CM-P00049203

Archives

Ref.TH.1731-CERN

LOW ENERGY $\pi\pi$ SCATTERING PARAMETERS *)

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A B S T R A C T

We use the Froissart-Gribov representation and derivatives thereof, to calculate scattering lengths and effective range parameters for the lowest (up to and including $\ell = 4$) $\pi\pi$ partial waves. The results are discussed.

*) Research partially supported by G.I.F.T.

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INTRODUCTION

In this note we will use consistently the Froissart-Gribov representation to calculate low energy scattering parameters for the $\pi\pi$ system. The results are then compared with what one gets from conventional calculations, and some conclusions are drawn ; in particular, we get that our results give support to the factorization hypothesis for the ρ trajectory and favour values of the S wave, zero isospin scattering length in the interval (0.13, 0.17). A striking result is that the $\ell=2, I=2$ wave varies quicker than generally expected.

We shall write

$$F^I(s, t) = \sum_{\ell=0}^{\infty} (2\ell+1) f_{\ell}^I(s) P_{\ell}(\cos \theta_s) \quad (1)$$

for the scattering amplitude with isospin I in s channel. The partial waves are normalized to

$$f_{\ell}^I(s) = \sqrt{\frac{s}{s-4}} \sin \delta_{\ell}^I(s) e^{i \delta_{\ell}^I(s)}; \quad (2)$$

we take units with $m_{\pi}^2 = 1$. The Froissart-Gribov representation then reads

$$f_{\ell}^I(s) = \frac{4}{\pi(s-4)} \sum_{I'} M_{II'} \int_4^{\infty} dt F_t^{I'}(t, s) Q_{\ell}\left(\frac{s-4+2t}{s-4}\right); \quad (3)$$

here we assume that I, ℓ are connected by $(-)^I = (-)^{\ell}$. M is the s $\rightleftharpoons$ t crossing matrix,

$$M = \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix}, \quad (4)$$

and F_t the absorptive part of F in t channel. We shall be interested in the quantities

$$\left. \begin{aligned} a_\ell^I &= \lim_{s \rightarrow 4} h_\ell^I(s) \\ b_\ell^I &= \lim_{s \rightarrow 4} (d/ds) h_\ell^I(s) \\ c_\ell^I &= \lim_{s \rightarrow 4} (d^2/ds^2) h_\ell^I(s), \end{aligned} \right\} \quad (5a)$$

where $h_\ell^I(s)$ is defined as follows :

$$h_\ell^I(s) = f_\ell^I(s) / (s/4 - 1)^\ell. \quad (5b)$$

a_ℓ^I are the scattering lengths, b, c related to the effective range parameters. In general we can prove that the quantities

$$\alpha_\ell^{I(n)} = \lim_{s \rightarrow 4} (d^n/ds^n) h_\ell^I(s)$$

are finite when $\ell \geq n$. Furthermore, for $\ell \geq 1$ ^{*}, the α can be obtained by differentiating the Froissart-Gribov representation, (3), and taking then the limit as $t \rightarrow 4$. (For the rigorous proof of these last statements, see the Appendix.) Such calculation is best performed by using the power expansion for Q_ℓ ,

$$Q_\ell(z) = \sum_{\nu=0}^{\infty} z^{-\ell-2\nu-1} \frac{\Gamma(3/2) \Gamma(\ell+1) \Gamma(\ell/2+\nu+1) \Gamma(\ell/2+\nu+1/2)}{2^\ell \Gamma(\ell/2+1) \Gamma(\ell/2+1/2) \Gamma(\ell+\nu+3/2) \Gamma(\nu+1)}. \quad (6)$$

^{*}) The case $\ell = 1$ necessitates, however, some assumption ; essentially that the charge-exchange cross-section vanishes at infinity.

The result is, for a, b, c

$$a_l^I = \frac{4^{l+1}}{\pi} \gamma_{l_0} \sum_{I'} M_{II'} \int_4^\infty dt F_t^{I'}(t, 4) / (2t)^{l+1}, \quad (7a)$$

$$b_l^I = \frac{4^{l+1}}{\pi} \sum_{I'} M_{II'} \int_4^\infty dt \left\{ \left[\gamma_{l_0} / (2t)^{l+1} \right] \frac{d}{ds} F_t^{I'}(t, s) \right|_{s=4} + \left[\gamma_{l_1} / (2t)^{l+2} \right] F_t^{I'}(t, 4) \right\}, \quad (7b)$$

$$c_l^I = \frac{4^{l+1}}{\pi} \sum_{I'} M_{II'} \int_4^\infty dt \left\{ \left[\gamma_{l_0} / (2t)^{l+1} \right] \frac{d^2}{ds^2} F_t^{I'}(t, s) \right|_{s=4} + \left[2 \gamma_{l_1} / (2t)^{l+2} \right] \frac{d}{ds} F_t^{I'}(t, s) \right|_{s=4} + \left[2 \gamma_{l_2} / (2t)^{l+3} \right] F_t^{I'}(t, 4) \right\}. \quad (7c)$$

Here

$$\gamma_{l_0} = \Gamma_{l_0}$$

$$\gamma_{l_1} = - (l+1) \Gamma_{l_0}$$

$$\gamma_{l_2} = \frac{(l+1)(l+2)}{2} \Gamma_{l_0} + \Gamma_{l_2}, \dots,$$

where

$$\Gamma_{l_v} \equiv \frac{\Gamma(3/2) \Gamma(l+1) \Gamma(l/2 + v + 1) \Gamma(l/2 + v + 1/2)}{2^l \Gamma(l/2 + 1) \Gamma(l/2 + 1/2) \Gamma(l + v + 3/2) \Gamma(v+1)}.$$

Equations (7) allow us to calculate s channel low energy parameters. The reasons why such calculation is not circular are, of course, crossing symmetry (we calculate s channel quantities in terms of the t channel ones) and unitarity : for example, at low energy $\text{Im } f_{\ell}^I(t)$ is proportional to the square of the corresponding scattering length, $[a_{\ell}^I]^2$. For the sake of completeness we have, besides Eqs. (7), calculated Olsson's sum rule,

$$a_0^0 - \frac{5}{2} a_0^2 = \frac{12}{\pi} \int_4^{\infty} dt F_t^{(I_s=1)}(t, 0) / t(t-4). \quad (8)$$

1. - CALCULATION OF EQUATIONS (7), (8)

We assume that the contribution to F_s and its derivatives appearing in Eqs. (7), (8) of waves higher than F is negligible. This is inferred from the fact that, within all reasonable estimates, the contribution of F and G waves is indeed negligible. The range of integration, 4 to ∞ , is split in four parts.

(i) - Very low energy region ($t < 14$)

We parametrize

$$f_0^0(t) = \frac{1}{-i [(t-4)/t]^{1/2} + \psi_0^0(t)}, \quad (9a)$$

$$\psi_0^0(t) = \alpha_0 + \alpha_1 \frac{t-4}{t-t_z} \quad (9b)$$

α_0 is related to the scattering length, t_z to the location of the Adler zero. Finally α_1 is fixed by requiring smooth junction with higher energies. All other waves are given by the same parametrization as in region (ii) below.

(ii) - Low energy region ($14 < t < 62$)

For the $I = 0$ S wave, we use the parametrization of Ref. 1), for the $I = 2$ S wave, the one of Ref. 2). The P , D ($I=0$) and F waves are parametrized by a relativistic Breit-Wigner formula ³⁾

$$f_{\ell}^I(t) = \sqrt{\frac{t}{t-4}} \cdot \frac{m \Gamma K(t)}{m^2 - t - i m \Gamma K(t)}, \quad (10a)$$

where m is the mass of the corresponding resonance, Γ its width, and K is a kinematical factor

$$K(t) = \left(\frac{t-4}{m^2-4} \right)^{\ell+1/2} \left(\frac{m^2+t_0}{t+t_0} \right)^{\ell+1/2}. \quad (10b)$$

The value of t_0 is related to the corresponding scattering length by

$$a_{\ell} = \frac{2^{\ell+1} m \Gamma}{(m^2-4)^{\ell+3/2}} \left(\frac{m^2+t_0}{4+t_0} \right)^{\ell+1/2}. \quad (10c)$$

The $I = 2$ D wave is found to contribute a negligible amount.

(iii) - Medium energy region ($62 < t < 115$)

All waves retain the parametrizations of region (ii) except the $I = 0$ S wave. As this wave is not well known in the present range, we have allowed it to vary between the unitarity limits, $0 \leq \text{Im } f_0^0(t) \leq \sqrt{(t/t-4)}$. This is seen to amount to an error of $\sim 4\%$, for the quantity a_1^1 and for the Olsson's sum rule, and totally negligible for all other quantities.

(iv) - High energy region ($t > 115$)

This region corresponds to a lab. energy of $E_{\text{lab}} > 8$ GeV, and therefore we will use a Regge formula. Defining

$$\sum_{I'} M_{II'} F_t^{I'}(t, s) = \text{Im } T^I(s, t),$$

we set ⁴⁾

$$\left. \begin{aligned} \text{Im } T^0(t, s) &= \gamma(s) t^{\alpha_2(s)} \\ \alpha_2(s) &= 1 + (0.0057)s, \gamma(s) = (0.05) \exp[(0.016)s], \end{aligned} \right\} \quad (11a)$$

and 3)

$$\begin{aligned} \text{Im } T^1(t, s) &= (0.53) [\sin \pi \alpha_p(s)] \left(\frac{t}{s_0}\right)^{\alpha_p(s)} \Gamma(1 - \alpha_p(s)), \\ \alpha_p(s) &= 0.5 + (0.017)s. \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{Im } T^1(t, s) &= (0.53) [\sin \pi \alpha_p(s)] \left(\frac{t}{s_0}\right)^{\alpha_p(s)} \Gamma(1 - \alpha_p(s)), \\ \alpha_p(s) &= 0.5 + (0.017)s. \end{aligned}} \right\} \quad (11b)$$

The values of the parameters are those implied by factorization. We also take

$$\text{Im } T^2(t, s) = 0. \quad (11c)$$

A few comments on the parametrizations used are in order. Firstly the precise form of the parametrizations is essentially irrelevant (provided of course they fit the data), as they enter into non-singular integrals. Secondly, while some of the parameters are well established, others are not so. In particular, we have allowed for variations in the t_0 parameters of (10) while fixing the others at $m_\rho = 5.7$, $\Gamma_\rho = 1.1$; $m_{f_0} = 9.4$, $\Gamma_{f_0} = 1.2$; $m_g = 12.45$, $\Gamma_g = 1.2$. However, t_0 have been centered around $t_{p_0} = 20$, $t_{D_0} = 30$, $t_{F_0} = 45$.

Thirdly, the most troublesome waves (both S waves) have been allowed to vary in region (i). In region (ii) they are fairly well known, in region (iii) their values are irrelevant, in region (iv) they are just part of the Regge formula. Of the two free parameters (α_0 and t_z) for $I = 0$, and a_0^2 for $I = 2$, the results are rather insensible to the variation of t_z in the interval $[0, 2]$. Therefore we have fixed $t_z = 1$. The other two are then allowed to vary.

2. - RESULTS

We begin with a few words on the ranges of variation allowed for all variable parameters in (9), (10). We allow values of the scattering lengths as follows

$$\left. \begin{aligned} a_0^0 &\in [0.10, 0.60], \quad a_0^2 \in [-0.10, -0.03] \\ a_1^1 &\in [0.032, 0.040] \\ a_2^0 &\in [1.4 \times 10^{-3}, 1.8 \times 10^{-3}], \quad a_2^2 \in [-2 \times 10^{-4}, 3 \times 10^{-4}]. \end{aligned} \right\} \quad (12)$$

These values are reasonable extrema consistent with present evidence [see, e.g., Refs. 1), 5)]. Another point to be noted is that direct channel low energy parameters can be calculated directly if we assume Breit-Wigner parametrizations. Although, clearly, one has to extrapolate such parametrizations a long way from the resonance region (ρ, f_0) to threshold, the result of such calculations may serve as a check of our results. The values of $a_1^1, b_1^1, a_2^0, b_2^0, c_2^0$ (the parameters for which one would expect more reliable results) from this direct channel calculation are :

$$\left. \begin{aligned} a_1^1 &= 0.036 \pm 0.004 \\ b_1^1 &= (1.10 \pm 0.05) \times 10^{-3} \\ a_2^0 &= (1.6 \pm 0.2) \times 10^{-3} \\ b_2^0 &= (-0.65 \pm 0.32) \times 10^{-4} \\ c_2^0 &= (0.8 \pm 0.5) \times 10^{-5} \end{aligned} \right\} \quad (13)$$

The results are summarized in Table I. We have plotted them against variations in a_0^0 for it is the only parameter to which the sensitivity of the results is higher than 10%. We can draw the following conclusions.

(i) - The results of the Olsson sum rule and those for a_1^1 are compatible with Ref. 5) as well as with the direct channel calculation of a_1^1 [Eq. (13)]. As these results depend only slightly on the very low energy region, they show the correctness of the parametrizations of $\text{Im } T^{(1)}(t, s)$ for $t > 14$.

(ii) - the value for b_1^1 is compatible with (13) only if $a_0^0 \in [0.10, 0.17]$.

- (iii) - The results for b_2^0 are compatible with (13) only if $a_0^0 \in [0.13, 0.38]$; if we require also compatibility for c_2^0 , this forces $a_0^0 < 0.17$.
- (iv) - An unexpected result is that b_2^2 and c_2^2 are of the same order of magnitude as b_2^0 and c_2^0 , respectively. This seems to imply that although small compared to f_2^0 , the f_2^2 wave presents as much structure. We cannot think of a physical explanation for this fact.
- (v) - a_4^0 , a_4^2 and, to a lesser extent, a_3^1 , depend very strongly on a_0^0 . Therefore, even a rough direct measurement of these quantities would give a precise determination for a_0^0 .
- (vi) - All other parameters are in agreement with the determination of Refs. 1) and 5).

We end this paper with four comments. First of all, it is clear that the consistency requirements used to get our results have to be taken cum granum salis. The point is, although the Froissart-Gribov calculations are fairly stable (the reason being the non-singular character of the equations), it is clear that the Breit-Wigner extrapolations are much less reliable. It is with this proviso in mind that all our comparisons have to be regarded. Second, if we take seriously the consistency requirements between direct and crossed channel calculations [which is supported by conclusion (i)], they give three compatible constraints on a_0^0 , fixing it in the range $[0.13, 0.17]$. This in turn fixes all other parameters within small error bands ; the results are summarized in Table II. Third, in the same spirit, the agreement of the calculations for a_1^1 , b_1^1 and Olsson's sum rule may be interpreted as confirmation of the factorization hypothesis. Lastly, the self-consistent values of b_2 , c_2 , a_4 , etc., may be used to derive bounds on total cross-sections, along the lines of Ref. 6). This work is currently in progress.

ACKNOWLEDGEMENTS

The authors are grateful to Dr. C. López for several useful discussions and to G.I.F.T. (Spain) for financial support. One of us (F.J.Y.) is grateful to CERN for the hospitality extended to him at the Theoretical Study Division, where this work was completed.

A)	a_0^0	0.10	0.13	0.17	0.22
	a_1^1	0.0336	0.0339	0.0343	0.0348
	b_1^1	0.113×10^{-2}	0.109×10^{-2}	0.104×10^{-2}	0.97×10^{-3}
	a_2^0	0.143×10^{-2}	0.146×10^{-2}	0.150×10^{-2}	0.156×10^{-2}
	b_2^0	-0.27×10^{-4}	-0.34×10^{-4}	-0.43×10^{-4}	-0.56×10^{-4}
	c_2^0	0.71×10^{-5}	0.95×10^{-5}	0.131×10^{-4}	0.178×10^{-4}
	a_2^2	-0.87×10^{-4}	-0.53×10^{-4}	-0.9×10^{-5}	0.45×10^{-4}
	b_2^2	-0.33×10^{-4}	-0.40×10^{-4}	-0.49×10^{-4}	-0.62×10^{-4}
	c_2^2	0.85×10^{-5}	0.109×10^{-4}	0.145×10^{-4}	0.192×10^{-4}
	a_3^1	0.32×10^{-4}	0.36×10^{-4}	0.41×10^{-4}	0.48×10^{-4}
	a_4^0	0.38×10^{-5}	0.44×10^{-5}	0.52×10^{-5}	0.62×10^{-5}
	a_4^2	0.75×10^{-6}	0.13×10^{-5}	0.21×10^{-5}	0.31×10^{-5}
	$2a_0^0 - 5a_0^2$	0.59	0.61	0.63	0.66
B)	a_0^0	0.30	0.38	0.48	0.60
	a_1^1	0.0355	0.0361	0.0368	0.0376
	b_1^1	0.87×10^{-3}	0.77×10^{-3}	0.65×10^{-3}	0.51×10^{-3}
	a_2^0	0.164×10^{-2}	0.172×10^{-2}	0.181×10^{-2}	0.192×10^{-2}
	b_2^0	-0.76×10^{-4}	-0.96×10^{-4}	-0.122×10^{-3}	-0.151×10^{-3}
	c_2^0	0.26×10^{-4}	0.34×10^{-4}	0.45×10^{-4}	0.58×10^{-4}
	a_2^2	0.127×10^{-3}	0.21×10^{-3}	0.30×10^{-3}	0.41×10^{-3}
	b_2^2	-0.82×10^{-4}	-0.102×10^{-3}	-0.128×10^{-3}	-0.157×10^{-3}
	c_2^2	0.27×10^{-4}	0.36×10^{-4}	0.46×10^{-4}	0.59×10^{-4}
	a_3^1	0.60×10^{-4}	0.72×10^{-4}	0.86×10^{-4}	0.103×10^{-3}
	a_4^0	0.80×10^{-5}	0.99×10^{-5}	0.122×10^{-4}	0.151×10^{-4}
	a_4^2	0.49×10^{-5}	0.68×10^{-5}	0.91×10^{-5}	0.120×10^{-4}
	$2a_0^0 - 5a_0^2$	0.71	0.77	0.84	0.94

$$a_0^0 = 0.15 \pm 0.02$$

$$a_0^2 = -0.065 \pm 0.025$$

$$a_1^1 = 0.0341 \pm 0.0036$$

$$b_1^1 = (1.07 \pm 0.27) \times 10^{-3}$$

$$a_2^0 = (1.48 \pm 0.08) \times 10^{-3}$$

$$a_2^2 = (-3 \pm 8) \times 10^{-5}$$

$$b_2^0 = (-3.8 \pm 1.1) \times 10^{-5}$$

$$b_2^2 = (-4.4 \pm 1.1) \times 10^{-5}$$

$$c_2^0 = (1.13 \pm 0.36) \times 10^{-5}$$

$$c_2^2 = (1.27 \pm 0.36) \times 10^{-5}$$

$$a_3^1 = (3.8 \pm 0.5) \times 10^{-5}$$

$$a_4^0 = (4.8 \pm 0.8) \times 10^{-5}$$

$$a_4^2 = (1.7 \pm 0.8) \times 10^{-5}$$

- TABLE II -

APPENDIX

We prove here that, subject to the assumption that continuation across the elastic cut is possible for partial waves, one can differentiate the Froissart-Gribov representation (for definiteness, we consider $l = \text{even} \geq 2$),

$$f_l(s)/(s-4)^l = \frac{4}{\pi(s-4)^{l+1}} \int_4^\infty dt \frac{F(t,s)}{t} Q_l\left(\frac{s-4+2t}{s-4}\right), \quad s < 4, \quad (\text{A.1})$$

with respect to s l times inside the integration sign, even in the limit $s = 4$. For this, we use elastic unitarity to write

$$f_l(s)/(s-4)^l = \frac{a}{(s-4)^{l+1/2} + \varphi_l(s)} \quad (\text{A.2})$$

where a is a constant and φ_l is analytic at $s = 4$. Therefore, (A.2) tells us that

$$\frac{d^n}{ds^n} \left[f_l(s)/(s-4)^l \right]$$

is finite, as $s \rightarrow 4$, for $n \leq l$.

The representation (6) for $Q_l(z)$ is absolutely convergent for all $z = 1 + 2t/(s-4)$, $t \leq 4$, $4 \leq s < \infty$. Therefore, we can substitute it into (A.1). The theorem we want to prove is then a direct consequence of the following lemma.

Lemma

Let

$$\xi_N(s) = \lambda_N(s) \int_4^\infty dt \Phi_t(s,t) (s-4+2t)^{-N}, \quad (\text{A.3a})$$

$$\Phi_t \geq 0.$$

Then if $\xi_N(s) \rightarrow \xi_N(4) = \text{finite}$, $\lambda_N(s) \rightarrow \lambda_N(4) = \text{finite}$,

as $s \rightarrow 4$, $\mathcal{J}(4, t)$ exists and one has

$$\xi_N(4) = \lambda_N(4) \int_4^{\infty} dt \Phi_t(4, t) (2t)^{-N} \quad (\text{A.3b})$$

Proof

As shown in Ref. 7), a necessary and sufficient condition to have (A.3) is that the $\xi(s)$ satisfy the inequalities

$$\sum_{j=0}^m \binom{m}{j} (-1)^j (s+4)^{j-m} \xi_{k+j}(s) / \lambda_{k+j}(s) \geq 0; \quad (\text{A.4})$$

$m = 0, 1, \dots; \quad k = 0, 1, \dots$

Because of the finiteness of ξ, λ at $s = 4$, we can take the limit as $s \rightarrow 4$ of (A.4); the inequalities will be still satisfied, so the $\xi(4)$ will verify the representation (A.3b). That the kernel is really $\mathcal{J}(4, t)$ follows from the uniqueness of the representation ⁷⁾.

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