}) \quad (2.17)$$

Note that the SU(3) character of the leading exchanges in forward meson-baryon scattering are known; there is no ambiguity about the dominance of the vector-tensor nonets of exchange degenerate trajectories. However, because the relative strengths of the couplings of the singlet and octet to the $B\bar{B}(8 \otimes 8)$ and to the $\Delta\bar{\Delta}(10 \otimes \bar{10})$ channels are undetermined, and because the f/d ratio of the octet's coupling to the $B\bar{B}$ channel is also undermined, the known SU(3) structure of the leading exchange places no constraints on the direct channel resonances beyond those implied by the absence of exotic mesonic exchanges.

The constraints on amplitudes not required to vanish by the $M = 0, \bar{\eta}_P = 1$ or $M = 1/2$ nature of the leading Regge exchange can also be written in the inner product notation. If $\vec{A}_{\mu\nu}^{(m \otimes n)a}$ and $\vec{B}_{\lambda\sigma}^{(m \otimes n)a}$ are vectors of couplings to an external m-plet of meson and n-plet of baryons, the constraints on direct channel baryon resonances imposed by the absence of exotic exchanges (for $\mu - \nu = \lambda - \sigma$) are:

$$\frac{1}{8} \vec{A}_{\mu\nu}^{(8 \otimes 8)1} \cdot \vec{B}_{\lambda\sigma}^{(8 \otimes 8)1} - \frac{2}{5} \vec{A}_{\mu\nu}^{(8 \otimes 8)8_1} \cdot \vec{B}_{\lambda\sigma}^{(8 \otimes 8)8_1} + \frac{1}{4} \vec{A}_{\mu\nu}^{(8 \otimes 8)10} \cdot \vec{B}_{\lambda\sigma}^{(8 \otimes 8)10} = 0 \quad (no \ 10_t, \bar{10}_t) \quad (2.18)$$

$$\begin{aligned} \frac{1}{8} \vec{A}_{\mu\nu}^{(8 \otimes 8)1} \cdot \vec{B}_{\lambda\sigma}^{(8 \otimes 8)1} + \frac{1}{5} \vec{A}_{\mu\nu}^{(8 \otimes 8)8_1} \cdot \vec{B}_{\lambda\sigma}^{(8 \otimes 8)8_1} - \frac{1}{5} \vec{A}_{\mu\nu}^{(8 \otimes 8)8_a} \cdot \vec{B}_{\lambda\sigma}^{(8 \otimes 8)8_a} \\ - \frac{1}{12} \vec{A}_{\mu\nu}^{(8 \otimes 8)10} \cdot \vec{B}_{\lambda\sigma}^{(8 \otimes 8)10} = 0 \quad (no \ 27_t) \end{aligned} \quad (2.19)$$

$$\begin{aligned} \frac{1}{8} \bar{A}_{\mu\nu}^{(8 \otimes 8)_1} \cdot g \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 8)_1} - \frac{2}{5} \bar{A}_{\mu\nu}^{(8 \otimes 8)_2} \cdot g \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 8)_2} + \frac{3}{5} \bar{A}_{\mu\nu}^{(8 \otimes 8)_3} \cdot g \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 8)_3} \\ + \frac{3}{5} \bar{A}_{\mu\nu}^{(8 \otimes 8)_4} \cdot g \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 8)_4} - \frac{1}{4} \bar{A}_{\mu\nu}^{(8 \otimes 8)_{10}} \cdot g \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 8)_{10}} = 0 \quad (\text{no } \underline{10}_u) \end{aligned} \quad (2.20)$$

$$\begin{aligned} \frac{1}{8} \bar{A}_{\mu\nu}^{(8 \otimes 8)_1} \cdot g \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 8)_1} + \frac{1}{5} \bar{A}_{\mu\nu}^{(8 \otimes 8)_2} \cdot g \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 8)_2} + \frac{1}{5} \bar{A}_{\mu\nu}^{(8 \otimes 8)_3} \cdot g \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 8)_3} \\ + \frac{1}{12} \bar{A}_{\mu\nu}^{(8 \otimes 8)_{10}} \cdot g \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 8)_{10}} = 0 \quad (\text{no } \underline{27}_u) \end{aligned} \quad (2.21)$$

$$\frac{\sqrt{2}}{5} \bar{A}_{\mu\nu}^{(8 \otimes 8)_2} \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 10)_2} - \frac{3\sqrt{2}}{5} \bar{A}_{\mu\nu}^{(8 \otimes 8)_3} \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 10)_2} - \frac{1}{2} \bar{A}_{\mu\nu}^{(8 \otimes 8)_{10}} \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 10)_{10}} = 0 \quad (\text{no } \underline{10}_t) \quad (2.22)$$

$$\frac{2}{5} \bar{A}_{\mu\nu}^{(8 \otimes 8)_2} \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 10)_2} + \frac{2}{5} \bar{A}_{\mu\nu}^{(8 \otimes 8)_3} \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 10)_2} - \frac{\sqrt{2}}{6} \bar{A}_{\mu\nu}^{(8 \otimes 8)_{10}} \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 10)_{10}} = 0 \quad (\text{no } \underline{27}_t) \quad (2.23)$$

$$\begin{aligned} \frac{2}{5} \bar{A}_{\mu\nu}^{(8 \otimes 8)_2} \cdot g \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 10)_2} - \frac{2}{5} \bar{A}_{\mu\nu}^{(8 \otimes 8)_3} \cdot g \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 10)_2} + \frac{\sqrt{2}}{6} \bar{A}_{\mu\nu}^{(8 \otimes 8)_{10}} \cdot g \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 10)_{10}} = 0 \\ (\text{no } \underline{27}_u) \end{aligned} \quad (2.24)$$

$$\frac{2}{5} \bar{A}_{\mu\nu}^{(8 \otimes 10)_2} \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 10)_2} - \frac{1}{2} \bar{A}_{\mu\nu}^{(8 \otimes 10)_{10}} \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 10)_{10}} = 0 \quad (\text{no } \underline{27}_t) \quad (2.25)$$

$$\frac{2}{5} \bar{A}_{\mu\nu}^{(8 \otimes 10)_2} \cdot g \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 10)_2} - \frac{1}{2} \bar{A}_{\mu\nu}^{(8 \otimes 10)_{10}} \cdot g \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 10)_{10}} = 0 \quad (\text{no } \underline{27}_u, \underline{35}_u) \quad (2.26)$$

The exotic channel in which the imaginary part is eliminated by each constraint is shown in parentheses.

Note that the spin and SU(3) constraints do not factor. There is a linear combination of Eqa. (22) and (23) which is an orthogonality condition, and so should be especially noted:

$$[\bar{A}_{\mu\nu}^{(8 \otimes 8)_2} + 3 \bar{A}_{\mu\nu}^{(8 \otimes 8)_3}] \cdot \bar{B}_{\lambda\sigma}^{(8 \otimes 10)_2} = 0 \quad (\mu - \nu = \lambda - \sigma) \quad (2.27)$$

In summary, the constraints on forward and backward meson-baryon scattering are Eqs. (3), (4), and (17)-(27).

3. THE MINIMAL MESON SPECTRUM

The problem of satisfying the duality constraints on all meson-meson scattering reactions involving external mesons with spin up to L is simpler than it may appear. Once a solution to the duality spin constraints on reactions involving a spin L natural parity + meson and a spinless natural parity - meson is known, a solution to the apparently more complicated problem can be straightforwardly constructed without adding any additional resonances.

Let us carry out that construction. Call the spin L natural parity + meson a and the spinless natural parity - meson b . Denote the vectors of couplings to the four channels by

$$\begin{aligned}\vec{A}_{\mu\nu} &= \text{vector of couplings to the } aa \text{ channel} \\ \vec{B}_{\mu} &= \text{vector of couplings to the } ab \text{ channel} \\ \vec{C}_{\mu} &= \text{vector of couplings to the } ba \text{ channel} \\ \vec{D} &= \text{vector of couplings to the } bb \text{ channel} .\end{aligned}$$

Note that $\vec{B}_{\mu}$ and $\vec{C}_{\mu}$ are related by

$$\vec{C}_{\mu} = g_{\mu} \cdot \vec{B}_{-\mu} \quad (3.1)$$

where again g_{μ} is the metric

$$g_{nn'} = \delta_{nn'} \eta_{P_n}$$

We will take as given that $\vec{A}_{\mu\nu}$, $\vec{B}_{\mu}$, $\vec{C}_{\mu}$, and $\vec{D}$ satisfy the duality spin constraints on the forward reactions

$$\begin{array}{ll} aa \rightarrow aa & aa \rightarrow ba \\ aa \rightarrow ab & aa \rightarrow bb \end{array} \quad (3.2)$$

Denote by $\vec{X}_{\mu\nu}^{(\bar{\eta}_{P_1}, \bar{\eta}_{P_2})}$ the vector of couplings of the meson resonances to two external mesons with natural parities $\bar{\eta}_{P_1}$ and $\bar{\eta}_{P_2}$ whose spins do not exceed L . The following choice for $\vec{X}_{\mu\nu}^{(\bar{\eta}_{P_1}, \bar{\eta}_{P_2})}$ satisfies all the spin constraints:

$$\begin{aligned}\vec{X}_{\mu\nu}^{(+,+)} &= \vec{A}_{\mu\nu} ; & \vec{X}_{\mu\nu}^{(+,-)} &= \vec{A}_{\mu\nu} \quad \text{if } \nu \neq 0 ; \\ \vec{X}_{\mu 0}^{(+,-)} &= \vec{B}_{\mu} ; & \vec{X}_{\mu\nu}^{(-,+)} &= \vec{A}_{\mu\nu} \quad \text{if } \mu \neq 0 ; & \vec{X}_{0\nu}^{(-,+)} &= \vec{C}_{\nu} \\ \vec{X}_{\mu\nu}^{(-,-)} &= \vec{A}_{\mu\nu} \quad \text{if } \mu \neq 0 \text{ or } \nu \neq 0 ; & \vec{X}_{00}^{(-,-)} &= \vec{D}\end{aligned}$$

To obtain a minimal mesonic spectrum consistent with duality we are thus left with the task of solving the spin constraints on the forward reactions (3.2) which read

$$\vec{A}_{\mu\nu} \cdot \vec{A}_{\lambda\sigma} = 0 \quad (\mu - \nu = \lambda - \sigma, \mu \neq \lambda) \quad (3.3)$$

$$\vec{A}_{\mu\nu} \cdot \vec{B}_\lambda = 0 \quad (\mu - \nu = \lambda) \quad (3.4)$$

$$\vec{A}_{\mu\nu} \cdot \vec{C}_\lambda = 0 \quad (\mu - \nu = -\lambda) \quad (3.5)$$

$$\vec{A}_{\mu\mu} \cdot \vec{D} = 0 \quad (3.6)$$

$$\vec{B}_\mu \cdot \vec{C}_{-\mu} = 0 \quad (3.7)$$

$$\vec{B}_0 \cdot \vec{D} = 0 \quad (3.8) \quad \text{and} \quad \vec{C}_0 \cdot \vec{D} = 0 \quad (3.9)$$

Before going into the details of the solution to those constraints it may be useful to recall the restrictions on the coupling vectors $\vec{A}_{\mu\nu}$, $\vec{B}_\mu$, $\vec{C}_\mu$, and $\vec{D}$ which follow from parity conservation and Bose statistics. For any resonance n which can couple to one of the considered channels, invariance under parity imposes that

$$\begin{aligned} A_{-\lambda-\mu}^n &= \bar{\eta}_n A_{\lambda\mu}^n ; & B_{-\mu}^n &= -\bar{\eta}_n B_\mu^n ; \\ C_{-\mu}^n &= -\bar{\eta}_n C_\mu^n ; & D^n &= \bar{\eta}_n D^n \end{aligned} \quad (3.10)$$

where $\bar{\eta}_n$ is the natural parity of the resonance n . Note in particular that $\vec{A}_{00}$ and $\vec{D}$ will be built up only from positive natural parity resonances while $\vec{B}_0$ and $\vec{C}_0$ will receive contributions exclusively from negative natural parity resonances. Bose statistics gives further constraints on the coupling vectors $\vec{A}_{\lambda\mu}$ and $\vec{D}$: $\vec{A}_{\lambda\mu}$ will be built up from $PC = +1$ resonances only while components of $\vec{D}$ which are symmetric (antisymmetric) under interchange of the spins of the external mesons will receive contributions from $PC = +1$ ($PC = -1$) resonances only. This implies in particular that

$$\begin{aligned} A_{\lambda\mu}^n &= A_{-\mu-\lambda}^n \quad \text{for } PC = +1 \text{ resonances,} \\ A_{\lambda\mu}^n &= -A_{-\mu-\lambda}^n \quad \text{for } PC = -1 \text{ resonances.} \end{aligned} \quad (3.11)$$

The constraints (3.3-3.9) break up into subsets, each of which is a complete set of mutual orthogonality conditions among a number of coupling vectors. The subsets are disjoint, in the sense that each coupling vector appears in exactly one subset, and so they can be solved independently. The construction of the solutions then simply involves repeated use of the fact that the smallest positive

norm vector space in which n non-zero vectors can be mutually orthogonal has n dimensions. Within each subset, each linearly independent coupling vector must be associated with a distinct meson. In the following paragraphs, we effect the break-up of the constraints into progressively finer subsets and, by counting the minimum number of coupling vectors needed to satisfy each such subset, we obtain progressively more information about the meson spectrum.

All constraints (3.3-3.9) are among coupling vectors with the same total helicity. For each total helicity n it is straightforward to count the number of orthogonality conditions, or equivalently, the smallest number of resonances with spin n or greater, necessary to solve the duality constraints.

In the case $n = 0$, for example, which corresponds to the main diagonal of the coupling matrix $\vec{A}_{\mu\nu}$ (i.e. $\vec{A}_{\lambda\lambda}$ with $\lambda = -L, \dots, 0, \dots, L$) and to the coupling vectors $\vec{B}_0, \vec{C}_0$ and D , one has a total number of $2L + 4$ coupling vectors which are mutually orthogonal and, by assumption, non zero. Hence one needs at least $2L + 4$ resonances to solve these duality constraints. Similarly for helicity $= 1$ there are $2L$ coupling vectors $\vec{A}_{L,L-1} \dots \vec{A}_{-L+1,-L}$, mutually orthogonal and orthogonal to $\vec{B}_1$ and $\vec{C}_1$, which are themselves orthogonal. Since only resonances with spin at least one can contribute to these coupling vectors, $2L + 2$ resonances with spin ≥ 1 are required. Proceeding along successive diagonals of $\vec{A}_{\lambda\mu}$ one eventually obtains the following minimal spectrum of 18-plets (nonets with both values of C) of mesons required by duality:

$$\begin{aligned}
 & 2L + 4 \text{ resonances with spin } \geq 0 \\
 & 2L + 2 \text{ resonances with spin } \geq 1 \\
 & 2L + 1 \text{ resonances with spin } \geq 2 \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
 & L + 3 \text{ resonances with spin } \geq L \\
 & L \text{ resonances with spin } \geq L + 1 \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
 & 1 \text{ resonance with spin } \geq 2L .
 \end{aligned}$$

Beyond this minimal number of resonances the duality constraints give us information on their natural parities and CP eigenvalues. Along the main diagonal ($n = 0$) it is possible to isolate the contributions from resonances with opposite natural parities: $\vec{D}$ and the $L + 1$ coupling vectors $\vec{N}_\mu = \vec{A}_{\mu\mu} + \vec{A}_{-\mu,-\mu}$ are mutually orthogonal, non-zero, and receive contributions only from positive natural parity resonances; similarly $\vec{B}_0$ and $\vec{C}_0$ as well as the L coupling vectors $\vec{U}_\mu = \vec{A}_{\mu\mu} - \vec{A}_{-\mu,-\mu}$ are mutually orthogonal, non-zero, and built up only from negative natural parity resonances. Hence among the $2L + 4$ resonances with spin ≥ 0 required by duality,

$L + 2$ must have positive natural parity and $L + 2$ negative mutual parity. The $L + 2$ resonances with positive natural parity have all $CP = +1$ since according to Eqs. (3.10-3.11) positive natural parity resonances with negative CP cannot contribute to the main diagonal of the coupling matrix $\vec{A}_{\lambda\mu}$. Similarly the L coupling vectors $\vec{U}_\mu$ receive contributions exclusively from states with negative CP.

Furthermore, $\vec{B}_0$ and $\vec{C}_0$ must receive contributions from resonances with both eigenvalues of CP. To see this, denote by $\vec{B}_0^{\pm}$ and $\vec{C}_0^{\pm}$ the resonant contributions to $\vec{B}_0$ and $\vec{C}_0$ and with $CP = \pm 1$. Equation (3.7), $\mu = 0$, can be written as

$$\vec{B}_0^+ \cdot \vec{C}_0^+ + \vec{B}_0^- \cdot \vec{C}_0^- = 0 \quad (3.12)$$

If one then selects a specific charge conjugation eigenchannel by proper selection of the charges of the external particles, for example $C = +1$, then the components of $\vec{C}_0^{\pm}$ correspond to resonances with opposite parities and hence $\vec{C}_0^+ = +\vec{B}_0^+$ and $\vec{C}_0^- = -\vec{B}_0^-$ (see Eq. 3.1) Eq. (3.12) and the assumption that $\vec{B}_0$ is not a null vector, prove then that both $|\vec{B}_0^+|$ and $|\vec{B}_0^-|$ must be different from zero. The same reasoning applies of course *mutatis mutandis*, to the $C = -1$ eigenchannel.

To summarize, among the $2L + 4$ resonances necessary to satisfy the duality constraints, $L + 2$ must have positive natural parity and positive CP while the $L + 2$ others must have negative natural parity, $L + 1$ of those having $CP = -1$ and only one $CP = +1$.

Along the secondary diagonals it is not possible to disentangle positive and negative natural parity contributions, but one has still some information about the CP values of the resonances. Let us take, for instance, the secondary diagonal corresponding to $\vec{A}_{\lambda\mu}$ with $\lambda - \mu = L$ and denote by $\vec{n}_{L0}^+$ and $\vec{n}_L^+$ the contributions from positive natural parity resonances to $\vec{A}_{L0}$ and $\vec{B}_L$ respectively (we assume they all have positive CP eigenvalues); by $\vec{u}_{L0}^-$ and $\vec{u}_L^-$ the contributions from negative natural parity and negative CP resonances to $\vec{A}_{L0}$ and $\vec{B}_L$, and by $\vec{u}_{L0}^+$ and $\vec{u}_L^+$ the contributions from negative natural parity resonances with $CP = +1$ to the same $\vec{A}_{L0}$ and $\vec{B}_L$ coupling vectors. With this notation, the duality constraint $\vec{B}_L \cdot \vec{C}_{-L} = 0$ is then written symbolically as

$$\vec{n}_L \cdot \vec{n}_L (-)^{J_n} - \vec{u}_L^+ \cdot \vec{u}_L^+ (-)^{J_u} - \vec{u}_L^- \cdot \vec{u}_L^- (-)^{J_u} = 0 \quad (3.13)$$

where, e.g., $\vec{n}_L \cdot \vec{n}_L (-)^{J_n}$ stands for $\sum_i \vec{n}_L^i \cdot \vec{n}_L^i (-)^{J_n^i}$. From Eq. (3.13) it follows that if $\vec{u}_L^-$ were zero it would be impossible to solve the duality constraint $\vec{B}_L \cdot \vec{C}_{-L} = 0$ since for each charge conjugation eigenamplitude the remaining contributions to Eq. (3.13) have the same sign.

On the other hand from the orthogonality conditions $\vec{A}_{L0} \cdot \vec{B}_L = \vec{A}_{L0} \cdot \vec{C}_{-L} = 0$ and $\vec{A}_{0L} \cdot \vec{B}_L = \vec{A}_{0L} \cdot \vec{C}_L = 0$ and Eqs. (3.10) one derives the constraints

$$\vec{u}_{L0}^- \cdot \vec{u}_L^- = 0 \quad \text{and} \quad \vec{u}_{L0}^- \cdot \vec{u}_L^- (-)^J = 0$$

however, by the previous agreement $|\vec{u}_{L0}^-| \neq 0$; hence since u_L^- cannot be a null vector it must have at least two components. Thus either there exist positive natural parity resonances with negative CP or at least two negative natural parity $CP = -1$ resonances, with spin at least L .

In summary, a minimal spectrum of resonances needed to satisfy the duality spin constraints in the scattering of mesons with spin up to L from one another is:

- $L+1$ $PC = +1$, $P(-)^J = +1$, odd spin nonets with spins $2L+1, 2L-1, \dots, 1$.
- $L+1$ $PC = +1$, $P(-)^J = +1$, even spin nonets with spins $2L, 2L-2, \dots, 0$.
- L $PC = -1$, $P(-)^J = -1$, odd spin nonets with spins $2L-1, 2L-3, \dots, 1$.
- L $PC = -1$, $P(-)^J = -1$, even spin nonets with spins $2L-2, 2L-4, \dots, 0$.
- 1 $PC = +1$, $P(-)^J = -1$, odd spin nonet with spin 1 .
- 1 $PC = +1$, $P(-)^J = -1$, even spin nonet with spin 0 .

except for the fact that for low L some of the $PC = -1$, $P(-1)^J = -1$ nonets must have higher spins than indicated, so that at least two even spin and two odd spin nonets have spin at least L . For L above three the above summary suffices. This spectrum is minimal in the sense that one can add any additional resonances, or increase the spins of any of the above resonances by any integer, and one will still have a solution to the constraints, but if any mesons were removed (even while increasing the spins of others) or if some of their spins were decreased, it would no longer be able to non-trivially satisfy the constraints. However, to some extent one could decrease the spins of some resonances while increasing the spins of others and still have a solution.

Both as a mnemonic and because it accords well with the ideas of duality, it is instructive to see how our predicted resonances fit on rising Regge trajectories. It should be stressed though that the duality equations to which we restricted ourselves in this paper do not give any information on the masses of the states which have to exist. Hence the degeneracy in masses that a literal interpretation of degenerate trajectories would imply should not be taken seriously: one does expect masses to be shifted around by as much as $1-2 \text{ GeV}^2$ which is approximately the size of the region where local duality might be expected to be valid.

A) Resonances with positive natural parity and $CP = +1$

Suppose one selects the $C = +1$ eigenchannel; starting with the scattering of 0^- mesons off $L = 1^-$ mesons, our minimal spectrum requires three such resonances one of which has spin 2. Since we do not expect the leading vector-tensor trajectory to be doubled the minimal solution to the duality constraints would thus consist of 1 nonet of resonances with spin 2^{++} and two 0^{++} nonets. Scattering of a 2^{++} nonet would however require in the $C = 1^{+-}$ eigenchannel the existence of one nonet with spin ≥ 4 , one nonet with spin ≥ 2 and two nonets with spin ≥ 0 . Consistency of this $L = 2$ solution with the $L = 1$ solution and rising Regge trajectories requires that one replaces one of the spin 0^{++} nonets with a spin 2^{++} nonet. Repeating this procedure for successive L 's gives us for the positive natural parity $C = +1$, $P = +1$ nonets a trajectory pattern consisting of a leading trajectory with an infinite set of subsidiary trajectories spaced by even integers, the first of which is doubled. Clearly a solution for positive natural parity nonets with $C = -1$ and hence $P = -1$ will consist of the set of trajectories exchange degenerate with the preceding ones.

B) Resonances with negative natural parity and $CP = +1$

Similarly for negative natural parity states with $PC = +$ one has found that for each scattering of a 0^- nonet off a $L(-)^L$ nonet at least one nonet of such states is required. By demanding that successive nonets lie on a rising Regge trajectory, one obtains a single exchange degenerate trajectory of such nonets of resonances.

C) Resonances with negative natural parity and $CP = -1$

Finally, for negative natural parity resonances with $CP = -1$ duality requires in each C eigenchannel in the scattering of 0^- mesons off spin L , natural parity $+$ mesons, at least $L + 1$ such resonances with spin ≥ 0 , with two of these having spin $\geq L$. From the scattering of 0^- mesons and 1^- mesons one needs thus, for each C eigenchannel, two resonances with spins ≥ 1 . The simplest choice is of course a doubled L^{+-} nonet and a doubled 2^{-+} nonet. From the scattering of 0^- mesons and 2^+ mesons duality imposes the existence, for each eigenchannel of C of 3 negative natural parity, $CP = -1$ resonances two of which at least with spins ≥ 2 . With linearly rising trajectories one expects for negative C , two 3^{+-} nonets to which one should add a 1^{+-} nonet. Extending this reasoning to higher and higher values of L one obtains eventually a doubled exchange degenerate nonet of trajectories with an infinite set of non-degenerate subsidiaries spaced by even integers.

4. OUTLOOK

A realistic treatment of duality for reactions involving spinning particles, involves imposing, beyond the kinematical constraints imposed here, the dynamical analytic constraint that at asymptotic energies amplitudes built from narrow resonances have Regge asymptotic behaviour. In this procedure there are, however, further kinematical constraints of which one should be aware.

One set of these constraints arises because an amplitude dominated by an $M = 0$ exchange vanishes at asymptotic energies like $t^{|\Delta\lambda|+|\Delta\mu|}$ where $\Delta\lambda$ and $\Delta\mu$ are the helicity flips at each vertex, while such an amplitude is required to vanish only as fast as $t^{|\Delta\lambda+\Delta\mu|}$ by angular momentum conservation. This requirement can be formulated as a series of algebraic constraints on multiple derivatives with respect to t of the imaginary parts of appropriate helicity amplitudes at $t = 0$. Derivatives of amplitudes with too much helicity flip vanish at $t = 0$. Note also that the leading trajectory decouples, at all t , from zero helicity vertices involving a change of natural parity.

There are additional constraints at the special values of t corresponding to mass squares of the resonances lying on the leading exchange degenerate trajectory. The constraints are that helicity amplitudes with more helicity flip than the spin of the resonance must vanish at $t = m_{\text{resonance}}^2$.

A further problem which will have to be faced in a realistic treatment of duality is the famous difficulty associated with baryon-antibaryon scattering. As is well known, there are no purely non exotic solutions to the duality constraints. Presumably this will require the introduction of both exotic meson and baryon particles and trajectories in order to construct a successful dual model of all scattering amplitudes.

REFERENCES

- 1) K. Igi, Phys. Rev. Letters 9, 76 (1962)
A. Logunov, L.D. Soloviev and A.A. Tavkhelidze, Phys. Letters 24B 181 (1967).
K. Igi and S. Matsuda, Phys. Rev. Letters 18, 625 (1967).
R. Dolen, D. Horn and C. Schmid, Phys. Rev. 166, 1768 (1968).
For a review, see e.g. M. Jacob, Lectures given at the Brandeis Summer
Institute in Theoretical Physics (1970).
- 2) P.G.O. Freund, Phys. Rev. Letters 20, 235 (1968).
H. Harari, Phys. Rev. Letters 20, 1395 (1968).
- 3) See, e.g. the contributions of G. Veneziano and M. Virasoro in Proceedings
of the Tel Aviv Conference on Duality and Symmetry (1971).
- 4) See e.g. R.H. Capps, Phys. Rev. Letters 22, 215 (1969).
V. Barger and C. Michael, Phys. Rev. 186, 1592 (1969).
J. Mandula, C. Rebbi, R. Slansky, J. Weyers and G. Zweig, Phys. Rev.
Letters 22, 1147 (1969).
J. Mandula, J. Weyers and G. Zweig, Annual Review of Nuclear Science
Vol. 20, 289 (1970).
J. Rosner to be published.
- 5) J. Mandula, J. Weyers and G. Zweig, Phys. Rev. Letters 23, 627 (1969).
See also, e.g. R.H. Capps and V.V. Dixit, Phys. Letters 33 B, 596 (1970).
M.J. King and K.C. Wali, Phys. Rev. D3, 1602 (1972).
M.J. King and P.M. Shah and K.C. Wali, Phys. Rev. D6, 229 (1972).
- 6) C. Rebbi and R. Slansky, Rev. Mod. Phys. 42, 68 (1970).
- 7) J.L. Rosner Phys. Rev. Letters 21, 950 (1968).
- 8) J. Mandula, J. Weyers and G. Zweig, Phys. Rev. Letters 23, 266 (1969).
- 9) G. Zweig, CERN preprint TH 412, unpublished.