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RECENT PROGRESS OF THE COMPUTATIONAL METHODS IN FIELD THEORY

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A B S T R A C T

Remarkable improvements on the convergence of the Padé approach in strong coupling field theory are achieved by working with the Green functions instead of the S matrix. Two methods are considered : the off-shell variational and the matricial ones. For the latter case applications to the pion-nucleon and to the nucleon-nucleon elastic amplitudes are sketched.

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The search of an adequate approximation method is important in theoretical physics, if people have to compare complicated formulas, derived from general physical theories, with reality. It becomes a necessity whenever the exact mathematical solution is missing and the only information available is fragmentary and consists, for instance, of a formal series expansion in a suitable parameter. This situation exists in field theory and the Padé approximants (P.A.) have been proposed long ago ¹⁾ as a convenient mathematical tool one can use to derive quantitative results from perturbation theory, particularly in the case of strong coupling. More specifically we are thinking of those realistic field theoretical models which are renormalizable and allow a power series in the renormalized coupling constant.

In this note we are not going to review the properties of the P.A. and their physical applications since this was the object of specialized conferences and is summarized in a very approachable collection of review papers and books ²⁾. We rather want to focus the attention of the reader on some peculiarities that make the Padé method valid along the present line of research in field theory.

We recall the definition of P.A. : given a power series

$$\sum_n a_n z^n$$

the approximant $|N/M|(z)$ is defined as the ratio of two polynomials of degrees N and M such that

$$\sum_n a_n z^n - P_N(z)/Q_M(z) = O(z^{N+M+1})$$

The series may be formal, with zero radius of convergence and the set of P.A. $|N/M|(z)$ is shown to be unique by the previous definition.

The fundamental problem is the existence of sequences (in the Padé table) converging towards a unique analytic function $f(z)$. This question is dealt quite extensively in the literature ³⁾. We remind that under general conditions diagonal sequences are assumed (and in some cases proved) to converge to $f(z)$, when this has an asymptotic expansion, and also to provide the analytic continuation if $f(z)$ is regular at the origin (consequently its Taylor series converges). The procedure has been extended to treat the case of functions which are matrices or operators in a Banach space.

How can a field theorist use the P.A. approach and when ? In principle any time he deals with a function that is available as a power series in one parameter. In practice the expansion parameter is the coupling constant and the use of P.A. is only limited by the difficulties of computing high orders. In the case of trilinear local field interaction only the second order (one loop approximation) can be treated by semi-analytic methods which allow unlimited numerical accuracy. For scalar particles pure quadrilinear couplings are admitted and higher orders can be computed ; the introduction of fermions requires trilinear interactions, for renormalizable theories, and limit to the $|1/1|$ the only significant P.A. Since it is extremely hard to push further the perturbative expansion, a significant improvement can be achieved by working in a wider space, where one deals with the Green functions rather than the S matrix.

In the Green functions we may distinguish the discrete spin variables from the continuous momenta. If we fix the momenta to their on-shell values, we are left with a matrix, a submatrix of which is the physical amplitude. We call the P.A. of this matrix Green function approximants and notice that they reduce to the usual S matrix approximants in the case of scalar particles. On the other hand the momenta can also be varied and a stationary solution looked for ; we refer to this procedure as variational approximation.

For the variational approximation rigorous convergence theorems have been proved by Alabiso, Butera and Prosperi ⁴⁾, both in potential scattering and for the Bethe-Salpeter equation (ladder approximation for scalar particles). It is well known that the bound state poles of the Green function (off-shell T matrix) are independent of the momentum variables for fixed angular momentum :

$$\langle k' | T(E, g, J) | k \rangle \quad \text{poles at} \quad g = g(E, J)$$

This property is lost when T is replaced by any of its P.A. ; hence one can try to impose the independence from these parameters k, k' , thus obtaining the best approximation in the search of bound states

$$|N/M| \langle k' | T(E, g, J) | k \rangle \quad \text{poles at} \quad g = g^{N,M}(E, J, k, k')$$

To this end one looks for the stationary value of g (the coupling) when varying k and k' , usually with the constraint $k=k'$

$$\delta g^{N,M} = 0 \quad K = K(E, J) \quad g_{\text{stat.}} = g^{N,M}(E, J)$$

The value of k at the stationary point is found to be very far from the mass shell. This variational approach was also applied to scattering states in potential theory and the Bethe-Salpeter equation. The extension to field theory has not yet been fully investigated.

The Green function approximation has been devised for field theoretical applications. Since the convergence of the P.A. critically depends on the dimensionality of the domain in which the function, that is going to be approximated, is defined, it becomes evident that an improvement has to be achieved, passing from an S matrix element to the whole Green function.

The basic steps required by this method are the choice of a basis, the angular momentum expansion and the reduction due to the symmetry properties. Finally one is left with a set of matrices P^J which fulfil the extended unitarity conditions:

$$P^J - P^{J+} = i \quad P^J \Pi P^{J+}$$

where Π is the projector on the physical states. The $|N/M|$ $M \geq N$ P.A. fulfil the same equation and their physical amplitudes are exactly unitary. The analyticity properties in the energy variable are the perturbative ones except for additional poles which are good candidates to represent the spectrum. The relevant technical points are here briefly exposed for the πN and the NN systems ^{5),6)}.

The πN Green function $T(p, q, p', q')$ is a 4×4 matrix which decomposes on the set of Dirac matrices. Parity invariance reduces the number of independent amplitudes from 16 to 8, time reversal from 8 to 6. A suitable choice of basis vectors is the following

$$|\epsilon, \lambda\rangle = \Gamma_\epsilon u(p, \lambda) \quad \Gamma_\epsilon = \begin{cases} 1 & \epsilon = +1 \\ \gamma_5 & \epsilon = -1 \end{cases}$$

where $\epsilon = \pm 1$ labels the parity of the state and $\lambda = \pm \frac{1}{2}$ the helicity, u being the usual Dirac spinor. On such a basis T splits into four 2×2 matrices,

$$\tilde{u}(p') \Gamma_{\epsilon} T \Gamma_{\epsilon} u(p)$$

which can be treated as physical amplitudes of fictitious Green functions $\Gamma_{\epsilon} T \Gamma_{\epsilon}$. The kinematics is developed following this picture and the angular momentum expansion obtained following Jacob and Wick. We may formally introduce an angular momentum basis for T by defining $|\epsilon, L; J\rangle = \Gamma_{\epsilon} |L; J\rangle$ where ϵ appears as an additional quantum number (the nucleon parity). The space reflection invariance of T implies the parity of the initial and final states to be the same

$$\langle \epsilon', L', J | T | \epsilon, L, J \rangle \neq 0 \quad \text{only if } \epsilon (-1)^L = \epsilon' (-1)^{L'}$$

We are left with two 2×2 matrices $P_{\pm}^J$ (connecting the states $|+, J \pm \frac{1}{2}\rangle$, $|-, J \pm \frac{1}{2}\rangle$) which are symmetric if time reversal invariance holds and fulfils the extended unitarity equation.

For the NN case T is a 16×16 matrix for which beyond parity, we must account for the exchange symmetry due to the presence of identical particles. The basis vectors are in this case

$$|\alpha\rangle = \Gamma_{\alpha} |1, 2\rangle \quad \alpha = +, -, e, o \quad |1, 2\rangle = u(p_1) \otimes u(p_2)$$

where $\Gamma_+ = I \otimes I \quad \Gamma_- = \gamma_5 \otimes \gamma_5$

$$\Gamma_e = \frac{1}{\sqrt{2}} (I \otimes \gamma_5 + \gamma_5 \otimes I) \quad \Gamma_o = \frac{1}{\sqrt{2}} (I \otimes \gamma_5 - \gamma_5 \otimes I)$$

and again T splits into sixteen 4×4 matrices for which relevant kinematics is worked out. The exchange operator E (defined by $E|1, 2\rangle = |2, 1\rangle$) commutes with T and furthermore $TE = ET = -T$ if the Pauli principle has to hold for the physical amplitude. Let us notice that if P is the parity operator we have

$$E \Gamma_\alpha E^{-1} = \omega_\alpha \Gamma_\alpha$$

$$\omega_\alpha = \begin{cases} +1 & \alpha = +, -, e \\ -1 & \alpha = o \end{cases}$$

$$P \Gamma_\alpha P^{-1} = \pi_\alpha \Gamma_\alpha$$

$$\pi_\alpha = \begin{cases} +1 & \alpha = +, - \\ -1 & \alpha = e, o \end{cases}$$

The selection rules on the angular momentum states $|\alpha, L, S, J\rangle = \Gamma_\alpha |L, S, J\rangle$ (where S is the total spin and L the orbital momentum) issue from P and E invariance

$$P |\alpha, L, S, J\rangle = \pi_\alpha (-1)^L |\alpha, L, S, J\rangle$$

$$E |\alpha, L, S, J\rangle = \omega_\alpha (-1)^{L+S+I} |\alpha, L, S, J\rangle$$

where I is the isospin, a good quantum number like J .

The final result is that T splits into two 4×4 matrices and one 6×6 matrix having the singlet, the uncoupled triplet and the coupled triplet as physical amplitudes, respectively.

To conclude we notice that for πN we go from two amplitudes for S to six for T , while the ratio becomes 5 to 41 for NN . This increase of information should allow a better convergence, if any ; this has indeed been observed on the ladder series for NN , obtained by iterating the Bethe-Salpeter equation, where a sequence of diagonal P.A. was computed ⁷⁾.

One easily recognizes how the previous procedure can be extended to meson-baryon and baryon-baryon for the general mass case within the broken $SU(3)$ symmetry.