We examine interferometric measurements of the topological charge of (possibly non-Abelian) anyons. The anyons are placed in a Mach-Zehnder interferometer and their topological charge is determined from the effect it has on the interference of probe particles sent through the interferometer. We find that superpositions of distinct anyonic charges $a$ and $a'$ in the target decohere when the probe particles have nontrivial monodromy with the charges that may be fused with $a$ to give $a'$.

PACS numbers: 03.65.Ta, 03.65.Vf, 05.30.Pr, 03.67.Lx

Quantum physics in two spatial dimensions allows for the existence of particles which are neither bosons nor fermions. Instead, the exchange interactions of such “anyons” are described by representations of the braid group [1–3], which may even be non-Abelian [4, 5]. Recently, there has been a resurgence of interest in anyons, due to increased experimental capabilities in systems believed to harbor them, and also their potential application to topologically protected quantum computation [6–8]. In this quantum computing scheme, qubits are encoded in non-localized, topological charges carried by clusters of non-Abelian anyons. Topological charges decouple from local probes, affording them protection from decoherence, but also making their measurement difficult, typically requiring interferometry. The most promising candidate system for discovering non-Abelian statistics is the fractional quantum Hall (FQH) state observed at filling fraction $\nu = 5/2$ [9, 10], which is widely expected to be described by the Moore-Read state [11, 12]. Interference experiments, similar to that proposed [13] and only recently implemented [14, 15] for Abelian FQH states, may soon verify the braiding statistics of the $\nu = 5/2$ state [16–19]. The analyses in these treatments assume the target particle to be in an eigenstate of topological charge. We show that, when this is not the case, the density matrix of the target particle is diagonalized in the charge basis during the experiment if a simple criterion on the braiding of source and target particles is satisfied: superpositions of distinct anyonic charges $a$ and $a'$ decohere as long as the probe particles have nontrivial monodromy with the charge differences between $a$ and $a'$, that is, with the charges that fuse with $a$ to give $a'$.

We consider a Mach-Zehnder type interferometer (see Fig. 1), though the same methods may be applied to other types of interferometers. A target “particle” $A$ carrying a superposition of anyonic charges [26] is located in the region between the two paths of the interferometer. A beam of probe particles $B_k$, $k = 1, \ldots, N$ may be sent into two possible input channels, is passed through a beam splitter $T_1$, reflected by mirrors around the central region, passed through a second beam splitter $T_2$, and finally detected at two possible output channels. The system acquires a phase $e^{i\theta_k}$ or $e^{i\theta_k}$ when a probe particle passes through the bottom or top path around the central region (this may come from background flux, path length differences, phase shifter, etc.) and a separate, independent contribution strictly from the braiding of the probe and target particles, which, for non-Abelian anyons, will be more complicated than a mere phase. If the phases $e^{i\theta_k}$ and $e^{i\theta_k}$ are fixed, or closely monitored, this provides a non-demolitional measurement of the anyonic charge of $A$ [27]. This admittedly idealized setup is similar to one experimentally realized for quantum Hall systems [20], the primary difference being that the number of quasiparticle excitations in the central interferometry region is not fixed in that experiment. This situation, while unsuitable for measuring a target charge, could still be used to detect the presence of non-Abelian statistics [21].

The experiment we describe was also considered in the paper [22], where it was referred to as the “many-to-one” experiment. In that paper, the authors use a quantum group inspired approach, where individual particles are assumed to have internal Hilbert spaces, and they study what happens to the internal state of the target particle. In our descriptions of the systems examined, we use the theory of general anyon models, which does not ascribe individual particles internal degrees of freedom. Instead, the relevant observables are the overall anyonic charges of groups of particles (our main result will be stated in terms of the density matrix of an anyon pair $A\sim \overline{A}$). This is the situation relevant to the topological systems (e.g. FQH states) that we have in mind. We also remove some constraints imposed in [22], specifically, that the probe particles...
are all identical and have trivial self-braiding.

Let us recall some information about anyon models (see e.g. [23, 24] for additional details). States in these models may be represented by superpositions of oriented worldline diagrams that give a history of splitting and fusion of particles carrying an anyonic charge. Each allowed fusion/splitting vertex is associated with a (possibly multi-dimensional) vector space containing normalized bra/ket vectors

\[
(d_a d_b d_c)^{-1/4} = |a, b, c, \mu\rangle \in V_{ab}^c
\]

\[
(d_a d_b d_c)^{-1/4} = |a, b, c, \mu\rangle \in V_{ab}^{ac}
\]

where \(\mu\) labels the basis states of the splitting space \(V_{ab}^c\) of the charges \(a\) and \(b\) from charge \(c\) and the number \(d_a \geq 1\) is the quantum dimension of \(a\). The factors of \((d_a d_b d_c)^{-1/4}\) are necessary for isotopy invariance, i.e. so the meaning of the diagrams is not changed by continuous deformation. The vacuum is labeled 1, and has \(d_1 = 1\). Since \(\dim V_{ab}^c = 1\) when any of \(a, b, c\) equals 1, the basis label is redundant and will be dropped. In fact, the meaning of diagrams is invariant under addition/removal of vacuum lines, so we may drop them and smooth out their vertices. The charge conjugate, or antiparticle, of \(a\) is denoted \(\overline{a}\), and may also be denoted by reversing the arrow on a line labeled by \(a\). Diagrams with multiple vertices correspond to tensor products of vertex spaces. Density matrices may be represented by diagrams with the same numbers of lines emerging at the top and bottom (being combinations of kets and bras). Conjugating of states and operators corresponds to reflecting their diagrams in the horizontal plane. One may diagrammatically trace out a charge that enters and exits a diagram at the same spatial position by connecting the lines at these positions with an arc that does not interfere with the rest of the diagram. (If the charges at the bottom and top do not match, the trace is zero.) Here are some important diagrammatic relations:

\[
\begin{aligned}
R_{ab} & = a \otimes b, & R_{ab}^{-1} = b \otimes a, \\
S_{ab} & = \frac{1}{D} a \otimes b, & S_{ba}^{-1} = b \otimes a
\end{aligned}
\]

where \(d_a = DS_{ab}\) and the total quantum dimension \(D = \sqrt{\sum d_a^2}\). Another useful quantity, especially for interference experiments [25], is the monodromy matrix element \(M_{ab} = \frac{S_{ab} S_{1b}}{S_{1a} S_{ab}}\). It has the property \(|M_{ab}| \leq 1\), with \(M_{ab} = 1\) corresponding to trivial monodromy, i.e. the state is unchanged by taking the charges \(a\) and \(b\) all the way around each other.

Using this formalism, it is important to keep track of all particles involved in a process. We invoke the physical assumption that the particles \(A\) and all \(B_k\) are initially unentangled. This means there are no non-trivial charge lines connecting them, and to achieve this, they must each be created separately from vacuum, with their own antiparticles [28]. We write the initial state of the \(A\overline{A}\) system as

\[
|\Psi_0\rangle = \sum_a A_a |a, \overline{a}; 1\rangle
\]

and that of each \(B_k\) system as

\[
|\varphi_k\rangle = \sum_{b,s} B_{b,s}^{(k)} |b, b; 1; s\rangle
\]

where \(s = \uparrow, \downarrow\) indicates in which direction the probe particle is traveling. The probes’ antiparticles, \(\overline{B}_k\), will be taken off to the left and do not participate in the interferometry. The location of the target’s antiparticle \(\overline{A}\) with respect to the interferometer is important and we will let it be located below the central region, as in Fig. 1.

Utilizing the two-component vector notation \((\uparrow) = |\rightarrow\rangle\), \((\downarrow) = |\leftarrow\rangle\) to indicate whether a probe particle is traveling in the horizontal or vertical direction, the two beam splitters, which (along with the mirrors) are assumed to be lossless, are represented by the unitary operators \(T_j = \begin{bmatrix} t_j & r_j^* \\ r_j & t_j^* \end{bmatrix}\). The evolution operator for passing the probe particle \(B_k\) through the interferometer (after the first \(k - 1\) particles) is

\[
U_k = T_2 \Sigma_k T_1
\]

\[
\Sigma_k = \begin{bmatrix} 0 & e^{i\theta_k} R_{1B_k}^{-1} \\ e^{i\theta_k} R_{2B_k} & 0 \end{bmatrix}
\]

Diagrammatically, this takes the form

\[
\begin{bmatrix} A \\ B_k \end{bmatrix} = e^{i\theta_k} \begin{bmatrix} t_1 r_2^* & r_1^* r_2 \\ -t_1 r_2^* & r_1 r_2^* \end{bmatrix} \begin{bmatrix} B_k \\ A \end{bmatrix} + e^{i\theta_k} \begin{bmatrix} t_2 & -r_2^* r_1 \\ r_2 & t_2^* \end{bmatrix} \begin{bmatrix} B_k \\ A \end{bmatrix}
\]

Keeping track of antiparticles, we need \(V_k = R_{A,B_k}^{-1}\) for braiding the probe particles with \(\overline{A}\) [29], and, adding in each successive \(\varphi_k\) from the left, we also need the operators

\[
W_k = R_{B_k \overline{B}_{k-1}} R_{B_k \overline{B}_{k-1}} \ldots R_{B_k \overline{B}_1} R_{B_k \overline{B}_1}
\]

\[
(W_1 = 1), \text{ which move the } \overline{B}_k B_k \text{ pair from the left to the center of the configuration } B_1 \ldots \overline{B}_{k-1} B_{k-1} \ldots B_1. \text{ This may be viewed either as spatial sorting after creation, or, as shown suggestively in Eq. (13), as the temporal condition that each } \overline{B}_k B_k \text{ pair is utilized before the next one is created.}

The state of the combined system after \(N\) probe particles have passed through the interferometer (but have not yet been detected) may now be defined iteratively as

\[
|\Psi_N\rangle = V_N U_N W_N |\varphi_N\rangle \otimes |\Psi_{N-1}\rangle
\]
Focusing on the $A-\overline{A}$ system, the reduced density matrix, $\rho_N^A = Tr_{R \otimes S} |\Psi_N\rangle \langle \Psi_N|$, is obtained by tracing over the $B_k$ and $\overline{B}_k$ particles. This may be interpreted as ignoring the detection results. Given the placement of $A$, one sees that this averaging over detector measurements makes the second beam splitter irrelevant. If we kept track of the measurement outcomes $s_k$, we would project with $P_{s_k} = |s_k⟩⟨s_k|$ after the $k^{th}$ probe particle. In $|\Psi_N⟩$, we did not include braiding between the $B_k$ (or the $\overline{B}_k$), but they may be added without changing the results, as they drop out of $\rho_N^A$ [30].

We will first assume that the probe particles all have the same, definite anyonic charge $b$ and enter through the horizontal leg, so that $|ψ_k⟩ = |b; 1; →⟩$ for all $k$, and then later return to the general case. Expressed diagrammatically (with direction indices left implicit), this results in the state

$$|ψ_N⟩ = \sum_a A_a \frac{1}{\sqrt{d_a d_b^N}}.$$

We first consider the case $N = 1$. Tracing out the $b$ and $\overline{b}$ lines of $|ψ_1⟩ ⟨ψ_1|$, and using Eq. (10), one finds that terms cancel to give

$$\rho_1^A = \sum_{a,a'} \frac{A_a A_{a'}^*}{\sqrt{d_a d_b}} \times \begin{bmatrix} |t_1|^2 + |r_1|^2 M_{bc}^N \end{bmatrix}$$

(14)

This result should be obvious, since all that matters after tracing over measurement outcomes is that the probe particle passes between $A$ and $\overline{A}$ with probability $|t_1|^2$. Since they are initially unentangled, each additional probe particle has the same analysis as the first, and just provides an additional loop that passes between $A$ and $\overline{A}$ with probability $|t_1|^2$. Noting that an unlinked $b$ loop may be replaced by a factor $d_b$, we see that the reduced density matrix for $A$ after passing $N$ probe particles through the interferometer is

$$\rho_N^A = \sum_{a,a'} \frac{A_a A_{a'}^*}{\sqrt{d_a d_b}} \sum_{n=0}^N \binom{N}{n} |r_1|^{2(N-n)} |t_1|^{2n} \frac{1}{d_b^n} \prod_{1, (e, \alpha, \beta)} \sum_{n=0}^N \binom{N}{n} |r_1|^{2(N-n)} |t_1|^{2n} \frac{1}{d_b^n} \prod_{1, (e, \alpha, \beta)}$$

(15a)

$$= \sum_{a,a'} \frac{A_a A_{a'}^*}{\sqrt{d_a d_b}} \sum_{(e, \alpha, \beta)} \left[ (F^a_{\alpha, \beta})^{-1} \right]_{1, (e, \alpha, \beta)} \sum_{n=0}^N \binom{N}{n} |r_1|^{2(N-n)} |t_1|^{2n} \frac{1}{d_b^n} \prod_{1, (e, \alpha, \beta)}$$

(15b)

$$= \sum_{a,a'} \frac{A_a A_{a'}^*}{\sqrt{d_a d_b}} \sum_{(e, \alpha, \beta), (f, \mu, \nu)} \left[ (F^a_{\alpha, \beta} F^{a'}_{\alpha', \beta'})^{-1} \right]_{1, (e, \alpha, \beta), (f, \mu, \nu)} \left[ (F^a_{\alpha, \beta} F^{a'}_{\alpha', \beta'})^{-1} \right]_{1, (e, \alpha, \beta)}$$

(15c)

where the relations in Eq. (5) were used to remove all the $b$ loops, allowing us to perform the sum over $a$, before applying $F$ in the last step. The intermediate charge label $e$ represents the difference between the charges $a$ and $a'$, taking values that may be fused with $a'$ to give $a$. Notice the potential for this process to transfer an overall anyonic charge $f$ to the $A-\overline{A}$ system.

From this result we see, noting $|t_1|^2 + |r_1|^2 = 1$, that taking the limit $N \to \infty$ will kill off the $e$-channels with $M_{be} \neq 1$, and preserve only those which have trivial monodromy with $b$, $M_{be} = 1$. The interpretation of $M_{be} = 1$ is that $a$ and $a'$ have a difference charge $e$ that is invisible (in the sense of monodromy) to the charge $b$, and the corresponding fusion channel remains coherent and unmeasured. In general, the only $e$-channels guaranteed always to survive this process (even for the most general $B_k$ states) have trivial monodromy with all charges. This always includes $e = 1$ (and is the only such charge for modular theories/TQFTs), which requires that $a = a'$. Tracing over the $A$ and $\overline{A}$ particles gives $Tr \rho_N^A = 1$ (as one would expect for the total probability), but by considering the intermediate channels, one also finds that the entire contribution to this trace is from $e = 1$. This means that the diagonal terms of the reduced density matrix $\rho_N^A$ are those with not just $a = a'$, but more specifically the $e = 1$ contribution to these terms. We should also note that some terms may alternatively be killed off due to their corresponding $F$-symbols having zero values.

Denoting the charges $e$ with $M_{be} = 1$ (the channels not decohered away) as $e_b$ and $\rho_N^A = \lim_{N \to \infty} \rho_N^A$, we get the
\[ \rho^A = \sum_{a,a'} A_a A_{a'}^* \sum_{(e_a, \alpha, \beta), (f, \mu, \nu)} \left( (F_{e_a, \alpha}^{a, \beta})^{-1} \right)_{1, (e_a, \alpha, \beta)} \times \left( F_{e_a, \alpha}^{a, \beta} \right)_{(e_a, \alpha, \beta), (f, \mu, \nu)} \sqrt{d_f} (a', \alpha; f, \mu) \langle a', \alpha; f, \nu |. \]  

We now return to the case of general probe particle states as given in Eq. (7). Since tracing requires the charge on a line to match up, a similar analysis as before applies. For the result, we simply replace \( \left( |r_1|^2 + |t_1|^2 M_{be} \right)^N \) in Eq. (15c), with

\[ \prod_{k=1}^N \left[ 1 - \sum_b \left| B_{b, t_1}^{(k)} + B_{b, t_1}^{(k)} \right|^2 (1 - M_{be}) \right]. \]  

This term determines the rate at which each probe decoheres the \( A \) system, and, for generic probes, will vanish as \( N \to \infty \) unless \( e \) has trivial monodromy. In some cases, complete decoherence may even be achieved with a single probe step. One may also consider completely general initial states for the \( A \) and \( B_k \) systems described by density matrices, but as long as they are all unentangled amongst each other, the analysis and resulting behavior is essentially the same.

The decoherence, as described above, is due to measurements being made by probe particles, and suggests that, when the measurement outcomes are kept track of, they will indicate collapse of the target system state into the subspaces where difference charge has trivial monodromy with the probes. If this includes only the \( e = 1 \) subspaces, the target collapses into a state of definite charge. On the other hand, by setting \( |r_1| = 0 \) and \( |t_1| = 1 \) in Eq. (15c), we may do away with the interferometer and interpret the result as decoherence from stray anyons passing between \( A \) and \( \overline{A} \), which is important to consider as a source of errors in a quantum computation.

As a practical example, we apply the results to the Ising anyon model, which captures the essence of the Moore-Read state’s non-Abelian statistics. For the initial state \( |\Psi_0\rangle = A_1 |1, 1; 1 \rangle + A_\psi |\psi, \psi; 1 \rangle + A_\sigma |\sigma, \sigma; 1 \rangle \), using \( b = \sigma \) probes (which have trivial monodromy only with \( e = 1 \)) gives

\[ \rho^A = |A_1|^2 |1, 1; 1 \rangle \langle 1, 1; 1 | + |A_\psi|^2 |\psi, \psi; 1 \rangle \langle \psi, \psi; 1 | + |A_\sigma|^2 \frac{1}{2} (|\sigma, \sigma; 1 \rangle \langle \sigma, \sigma; 1 | + |\sigma, \sigma; 1 \rangle \langle \sigma, \sigma; 1 |) \]  

which exhibits loss of all coherence. For \( b = \psi \) probes (which have trivial monodromy with both \( e = 1 \) and \( \psi \)) the result

\[ \rho^A = |A_1|^2 |1, 1; 1 \rangle \langle 1, 1; 1 | + A_\psi A_\psi^* |\psi, \psi; 1 \rangle \langle 1, 1; 1 | + A_1 A_\psi^* |\psi, \psi; 1 \rangle \langle 1, 1; 1 | + |A_\psi|^2 |\psi, \psi; 1 \rangle \langle \psi, \psi; 1 | + |A_\sigma|^2 |\sigma, \sigma; 1 \rangle \langle \sigma, \sigma; 1 | \]  

shows decoherence only between \( \sigma \) and the other charges.

We thank I. Klich and especially J. Preskill for illuminating discussions, and the organizers and participants of the KITP Workshop on Topological Phases and Quantum Computation where this work was initiated. We would also like to acknowledge the hospitality of the IQI, the KITP, and Microsoft Project Q. This work was supported in part by the NSF under Grant No. PHY-0456720 and PHY99-07949, and the NSA under ARO Contract No. W911NF-05-1-0294.

[13] ...
If $\overrightarrow{A}$ is located above, rather than below, the central region of the interferometer, we would instead use $V_k = R_{B_k} \overrightarrow{\tau}$. This essentially interchanges $r_1$ with $t_1$ and conjugates $M_{he}$ in the result, Eq. (15c). If however, $\overrightarrow{A}$ is placed between the two output legs, the situation is complicated by the resulting $V_k = \begin{bmatrix} R_{B_k} \overrightarrow{\tau} & 0 \\ 0 & R_{M_{he}} \end{bmatrix}$, which makes evaluation more difficult, and gives a different limiting behavior. If $\overrightarrow{A}$ is situated in the central region (with $A$), there will, of course, be no effect.

Superpositions of braiding may however change these results.