Sum rules for total cross-sections of hadron photoproduction on pseudoscalar mesons

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I. INTRODUCTION

In the paper [1], considering the very high energy electron-nucleon scattering with peripheral production of a hadronic state $X$ moving closely to a direction of initial nucleon and utilizing analytic properties of the forward Compton scattering amplitudes on the nucleons, the new sum rule, relating proton Dirac radius and anomalous magnetic moments of the proton and the neutron to the convergent integral of the difference of total proton and neutron photoproduction cross-sections, was derived. In this paper we extend the previous method deriving sum rules for various suitable couples of the members of the pseudoscalar meson nonet, giving into a relation mean squared charge radii with the convergent integral of the difference of total hadron photoproduction cross-sections on the considered pseudoscalar mesons.

II. RELATION BETWEEN DIFFERENTIAL MESON ELECTROPRODUCTION AND TOTAL HADRON PHOTOPRODUCTION CROSS-SECTIONS

Let us consider a very high energy peripheral electroproduction process on pseudoscalar mesons $P$

$$e^-(p_1) + P(p) ightarrow e^-(p'_1) + X,$$

(1)

where the produced pure hadronic state $X$ is moving closely to the direction of the initial meson. Its matrix element in the one photon exchange approximation takes the form

$$M = i \frac{\sqrt{4\pi\alpha}}{q^2} \bar{u}(p'_1)\gamma_\mu u(p_1) < X | J_\mu | P > g^{\mu\nu}$$

(2)

and $m_X^2 = (p + q)^2$.

Now, by means of the method of equivalent photons [2], examining the pseudoscalar meson in the rest, the electron energy to be very high and the small photon momentum transfer, one can express the differential cross-sections of the processes [1] as a function of $q^2$ through integral over the total hadron photoproduction cross-sections on pseudoscalar mesons.

Really, applying to (2) the Sudakov expansion [3] of the photon transferred four-vector $q$

$$q = \beta q_1 + \alpha q + q_\perp, \quad q_\perp = (0, 0, q), \quad q^2 = -q_\perp^2$$

(3)

into the almost light-like vectors

$$\tilde{p}_1 = p_1 - m_0^2 p/(2p_1 p), \quad \tilde{p}_1 = p - m_0^2 p_1/(2p_1 p),$$

(4)

then using the Gribov prescription [4] for the numerator of the photon Green function

$$g_{\mu\nu} = g_{\mu\nu}^\parallel + \frac{2}{s} (\tilde{p}_\mu\tilde{p}_1\nu + \tilde{p}_\nu\tilde{p}_1\mu) \approx \frac{2}{s} \tilde{p}_\mu\tilde{p}_1\nu,$$

(5)

where $s = (p_1 + p)^2 \approx 2p_1 p \gg Q^2 = -q^2$, as a consequence of the electron energy in [1] to be very high and the photon momentum transfer squared $t = q^2 = -Q^2 = -q_\perp^2$ to be small one, one obtaines for the corresponding cross-section

$$d\sigma e^- p - e^- X = \frac{4\pi\alpha}{s (q^2)^2} p_1^\mu p_1^\nu \sum_{X \neq P} \langle P | J_{EM}^\mu | X \rangle^* \times$$

$$\times \langle X | J_{EM}^{\nu} | P \rangle d\Gamma$$

(6)

with a summation through the created hadronic states $X$.

Further, if the phase space volume of the final electron is adjusted

$$\frac{1}{(2\pi)^3} d^4 p_1^\mu = \frac{1}{(2\pi)^3} d^4 q_\perp [\langle p_1 - q \rangle^2] = \frac{1}{(2\pi)^3} 2s d^2 q_\perp,$$

(7)

in order to rewrite the final state phase-space volume in [1] into the form

$$d\Gamma = \frac{ds_1}{2s(2\pi)^3} d^2 q_\perp d\Gamma_X;$$

(8)
with
\[ s_1 = 2(qp) = m_X^2 + q^2 - m_P^2 = s\beta_q, \]
then the current conservation condition \((\alpha_q \tilde{p} \text{ gives a negligible contribution})\)

\[ q^\mu \langle X \mid J_{\mu}^{EM} \mid P \rangle \approx (\beta_q \tilde{p}_1 + q_\perp)^\mu \langle X \mid J_{\mu}^{EM} \mid P \rangle = 0, \]
is used in order to utilize in \(I\) the expression

\[
\int \tilde{p}_1^\mu \tilde{p}_1^\nu \sum_{X \neq P} \langle P \mid J_{\mu}^{EM} \mid X \rangle \langle X \mid J_{\nu}^{EM} \mid P \rangle d\Gamma_X = \\
= 2i\frac{s^2}{s_1^2} q^2 Im\tilde{A}^{(P)}(s_1, q),
\]
(10)

with the \(\tilde{A}^{(P)}(s_1, q)\) to be by a construction only a part of the total forward virtual Compton scattering amplitude \(\tilde{A}^{(P)}(s_1, q)\) on pseudoscalar mesons, which does not contain any crossing Feynman diagram contributions, for a difference of corresponding differential cross-sections of the electroproduction on \(P\) and \(P'\) (after integration in \(I\) over \(d\Gamma_X\), as well as over \(m_X^2\), i.e. over the variable \(s_1\) to be interested only for \(q\) distribution) one finds

\[
\left( \frac{d\sigma_{e^- P^- e^- X}}{d^2 q} - \frac{d\sigma_{e^- P'^- e^- X'}}{d^2 q} \right) = \\
= \frac{\alpha q^2}{4\pi^2} \int_{s_1^{th}}^{\infty} \frac{ds_1}{s_1^2[q^2 + (m_e s_1/s)^2]^2} \times \\
\times \left[ Im\tilde{A}^{(P)}(s_1, q) - Im\tilde{A}^{(P')} (s_1, q) \right].
\]

Finally, if one neglects the second term in square brackets of the denominator of the integral in \(I\) (owing to the small value of \(m_e\) and high \(s\) in comparison with \(s_1\)) and takes the limit \(q^2 \rightarrow 0\) along with the expressions \(d^2 q = \pi dq^2\) and \(Im\tilde{A}^{(P)}(s_1, q) = 4s_1 \sigma_{tot}^{P^- X}(s_1, q)\), one comes to the Weizsäcker-Williams like relation

\[
q^2 \left( \frac{d\sigma_{e^- P^- e^- X}}{d\Gamma_X} - \frac{d\sigma_{e^- P'^- e^- X'}}{d\Gamma_X} \right) \bigg|_{q^2 \rightarrow 0} = (12)
\]

between the difference of \(q^2\)-dependent differential cross-sections of the processes \(1\) and the convergent integral over the difference of the total hadron photoproduction cross-sections on pseudoscalar mesons.

III. SUM RULES FOR PSEUDOSCALAR MESONS

Now, let us investigate analytic properties of the forward Compton scattering amplitude \(\tilde{A}(s_1, q)\) in \(s_1\)-plane. They consist in meson intermediate state pole at \(s_1 = q^2\), the right-hand cut starting at the three meson threshold and the \(u_1\)-channel left-hand cut. Defining the path integral \(I\) (for more detail see \(2\)) in \(s_1\) plane

\[
I = \int_{C} ds_1 \frac{p_{1\mu} p_{1\nu}}{s^2} \left( \tilde{A}_{\mu\nu}^{(P)}(s_1, q) - \tilde{A}_{\mu\nu}^{(P')} (s_1, q) \right)
\]
(13)

from the gauge invariant light-cone projection \(p_{1\mu}^\mu p_{1\nu}^\nu \tilde{A}_{\mu\nu}\) of the part \(\tilde{A}_{\mu\nu}\) of the total Compton scattering tensor with photon first absorbed and then emitted along the meson world line as presented in Fig.1a and once closing the contour \(C\) to upper half-plane, another one to lower
half-plane (see Figs. 1b), the following sum rule

\[ \pi(\text{Res}(P') - \text{Res}(P)) = q^2 \int_{r.h.s.}^\infty \frac{ds_1}{s_1^2} [\text{Im}A^{(P)}(s_1, q) - \text{Im}A^{(P')}(s_1, q)] \]  
\[ = 2\pi\alpha F_M^2(-q^2) \]  
(14)

appears with

\[ \text{Res}^{(M)} = 2\pi\alpha F_M^2(-q^2) \]  
(15)

to be the residuum of the meson intermediate state pole contribution expressed through the pseudoscalar meson charge form factor \( F_M(-q^2) \) and the left-hand cut contributions expressed by an integral of the difference \( [\text{Im}A^{(P)}(s_1, q) - \text{Im}A^{(P')}(s_1, q)] \) are mutually annulled. Then, substituting \( 15 \) into \( 14 \) and taking into account \( 14 \) with \( d^2q = \pi dq^2 \), one comes to the meson sum rules

\[ [F_p^2(-q^2) - F_{p'}^2(0)] - [F_p^2(-q^2) - F_{p'}^2(0)] = \frac{2}{\pi^2\alpha} \int_{\omega_p}^\infty \frac{d\omega}{\omega} [\gamma_{p}\omega - \gamma_{p'}\omega] \]  
(16)

where the left-hand side was renormalized in order to separate the pure strong interactions from electromagnetic ones. Moreover, substituting here for small values of \( q^2 \) the relation \( 12 \) and using the laboratory coordinate system by \( s_1 = 2m_{\pi\omega} \) and finally taking a derivative according to \( q^2 \) of both sides for \( q^2 = 0 \), one comes to the new sum rule relating meson mean squared charge radii to the convergent integral of the difference of corresponding total hadron photoproduction cross-sections on mesons

\[ \frac{1}{3} \left( F_p(0)\langle r_{K^+}^2 \rangle - F_{p'}(0)\langle r_{K^+}^2 \rangle \right) = \frac{2}{\pi^2\alpha} \int_{\omega_p}^\infty \frac{d\omega}{\omega} \left[ \gamma_{p}\omega - \gamma_{p'}\omega \right] \]  
(17)

in which just a mutual cancelation of the rise of the latter cross sections for \( \omega \to \infty \) is achieved.

IV. APPLICATION TO VARIOUS COUPLES OF MESONS

According to the SU(3) classification of existing hadrons there are the following members of the ground state pseudoscalar meson nonet \( \pi^- , \pi^0 , \pi^+ , K^- , \bar{K}^0 , K^0 , K^+ , \eta , \eta' \). However, in consequence of CPT invariance the meson electromagnetic form factors \( F_p(-q^2) \) hold the following relation

\[ F_p(-q^2) = -F_{p'}(-q^2) \]  
(18)

where \( p' \) means antiparticle.

Since \( \pi_0 , \eta \) and \( \eta' \) are true neutral particles, their electromagnetic form factors are owing to the \( 18 \) zero in the whole region of a definition and therefore we exclude them from further considerations.

If one considers couples of particle-antiparticle like \( \pi^\pm , K^\pm , K^0 , K^\bar{0} \), the left-hand side of \( 18 \) is owing to the relation \( 18 \) equal zero and we exclude couples \( \pi^\pm , K^\pm , K^0 , K^\bar{0} \) from further considerations as well.

If one considers a couple of the isodoublet of kaons \( K^+, K^0 \) and \( K^- , \bar{K}^0 \), the following Cabibbo-Radicati \( 6 \) like sum rules for kaons can be written

\[ \frac{1}{6} \pi^2 \alpha \langle r_{K^+}^2 \rangle = \int_{\omega_{K^+}}^\infty \frac{d\omega}{\omega} \left[ \gamma_{\omega\gamma}K^+X(\omega) - \gamma_{\omega\gamma}K^0\bar{X}(\omega) \right] \]  
(19)

\[ \frac{1}{6} \pi^2 \alpha \langle -1 \rangle \langle r_{K^-}^2 \rangle = \int_{\omega_{K^-}}^\infty \frac{d\omega}{\omega} \left[ \gamma_{\omega\gamma}K^-X(\omega) - \gamma_{\omega\gamma}K^0\bar{X}(\omega) \right] \]  
(20)

in which the relation \( \langle r_{K^+}^2 \rangle = -\langle r_{K^-}^2 \rangle \) for kaon mean squared charge radii, following directly from \( 18 \), holds and divergence of the integrals, due to an increase of the total cross-sections \( \gamma_{\omega\gamma}K^+X(\omega) \) for large values of \( \omega \), is taken off by the increase of total cross-sections \( \gamma_{\omega\gamma}K^-X(\omega) \) and \( \gamma_{\omega\gamma}K^0\bar{X}(\omega) \), respectively. If besides the latter, also the relations

\[ \gamma_{\omega\gamma}K^0\bar{X}(\omega) = \gamma_{\omega\gamma}K^-X(\omega) \]  
(21)

following from \( C \) invariance of the electromagnetic interactions, are taken into account, one can see the sum rule \( 21 \), as well as all other possible sum rules obtained by combinations \( K^+\bar{K}^0 , K^-K^0 \), to be contained already in \( 19 \).

The last possibility is a consideration of a couple of mesons taken from the isomultiplet of pions and the isomultiplet of kaons leading to the following less precise (in comparison with \( 18 \)) sum rules

\[ \frac{1}{6} \pi^2 \alpha \langle \pm 1 \rangle \langle r_{\pi^\pm}^2 \rangle = \int_{\omega_{\pi^\pm}}^\infty \frac{d\omega}{\omega} \left[ \gamma_{\omega\gamma}\pi^\pm X(\omega) - \gamma_{\omega\gamma}K^\pm\bar{X}(\omega) \right] \]  
(22)

\[ \frac{1}{6} \pi^2 \alpha \langle -1 \rangle \langle r_{\pi^-}^2 \rangle = \int_{\omega_{\pi^-}}^\infty \frac{d\omega}{\omega} \left[ \gamma_{\omega\gamma}\pi^- X(\omega) - \gamma_{\omega\gamma}K^-\bar{X}(\omega) \right] \]  
(23)

as there is no complete annulation of the left-hand cut contributions in \( 14 \) due to a larger difference in the masses of joining pairs of particles.

The latter assertion can be roughly confirmed as follows. The left-hand cut contribution in \( 14 \) has no direct
interpretation in terms of a cross-section. Nevertheless, it can be associated (for more detail see ref. [7]) with contribution to the cross-sections of $MM$ meson pair electroproduction on considered target meson $M$, arising from taking into account the identity of final state mesons. Then the left-hand cut contribution to the derivative according to $q^2$ at $q^2 = 0$ of scattering amplitudes entering sum rules have an order of magnitude

$$I = \frac{g^4}{(2\pi)^2 s_{1\text{max}}}, \quad s_{1\text{max}} = \max[m^2_\rho, 8m^2_M]$$  \hspace{1cm} (24)$$

where $g$ is the strong coupling constant of the $\rho$-meson to the considered meson and $m_M$ is the mass of the target meson. Taking the PDG [6] typical value for the total cross-section of scattering of a pion on proton to be $\sigma_{\rho N}^{\text{tot}} \equiv 20 \text{ [mb]}$, then $\rho$-meson $t$-channel contribution and $s_{1\text{max}} = m^2_\rho$ in the case of the target meson is charged pion and $s_{1\text{max}} = 8m^2_K$ in the case of the target meson is kaon, we have $I_\pi \approx 0.081 \text{ [mb]}$ and $I_K \approx 0.024 \text{ [mb]}$ respectively, which confirm above mentioned statement.

Now taking the experimental values [8]

$(\pm 1)(q^2_{\pi \pm}) = +0.4516 \pm 0.0108 \text{ [fm}^2\text{]}$ and

$(\pm 1)(q^2_{K \pm}) = +0.3136 \pm 0.0347 \text{ [fm}^2\text{]}$

one comes to the conclusion that in average

$$[\sigma_{\rho N}^{\pi^+\pi^- - X}(\omega) - \sigma_{\rho N}^{K^+K^- - X}(\omega)] > 0$$

(25)

and

$$[\sigma_{\rho N}^{\pi^-\pi^+ - X}(\omega) - \sigma_{\rho N}^{K^-K^0 - X}(\omega)] > 0$$

from where the following inequalities for finite values of $\omega$ in average follow

$$\sigma_{\rho N}^{\pi^+\pi^- - X}(\omega) > \sigma_{\rho N}^{K^+K^- - X}(\omega) > \sigma_{\rho N}^{K^-K^0 - X}(\omega) > 0.$$  \hspace{1cm} (26)$$

Subtracting up [10] or [20] from the relation [28] the sum rule [29] is obtained, what demonstrates a mutual consistency of all considered sum rules. They have been derived in analogy with a derivation [11] of the sum rule for a difference of proton and neutron total photoproduction cross-sections, which are fulfilled with a very high precision. Therefore we believe that also the sum rules for total cross-sections of hadron photoproduction on pseudoscalar mesons presented in this paper are correct. However, the final word is always given by experimental tests.

The experimental test of the derived sum rules can be practically carried out if there are known the total hadron photoproduction cross-sections on pions and kaons as a function of energy, which, however, are missing till now. Nevertheless, the idea of a conversion of the electron beams of linear $e^+e^-$ colliders into photon beams, using the process of the backward Compton scattering of laser light off the high energy electrons, which is known already for few decades, together with a real possibility of a production of enough intensive beams of pions [10] provide a real chance for measurements of the total hadron photoproduction cross-sections on charged pions and kaons, and as a result also the experimental test of the sum rules derived in this paper.

V. CONCLUSIONS

Considering the very high energy peripheral electron pseudoscalar meson scattering with a production of a hadronic state $X$ moving closely to the direction of initial meson, then utilizing analytic properties of the forward Compton scattering amplitude on the same meson, for the case of small transferred momenta new Cabibbo-Radicati [6] like sum rules, relating the corresponding meson mean squared charge radii with the convergent integral over a difference of the total hadron photoproduction cross-sections on mesons are derived. Unlike the sum rules [11], [20], [24] and [29], derived in this paper, there could be difficulties with experimental verification of the Cabibbo-Radicati sum rule due to appearance of the sum of total photoproduction cross-sections on pions with the transition always to a hadronic state with a specific isospin state and moreover, the convergence of the corresponding integral is questionable.

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