Separability and Entanglement of Identical Bosonic Systems

Xiao-Hong Wang¹, Shao-Ming Fei¹, ² and Ke Wu¹

¹ Department of Mathematics, Capital Normal University, Beijing, China
² Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany

Abstract We investigate the separability of arbitrary $n$-dimensional multipartite identical bosonic systems. An explicit relation between the dimension and the separability is presented. In particular, for $n = 3$, it is shown that the property of PPT (positive partial transpose) and the separability are equivalent for tripartite systems.

Key words: Separability, Quantum entanglement, PPT state

PACS number(s): 03.67.Hk, 03.65.Ta, 89.70.+c

Quantum entanglement plays essential roles in quantum information processing and quantum computation. The entangled states provide key resources for a vast variety of novel phenomena such as quantum cryptography, quantum teleportation, super dense coding, etc [1]. An important problem in the theory of quantum entanglement is the separability. One of the famous separability criterion was given by Peres [2]. It says that all separable states necessarily have a positive partial transpose (PPT), which is further shown to be also sufficient for states on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$ [3, 4], where $\mathbb{C}^n$ denotes the $n$-dimensional complex space. There have been many results on the separability and entanglements of mixed states, see e.g., [5, 6, 7, 8, 9]. In particular, it is shown that every quantum states $\rho$ supported on $\mathbb{C}^M \otimes \mathbb{C}^N$, $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^N$ and $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^N$ with positive partial transposes and rank $r(\rho) \leq N$ are separable and have a canonical form [5, 6, 7].

Although the entanglement is extensively studied for distinguishable particle systems, the entanglement of identical particle systems has been less investigated. In fact in certain systems such as quantum dots [10], Bose-Einstein condensates [11] and parametric down conversion [12], the entanglement should be treated as the one of identical particle systems. Schliemann et al [10, 13] have discussed the entanglement in two-fermion systems. They found that the entanglement in two-fermion systems is analogous to that in a two-distinguishable particle system. The results for two-boson systems are quite different. Li et al. [14] and Paskauskas and You [15] have studied this problem of two-boson systems. For multipartite bosonic systems, there are very few discussions. Recently, the author in [16] obtained the canonical form for pure states of three identical bosons, and classified the
entanglement correlation into two types, the analogous GHZ and the W-types. In [17], it has been shown that rank \( n \) and rank \( \frac{n(n+1)}{2} - 2 \) PPT bosonic mixed states in the symmetrized tensor product space \( \mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n) \) are separable, and all three-qubit \((n = 2)\) bosonic PPT states are separable as well. For bosonic mixed state \( \rho \) in \( k \)-qubit system, \( k \geq 4 \), \( \rho \) is PPT implies that \( \rho \) is separable, except for the case of maximal rank.

In this letter, we investigate the separability of multi-partite identical bosonic systems with arbitrary dimension \( n \). Let \( \mathcal{H} = \mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n) \) denote the symmetrized tensor product space of \( k \) \( n \)-dimensional spaces associated with Alice, Bob, Charlie, etc. The dimension of the space \( \mathcal{H} \) is given by [18],

\[
I_n^k = \frac{(n+k-1)!}{k!(n-1)!} = C_{n+k-1}^k.
\]

We first consider the case of \( k = 3 \).

**Theorem 1** Let \( \rho \) be a bosonic mixed state in \( \mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n) \), with a positive partial transpose with respect to Alice. If the rank of \( \rho \), \( r(\rho) \leq n^2 \), then \( \rho \) is separable.

**Proof.** We first prove the case of \( n = 3 \). Suppose that the state \( \rho \) is a PPT state with respect to Alice and has a rank 9. We can treat it as a bipartite PPT state in a \( 3 \times 9 \) dimensional space of Alice-(Bob,Charlie). From the Theorem 1 in [5] (also Theorem 1 in [6]), such a state of rank 9 is necessarily separable and can be represented as \( \rho = \sum_{i=1}^{9} p_i |e_i, \Psi_i\rangle \langle e_i, \Psi_i| \), where the vectors \(|\Psi_i\rangle\) are generally entangled pure states associated with the spaces of Bob and Charlie. As \(|\Psi_i\rangle\) are mutually orthogonal, they belong to the range of the reduced density matrix (partial trace with respect to the space associated with Alice) \( \text{Tr}_A \rho \), and hence \(|\Psi_i\rangle \in \mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3)\). Moreover \(|e_i, \Psi_i\rangle\) belong to the range of \( \rho \). Therefore \(|e_i, \Psi_i\rangle \in \mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)\). According to Schmidt decomposition we can write \(|\Psi_i\rangle = a_i |00\rangle + b_i |11\rangle + c_i |22\rangle\) for some \(a_i, b_i, c_i \in \mathbb{C}\), where \(|0\rangle, |1\rangle, |2\rangle\), are the Schmidt basic vectors in \(\mathbb{C}^3\). The only possible forms of \(|\rho_{e, f}\rangle\) satisfying the above conditions are \(|000\rangle\) or \(|111\rangle\) or \(|222\rangle\). Therefore \( \rho \) is separable.

When the rank of \( \rho \) is strictly less then 9, \( \rho \) can be embedded into a smaller space. For instance, if \( r(\rho) = 8 \), \( \rho \) is supported on spaces \( 2 \times 8 \) or \( 3 \times 8 \). \( \rho \) is then separable in the partition Alice-(Bob,Charlie) and can be again written as \( \rho = \sum_{i=1}^{8} p_i |e_i, \Psi_i\rangle \langle e_i, \Psi_i| \). By using the same procedure as above, we can prove that \(|e_i, \Psi_i\rangle\) is fully separable, and hence \( \rho \) is separable. The general \( n \)-dimensional case can be proved similarly.

**Remark** From the theorem we see that a bosonic mixed state \( \rho \) in \( \mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)\) with a positive partial transpose is separable if \( r(\rho) \leq 9 \). As the dimension of the space of \( \mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)\) is 10, Theorem 1 says that almost all the PPT bosonic mixed states in \( \mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)\) are separable, except for the case \( r(\rho) = 10 \). Hence the rank of a bound entangled state in \( \mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)\) has to be 10.

When \( n = 4 \), we have \( I_4^3 = 20 \). As \( \rho \) is separable if \( r(\rho) \leq 16 \), all bound entangled states \( \rho \) in \( \mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n)\) satisfy \( 17 \leq r(\rho) \leq 20 \).
[Theorem 2] Let $\rho$ be a PPT bosonic mixed state in $\mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n)$ with $k$ subsystems ($k \geq 4$). If $r(\rho) \leq I_n^{k-1}$, then $\rho$ is separable.

**Proof.** We prove the case of $n = 3$ (the other cases can be proved similarly). Assume that $\rho$ is PPT, say with respect to the space associated with Alice, with rank $I_3^{k-1} = \frac{k(k+1)}{2}$.

If we consider $\rho$ as a bipartite state in the partition Alice - the rest, $\rho$ is supported on $\mathbb{C}^3 \otimes \mathcal{S}((\mathbb{C}^3)^{\otimes k-1})$. From [5] $\rho$ is separable with respect to this partition and has a form, $\rho = \sum_{i=1}^{\frac{k(k+1)}{2}} p_i |e_i, \Psi_i \rangle \langle e_i, \Psi_i|$, where $|e_i\rangle$ (resp. $|\Psi_i\rangle$) are vectors on the spaces associated to Alice (resp. the rest).

We prove result by induction. We illustrate the procedure by proving the case of $k = 4$.

As $|\Psi_i\rangle$ belong to the range of the reduced density matrix $\text{Tr}_A \rho$, they must belong to $\mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$. Since $\rho$ is PPT $|\Psi_i\rangle \langle \Psi_i|$ is a PPT state in $\mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$. However the rank $r(|\Psi_i\rangle \langle \Psi_i|) = 1$, from Theorem 1, $|\Psi_i\rangle$ is separable, and can written as $|\Psi\rangle = |f_i, f_i, f_i\rangle$ for some vectors $|f_i\rangle$ in $\mathbb{C}^3$. While the vectors $|e_i, \Psi_i\rangle$ belong to the range of $\rho$ and hence $|e_i, \Psi_i\rangle \in \mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$. Therefore the only possible forms of $|e_i, \Psi_i\rangle$ are $|f_i, f_i, f_i, f_i\rangle$. Therefore $\rho$ is separable. \(\square\)

We have presented some separability criteria for multipartite bosonic mixed states. For tripartite PPT states, all bound entangled states have necessarily rank greater than $n^2$. For general multipartite PPT bosonic states with $k$ subsystems ($k \geq 4$), if $r(\rho) \leq I_n^{k-1}$, $\rho$ is separable. The results can be used to construct possible bound entangled states of identical bosonic systems. For instance, if $k = 4$, $n = 3$, we have $I_3^4 = 15$. The rank of a bound entangled state has to be between $I_3^2 = 10$ and 15.

**References**


