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THE PHASE AND THE MODULUS OF THE SCATTERING AMPLITUDE

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1. PRELIMINARY REMARKS

1) There are, in particle physics, standing problems which, after many years, are still waiting for a solution. Professor Zichichi during this session has mentioned some of them. Some of the standing problems are so familiar that we forget that they are problems. Such is the case of the connection between the phase and the modulus of the scattering amplitude ¹⁾. Here I want to discuss three particular problems in which this connection plays a major rôle.

- First of all, what is called "phase shift analysis", which is an operation so familiar to all physicists that everybody believes that its only difficulty comes from experimental data.
- Second I want to discuss the asymptotic equality of differential cross-sections for reactions $A + B \rightarrow A + B$ and $A + \bar{B} \rightarrow A + \bar{B}$, a domain in which a theoretical prediction has been made several years ago and the experimental confirmation has been presented just now, at the Palermo Conference.
- Third I want to discuss why elastic differential cross-sections have a peak, called diffraction peak exactly in the forward direction. This is again one of those problems which is so familiar to everybody that it looks obvious, except, as you will see, when you think about it a little. As a supplement I shall discuss the theoretical possibility of a "scaling property" of the diffraction peak, since this has been proposed by phenomenologists and seems to have some experimental support since it appears that the proton-proton diffraction minimum is moving in with energy.

Most of the work I shall describe here has been done in collaboration with Henri Cornille. Other contributors are Hugh Burkhardt and Shashanka Roy.

2) I want to explain my rules of the game. I do not want - or I am not able - to make models. If you make a model, however clever it may be, you run the risk to find somebody telling you that he does not believe your model, that this model has nothing to do with physics. This happened a few days ago to a famous physicist here in this church.

Then what to do ? We can try to use exclusively what everybody is ready to believe !

- Locality and its corollary : analytic properties of scattering amplitudes, in particular dispersion relations. To be honest I confess that I make one illegal assumption, namely that one can consider a strong interaction world with no massless particle. My only excuse is that almost everybody makes this assumption, without which scattering amplitudes are more difficult to define and on-shell analytic properties extremely evasive.
- Unitarity, or conservation of probability, in particular positivity.

The only risk we take, with such a limited input, is that we may be unable to obtain really interesting results. The results may look rather trivial, especially to the outside observer. This is typical of mathematics and mathematical physics : it is not sufficient to obtain correct results, true theorems ; they must also have some substance. Here, year after year I have the recurrent thought that the field is exhausted. Still, year after year new, and - I believe - relevant results are coming out, but it is up to you to judge.

2. SOME BASIC FACTS ABOUT ELASTIC SCATTERING AMPLITUDES

What we shall say here applies, strictly speaking, only to spin 0 - spin 0 scattering amplitudes. However, most of it can be generalized to arbitrary spins.

The scattering amplitude for the reaction $A + B \rightarrow A + B$ is a complex function of two variables, which can be taken to be :

$$\begin{aligned} s & \text{ square of the centre-of-mass energy} \\ \cos \theta & \text{ cosine of the centre-of-mass scattering angle} \end{aligned} \quad (1)$$

We have

$$F(s, \cos \theta) = |F(s, \cos \theta)| \exp i \phi(s, \cos \theta) \quad (2)$$

$|F|$ is the modulus of the scattering amplitude,
 $\phi(s, \cos \theta)$ is the phase. Other auxiliary variables are

k centre-of-mass momentum, defined by

$$\begin{aligned} s &= \left(\sqrt{k^2 + M_A^2} + \sqrt{k^2 + M_B^2} \right)^2, \\ t &= -2k^2(1 - \cos \theta) \end{aligned}$$

square of the momentum transfer (notice that
 t physical is negative)

$$u = 2(M_A^2 + M_B^2) - s - t. \quad (3)$$

The normalization of F is fixed if we give the expression of the differential cross-section

$$\frac{d\sigma}{d\Omega} = \frac{4}{s} |F(s, \cos \theta)|^2 \quad (4)$$

Here we see that the measurement of the differential cross-section gives us the modulus of the amplitude, not the phase. This fact will, in some way, persist in more complicated cases; for instance in spin 0 - spin $\frac{1}{2}$ scattering (pion-nucleon) a complete set of measurements will give the moduli of the spin-flip and non-spin-flip amplitudes and their relative phase but not the over-all, angular dependent phase.

Can the phase, $\phi(s, \cos\theta)$, be measured ? First of all it has to be defined. It is defined by continuity (we assume that the scattering amplitude is a continuous function of s and $\cos\theta$) in any domain of s and $\cos\theta$ where $|F(s, \cos\theta)|$ does not vanish. Continuity is necessary to avoid the $2n\pi$ ambiguity. Now direct measurements of the phase seem to be, to say the least, difficult, in particle physics. You could imagine an interference experiment such as scattering of a particle on a frozen crystal, but the wavelengths are so small compared to the crystal lattice that it is strictly impossible to do. In systems where the interparticle distance is small enough, such as deuterons, the nucleons cannot be really considered as free and there are model-dependent corrections. The only known case where phase measurements are possible, because of exceptional circumstances, is $K_0 \bar{K}_0$ regeneration. My own conclusion is that even if, in principle, one can measure phases, it is most of the time impossible in practice and one should try to accumulate as much as possible theoretical information on the phase because of the lack of direct experimental information.

The theoretical tools that we shall use are unitarity and analyticity.

A) - Unitarity

A.1) - Elastic unitarity

If the centre-of-mass energy $\sqrt{s}$ is low enough, all inelastic channels are closed and unitarity takes an extremely simple form which can be expressed in terms of

$$f(s, \vec{n}_1, \vec{n}_2) = \frac{2k}{\sqrt{s}} F(s, \cos \theta) \quad (5)$$

$\vec{n}_1$ and $\vec{n}_2$ are unit vectors in the initial and the final directions of the colliding particles, so that $\cos \theta = \vec{n}_1 \cdot \vec{n}_2$. Unitarity is expressed by the equation

$$\text{Im } f(s, \vec{n}_1, \vec{n}_2) = \int \frac{d\Omega_3}{4\pi} f^*(s, \vec{n}_1, \vec{n}_3) f(s, \vec{n}_3, \vec{n}_2) \quad (6)$$

If you prefer a more familiar and pedestrian form you can write the partial wave expansion of F (which is known to be convergent from the results of local field theory) :

$$F(s, \cos \theta) = \sum (2\ell + 1) f_\ell(s) P_\ell(\cos \theta) \quad (7)$$

Then the unitarity equation becomes

$$\text{Im } f_\ell(s) = \frac{2k}{\sqrt{s}} |f_\ell(s)|^2 \quad (8)$$

or

$$f_\ell(s) = \frac{\sqrt{s}}{2k} \exp(i\delta_\ell) \sin \delta_\ell \quad (9)$$

δ_ℓ is called a phase shift. It is unitarity which guarantees that the phase shift is real. This is the origin of the name "phase shift analysis" which really means "amplitude analysis". Indeed to reconstruct the amplitudes from the differential cross-sections, it is not necessary to go through "phase shifts". It is sufficient to find the angular dependent phase $\phi(s, \cos\theta)$ of the whole amplitude, which is the solution of the non-linear integral equation

$$|f(\vec{n}_1 \vec{n}_2)| \sin \phi(\vec{n}_1 \cdot \vec{n}_2) = \frac{1}{4\pi} \int d\Omega_3 |f(\vec{n}_1 \vec{n}_3)| |f(\vec{n}_3 \vec{n}_2)| \cos [\phi(\vec{n}_1 \vec{n}_3) - \phi(\vec{n}_2 \vec{n}_3)]_{(10)}$$

which will be discussed later.

A.2) - Inelastic unitarity

Here there is no hope to fix the phase of the elastic amplitude by solving an integral equation. Many body channels are present in the unitarity equation. However, some simple information survives about the elastic amplitude : the partial wave amplitudes have to lie inside the unitarity circle :

$$\left| \frac{2k}{\sqrt{s}} f_\ell(s) - \frac{i}{2} \right| \leq \frac{1}{2} \quad (11)$$

in particular

$$\text{Im} f_\ell(s) \geq 0 \quad (12)$$

which means that

$$\text{Im} f(s, \vec{n}_1, \vec{n}_2) \quad \text{or} \quad \text{Im} F(s, \vec{n}_1 \cdot \vec{n}_2)$$

is what is called a function of positive type over the rotation group : for any reasonable function w of the direction $\vec{n}_1$ one has

$$\int w^*(\vec{n}_1) \operatorname{Im} F(s, \vec{n}_1 \cdot \vec{n}_2) w(\vec{n}_2) d\Omega_1 d\Omega_2 \geq 0 \quad (13)$$

This looks very fancy, but if you insert in (13) the partial wave expansion of $\operatorname{Im} F$ you just find inequality (13) for every ℓ by taking $w(\vec{n}_1) = P_\ell(\vec{n}_1 \cdot \vec{n}_0)$ where $\vec{n}_0$ is some fixed direction and P_ℓ a Legendre polynomial.

It is obvious and well known that positivity restricts the phase in the forward direction because for $\theta = 0$ we have

$$\operatorname{Im} F(s, \cos\theta=1) = \sum (2\ell+1) \operatorname{Im} f_\ell(s) \quad (14)$$

therefore $\operatorname{Im} F$, a sum of positive terms, is positive [incidentally, from what I said you could believe that it might be zero. However, more refined arguments using t channel unitarity and crossing show that $\operatorname{Im} F(s, \cos\theta=1)$ cannot vanish unless the scattering amplitude is identically zero at all energies and all angles]. Since $\operatorname{Im} F = \sin \phi |F|$ we conclude

$$0 < \phi(s, \cos\theta=1) < \pi \quad (15)$$

More exactly, if at one energy s_0 we decide that $0 < \phi(s_0, \cos\theta=1) < \pi$ to remove the $2\pi n$ ambiguity, the phase, defined by continuity from this energy s_0 will satisfy inequality (15).

For non-zero angles things are more complicated. In the physical region, $-1 < \cos\theta < +1$, the imaginary part of the amplitude is not necessarily positive, even very close to the forward direction. One well-known result, which is of little help for our problem is

$$|\operatorname{Im} F(s, \cos \theta)| < \operatorname{Im} F(s, \cos \theta = 1) \quad (16)$$

This follows from

$$|P_\ell(\cos \theta)| < P_\ell(\cos \theta = 1) = 1$$

A somewhat more elaborated and interesting result is ²⁾

$$\frac{\operatorname{Im} F(s, \cos \theta)}{\operatorname{Im} F(s, \cos \theta = 1)} > -1/2 \quad (17)$$

for $\cos \theta > -1/2$.

This result can be derived either from inequalities on Legendre polynomials ²⁾, or directly from condition (13) by making a special choice for w as was done by Glaser ³⁾ (one takes w consisting of the sum of three angular delta functions).

Now let me show why inequality (17) gives some new information on the phase. You measure $(d\sigma/d\Omega)(\cos \theta)$ and you find that it is a continuous function of $\cos \theta$. There is some angular interval $0 < \theta < \theta_0$ in which

$$\frac{d\sigma}{d\Omega}(\cos \theta) > \frac{1}{4} \frac{d\sigma}{d\Omega}(\cos \theta = 1) .$$

In this interval

$$|F(s, \cos \theta)| > \frac{1}{2} |F(s, 1)| > \frac{1}{2} \operatorname{Im} F(s, 1). \quad (18)$$

If $\theta_0 < 2\pi/3$ we also have

$$\operatorname{Im} F(s, \cos \theta) > -\frac{1}{2} \operatorname{Im} F(s, \cos \theta = 1) \quad (19)$$

From (18) and (19) we see that it is impossible to have $\operatorname{Re} F = 0$ $\operatorname{Im} F < 0$; indeed, if it were so we would have from (18) $\operatorname{Im} F = -|F| < -\frac{1}{2} \operatorname{Im} F(s, 1)$ which contradicts (19). Therefore the phase $\phi(s, \cos \theta)$ cannot be equal to $-\pi/2 + 2n\pi$. However from (15) we have $0 < \phi(s, 1) < \pi$. Using continuity in θ we conclude that

$$-\frac{\pi}{2} < \phi(s, \cos \theta) < \frac{3\pi}{2} \quad (20)$$

for $0 \leq \theta \leq \theta_0 \leq \frac{2\pi}{3}$

This information on the phase may be considered as very weak from a practical point of view. On the other hand, it has the merit of coming only from the knowledge of the modulus of the scattering amplitude and has, as we shall see later, important theoretical consequences.

You may ask the following question : what can I say about the phase if I know that in the interval $0 \leq \theta \leq \theta_0$ the differential cross-section is larger than $1/5$, $1/10$ of the forward differential cross-section instead of $1/4$. Cornille and I have found an answer to this question by using detailed properties of Legendre polynomials ⁴⁾. Here I just want to give the result which is

$$|\phi(s, \cos \theta)| < \pi + \frac{(\Gamma(1/4))^4}{\pi^3} \sup_{0 \leq \theta' \leq \theta} \frac{\frac{d\sigma}{d\Omega}(s, \cos \theta = 1)}{\frac{d\sigma}{d\Omega}(s, \cos \theta') \quad (21)$$

Here we no longer have the restriction $\theta < 2\pi/3$. Let us try to understand the meaning of (21). You plot the differential cross-section and want to find a bound on the phase for a particular angle θ . You look for the minimum of $d\sigma/d\Omega$ for $\cos\theta < \cos\theta' < 1$ and insert it into the denominator of the right-hand side of (21). This gives the desired bound :

In the case with spins, similar bounds can be obtained for the helicity conserving amplitudes if the moduli of these helicity amplitudes are known from a complete set of measurements ⁵⁾.

B) - Analyticity

We shall use only well-established results following from field theory and first of all ordinary dispersion relations, which are analyticity properties in s for fixed momentum transfer t . Originally, these dispersion relations were proved for $t=0$, i.e., forward dispersion relations, but now they have been extended ; with the help of positivity to any fixed t such that $|t| < 4m_\pi^2$ ⁶⁾ where m_π is the pion mass. For such a value of t $F(s,t)$ is analytic in a

twice cut plane. The boundary value above the right-hand cut $s \text{ real} > (M_A + M_B)^2$ is the amplitude for $A+B \rightarrow A+B$. The boundary value below the left-hand cut $u \text{ real} > (M_A + M_B)^2$, i.e., in the s variable

$$t + s < (M_A - M_B)^2,$$

is the $\bar{A} + B \rightarrow \bar{A} + B$ amplitude. The discontinuities across the cut are equal to $2i$ times the "absorptive parts" in the respective channels; the absorptive part in the s channel reduces to the imaginary part of the s channel amplitude for t real. In addition we have bounds on $F(s, t)$ for real and complex s ⁶⁾⁻⁸⁾:

for

$$|t| < 4m_\pi^2, \quad |F|/|s|^2 \rightarrow 0 \text{ for } s \rightarrow \infty, \quad (22)$$

for

$$t_{\text{real}} \leq 0 \quad |F(s, t)| < s (\log s)^2. \quad (23)$$

This allows to represent F by a contour integral which reduces to a dispersion integral, i.e., an integral over the discontinuities plus subtraction terms. From (22) one sees that there are at most two subtraction terms. So, apart from two subtraction constants, $\text{Re } F(s, t)$ for s physical can be expressed as a principal value integral over $\text{Im } F(s, t)$. Conversely the so-called "inverse" dispersion relations allow us to get $\text{Im } F$ from $\text{Re } F$. This means that in the traditional language of mathematicians, $\text{Re } F$ and $\text{Im } F$ are conjugate quantities. To connect the modulus and the phase of the amplitude we can notice that

$$\log F = \log |F| + i \phi$$

therefore $\log|F|$ and ϕ are conjugate quantities, which means that ϕ can be written as a dispersion integral over $\log|F|$. The only difficulty is that $\log F$ is an analytic function only outside the zeros of F . For this reason analyticity is not sufficient to guarantee that if $|F| = |G|$ it follows that $F = G$. Indeed take

$$G = F \frac{\alpha - \sqrt{(s_0 - s)(u_0 - u)}}{\alpha + \sqrt{(s_0 - s)(u_0 - u)}} \quad (24)$$

where s_0 and u_0 are the beginnings of the s cut and the u cut. Along the cuts, i.e., in both the s and u physical regions, $|F| = |G|$ because the square root is purely imaginary. Yet F and G are different! And you can multiply F by an arbitrary number of factors like the one appearing in the right-hand side of (24)! The difference between F and G comes from the fact that G has more or less zeros in the complex plane than F for $\alpha > 0$ or $\alpha < 0$, respectively.

The zeros influence the phase of the amplitude but the converse is true. Informations on the phase give information on the number and distribution of zeros in the complex plane. The oldest theorem of that kind ⁹⁾ says that the forward amplitude, for which the phases in the s channel (above the right-hand cut) and in the u channel (below the left-hand cut) both lie between 0 and π has at most two zeros (this theorem applies to processes with no unphysical threshold, like pion-nucleon scattering, but not necessarily to kaon-nucleon scattering!).

A generalization of this is the following theorem ¹⁰⁾.
Let ϕ_P be the phase of the $AB \rightarrow AB$ amplitude, ϕ_A the phase of the $\bar{A}B \rightarrow \bar{A}B$ amplitude. If

$$|\phi_A + \phi_P| < C \quad (25)$$

independent of energy, the number of zeros of F in the complex plane is finite.

A further generalization ¹⁰⁾ is obtained if $\phi_A + \phi_P$ grows as $s \rightarrow \infty$: if $|(\phi_A + \phi_P)|/\log s \rightarrow 0$ for $s \rightarrow \infty$ the number of zeros located inside the circle of radius $|s|$, $n(|s|)$ is such that

$$n(|s|)/\log s \rightarrow 0, \quad s \rightarrow \infty. \quad (26)$$

Therefore, from information on $|\phi_A + \phi_P|$ we keep the zeros under control. From where does this information come ? This will be discussed later. One can then factor out the zeros and write

$$F(s) = G(s) \prod \left(1 - \frac{s}{s_i}\right) \prod \left(1 - \frac{s}{s_i^*}\right) \times \\ \times \prod \left(1 - \frac{s}{s_j}\right) \quad (27)$$

where $G(s)$ has no zeros. The phase of $G(s)$ coincides with the phase of $F(s)$. We can now apply to $\log G$ the standard treatment of dispersion relations and this allows us to obtain in particular the following theorem ¹⁰⁾.

If $|\phi_P - \phi_A| < C$, or, more generally if $|\phi_P - \phi_A|/\log s \rightarrow 0$ for $s \rightarrow \infty$

the only possible limit of $|G(s)/G(-s)|$ for $s \rightarrow \infty$ is unity.

In the simple case where $|\phi_A + \phi_P| < \text{const}$ the number of factors in (27) is finite :

$$\frac{F(s)}{F(-s)} = \frac{G(s)}{G(-s)} \prod \frac{1 - s/s_i}{1 + s/s_i} \prod \frac{1 - s/s_i^*}{1 + s/s_i^*} \prod \frac{1 - s/s_j}{1 + s/s_j} \quad (28).$$

The multiplying factor approaches ± 1 for $s \rightarrow \infty$. Therefore the only possible limit of $|F(s)/F(-s)|$ is unity. Therefore if, separately

$$|\phi_P| < C \quad |\phi_A| < C \quad (29)$$

$$\lim \left| \frac{F(s)}{F(-s)} \right| = 1 \quad (30)$$

(if the limit exists !). This result, however, can be extended using (26) to the case

$$\phi_P / \log s \rightarrow 0 \quad \phi_A / \log s \rightarrow 0 \quad (31)$$

Conditions (31) are in a way optimal because, if they are violated, counter-examples can be built.

Equation (30) is fundamental in the discussion of the Pomeranchuk theorem for elastic cross-sections.

3. "AMPLITUDE" OR "PHASE SHIFT" ANALYSIS

We want to apply the tools of unitarity and analyticity to the fundamental problem of amplitude reconstruction from differential and total cross-sections.

A) - Amplitude analysis as one energy ¹¹⁾

There is only hope to fix the amplitude from the differential cross-section at all angles and one energy, if this energy is below the inelastic threshold. Only then we have the non-linear

equation (10) at our disposal. At higher energies it has been explicitly shown that one has a continuum of amplitudes ¹²⁾ producing the same modulus and satisfying the unitarity inequalities (11) ; the only positive thing is that sometimes the region of ambiguity is small.

In the elastic case we have first of all an obvious ambiguity : if

$$|f(s, \cos \theta)| \exp i \phi(s, \cos \theta)$$

is a solution of the unitarity equation,

$$|f(s, \cos \theta)| \exp i [\pi - \phi(s, \cos \theta)]$$

is also a solution. This is a trivial ambiguity. Are there others ? The most obvious thing to do is to try to solve equation (10) by iteration, starting for instance with $\phi(s, \cos \theta) = 0$ on the right-hand side. It is possible to prove that the iterative solution converges if ¹³⁾⁻¹⁶⁾

$$\sup_{\text{over all angles}} \frac{\int \frac{d\Omega_3}{4\pi} |f(\vec{n}_1, \vec{n}_3)| |f(\vec{n}_2, \vec{n}_3)|}{|f(\vec{n}_1, \vec{n}_2)|} < .79 \quad (32)$$

In mathematical terms one says that if (32) holds, one has a "contraction mapping" when one goes from the right-hand side of (10) to the left-hand side of (10). Then not only the iteration converges but it is independent of the starting trial function for ϕ and it is the only solution of the equation. This looks marvellous, except for the sufficient condition (32). At best one can hope to replace the number 0.79 by 1. Even then, if one investigates the consequences of this condition one finds that it is extremely restrictive ; in particular all phase shifts for $\ell \neq 0$ have to be such that ¹¹⁾

$$|\delta_\ell| \leq \pi/6$$

Therefore, in a situation where we are close to a resonance ($\delta_l \sim \pi/2$) we know in advance that condition (32) will be violated. We have therefore no way of knowing if an iterative solution of (10) will exist and if it will be the only one. Since nothing more could be done with Eq. (10), it has been attempted to inject new ingredients, in particular the fact that we know that partial wave amplitudes are exponentially decreasing¹⁷⁾. However, even then the existence and uniqueness of the solutions is not clear ; different sufficient conditions for contraction mapping are obtained ; in some cases they are stronger, in some cases they are weaker.

The only thing one can do is to investigate the simpler situation where the number of partial waves is finite. We know that this is not true in real life because of the exponential tail of the interaction between particles at large distances. However, it may be a good approximation. Even this case is not completely solved. If $L_{\max}$ the maximum angular momentum is 2¹⁸⁾, 3¹⁹⁾ or 4²⁰⁾, it has been shown that there are at most two essentially different solutions. What is a surprise is that the solution is not always unique, even after removing the trivial ambiguity. There are explicitly examples of ambiguities ; for instance the two sets of phase shifts

$$\begin{aligned} \delta_0 &= -23^\circ 20' & \delta_1 &= -43^\circ 27' & \delta_2 &= 20^\circ \\ \delta_0 &= 98^\circ 50' & \delta_1 &= -26^\circ 33' & \delta_2 &= 20^\circ \end{aligned}$$

give exactly the same differential cross-section.

Can there be more than two solutions for $L_{\max} > 4$? There is an argument, counting the number of equations, the number of parameters, which indicates that if the equations are really independent there are never more than two solutions but it is extremely difficult to prove that the equations are independent. One extreme case has been solved²¹⁾, in which we assume that the scattering

amplitude is a genuine entire function of $\cos\theta$. This means that there are infinitely many partial waves, but that the partial wave expansion converges for arbitrary values of $\cos\theta$ outside the physical range $-1 < \cos\theta < +1$. Then one can show again that there are never more than two essentially different solutions.

In summary, the problem is not solved completely, but I, personally, believe that there are never more than 2×2 solutions (the second factor 2 is the trivial ambiguity).

Let me repeat that above the inelastic threshold a continuum of solutions exists and Atkinson²²⁾ for instance, has shown this explicitly on nN scattering and shown that some resonances may disappear.

B) - Amplitude analysis using various energies and angles

First of all let me say that if you know the modulus of the scattering amplitude at all angles from threshold up to some energy with infinite accuracy, the amplitude is fixed completely because, at sufficiently low energy, condition (32) is always satisfied (normal threshold behaviour) and one is left with the trivial ambiguity. From one energy to the next, one uses continuity, so that once a choice is made for the trivial ambiguity at one energy, it is made at all energies. So in some domain of energies and angles we have at most two amplitudes fitting the data ; but the amplitudes can each be analytically continued in a unique way. To choose between the two amplitudes one can use dispersion relations. If one adds the further constraint that the imaginary part of the forward amplitude, which is proportional to the total cross-section, should be the same for both amplitudes, one can easily find a contradiction unless the amplitudes are trivial, without inelastic threshold (which, we know, is impossible !). The conclusion is that the only ambiguity left is eliminated¹¹⁾.

However, this result is not very useful because it requires a perfect knowledge of elastic cross-sections near the elastic threshold. "Perfect" means that you must not only know the S waves but all the tiny higher waves and then extrapolate to arbitrarily high energy. This is a highly unstable process and so the previous result has a value only from the point of view of principles.

This is why Burkhardt and myself ²³⁾ have tried to see what happens if one disregards elastic unitarity. First, is two-variable analyticity enough to remove the ambiguity if one knows the modulus of the amplitude at all energies and all angles in the various physical channels $AB \rightarrow AB$, $\bar{A}B \rightarrow \bar{A}B$, $A\bar{A} \rightarrow B\bar{B}$? The answer is no and the proof is that we can have $|F| = |G|$ with

$$\frac{F}{G} = \frac{\phi(s, t, u) + \sqrt{(s_0 - s)(u_0 - u)(t_0 - t)}}{\phi(s, t, u) - \sqrt{(s_0 - s)(u_0 - u)(t_0 - t)}} \quad (33)$$

ϕ is an arbitrary real meromorphic function of s, t, u . Equation (33) is in fact the necessary and sufficient form of F/G if no unphysical threshold is present (case of pion-pion scattering, for instance).

What we have noticed, however, is that the very weak consequence of unitarity which is positivity reduces enormously the freedom we have. We have seen that the forward amplitude, for $AB \rightarrow AB$, has at most two zeros in the complex plane because of the positivity of the imaginary parts of $AB \rightarrow AB$ and $\bar{A}B \rightarrow \bar{A}B$. We know that dispersion relations also hold for $0 < t < 4m_\pi^2$. Then the imaginary parts of $AB \rightarrow AB$ and $\bar{A}B \rightarrow \bar{A}B$ are again positive because the partial wave expansion

$$\sum (2l+1) \text{Im} f_l(s) P_l\left(1 + \frac{t}{2k^2}\right)$$

is still convergent and is a sum of positive terms. So, again, for $0 < t < 4m_\pi^2$, the amplitude has at most two zeros in the s complex plane. In this case $\phi(s, t, u)$, appearing in Eq. (33), reduces to the much simpler form

$$\phi = \frac{A(t) + s B(t) + s^2 C(t)}{\alpha(t) + s \beta(t)} \quad (34)$$

(here we no longer use the equality of the moduli in the t channel). What we have found then is that by demanding normal threshold behaviour $\text{Im } F \sim \text{Im } f_0 \sim ka_0^2$ where k is the scattering length, in both the s and u channels, we can eliminate the ambiguity completely. This result gives some credibility to those who claim, like Pietarinen²⁴⁾, that phase-shift analysis gives unique results in practice when dispersion relations and inelastic unitarity are used.

4. POMERANCHUK-LIKE THEOREM FOR ELASTIC CROSS-SECTIONS

One of the problems which can be solved by the techniques developed in Section 2 is the question of the asymptotic equality of the elastic differential cross-sections for relations $A + B \rightarrow A + B$ and $\bar{A}B \rightarrow \bar{A}B$. We call this a Pomeranchuk theorem though it was not proved by Pomeranchuk. Pomeranchuk was questioning the equality of total cross-sections for AB and $\bar{A}B$. Let me summarize here the theoretical situation for the classical Pomeranchuk theorem: if the total cross-sections are bounded it is not possible to prove that $\lim \sigma_T^{AB} - \sigma_T^{\bar{A}B} = 0$ without additional assumptions; a sufficient additional assumption is that the amplitude is not dominantly real, more specifically $(\text{Re } F / \text{Im } F \log s) \rightarrow 0$. If total cross-sections are tending to infinity as $s \rightarrow \infty$ (which might be the case) one can prove from unitarity and analyticity that the only possible limit of $\sigma_T^{AB} / \sigma_T^{\bar{A}B}$ is unity²⁵⁾.

Let us return to the elastic problem. We have seen in Section 2 that if we make the assumption that the phase ϕ_P of the $AB \rightarrow AB$ amplitude and ϕ_A of the $\bar{A}B \rightarrow \bar{A}B$ amplitude are such that

$$\begin{aligned} \phi_P / \log s &\rightarrow 0 \\ \phi_A / \log s &\rightarrow 0 \end{aligned} \quad (35)$$

for some value of t , then the only possible limit, for $s \rightarrow \infty$, of $|F(s)|/|F(-s)|$ is unity¹⁰⁾. Therefore

$$\lim_{s \rightarrow \infty} \frac{d\sigma_{AB \rightarrow AB}(s, t)}{d\Omega} / \frac{d\sigma_{\bar{A}B \rightarrow \bar{A}B}(s, t)}{d\Omega} = 1. \quad (36)$$

In the early discussions of the elastic cross-section problem²⁶⁾ the models taken for the amplitudes were such that assumption (35) was automatically satisfied (in fact the phases were approaching fixed limits at infinite energy). It is no wonder that the equality of the cross-sections was obtained.

The new fact, in the present analysis, is that the phases of the $A+B \rightarrow A+B$ and $\bar{A}+B \rightarrow \bar{A}+B$ amplitudes are under control because of inequality (21). What is the definition of the diffraction peak, in the high energy limit? It is reasonable to say²⁷⁾ that the diffraction peak is that interval of momentum transfer where the differential cross-section is larger than a finite fraction of the forward differential cross-section. Therefore, inside the diffraction peak, which we take to be such that, for instance,

$$\frac{d\sigma}{d\Omega} \geq \frac{1}{100} \frac{d\sigma}{d\Omega} \text{ forward}$$

the phase of the scattering amplitude is bounded.

In the simple case where we have a fixed width diffraction peak, i.e., if $T(s)$, defined by

$$\frac{d\sigma}{d\Omega}(s, -T(s)) = \frac{1}{100} \frac{d\sigma}{d\Omega}(s, t=0)$$

is such that

$$\lim_{s \rightarrow \infty} T(s) = T \neq 0$$

we can work with fixed t dispersion relations. Inside the diffraction peak

$$|\phi_A(s, t)| < C \quad |\phi_P(s, t)| < C$$

and therefore

$$\lim_{s \rightarrow \infty} \frac{d\sigma}{d\Omega}_{AB \rightarrow AB}(s, t) / \frac{d\sigma}{d\Omega}_{\bar{A}B \rightarrow \bar{A}B}(s, t) = 1$$

Hence we can prove that the "width" of the diffraction peaks for $AB \rightarrow AB$ and $\bar{A}B \rightarrow \bar{A}B$ are asymptotically equal. We can define the width by

$$\frac{d\sigma}{d\Omega}(s, -T_W(s)) / \frac{d\sigma}{d\Omega}(s, t=0) = \frac{1}{e} ,$$

and if

$$T_W^{AB \rightarrow AB}(s) \rightarrow T_W^{AB \rightarrow AB}$$

and

$$T_W^{\bar{A}B \rightarrow \bar{A}B}(s) \rightarrow T_W^{\bar{A}B \rightarrow \bar{A}B}$$

we must have

$$T_W^{AB \rightarrow AB} = T_W^{\bar{A}B \rightarrow \bar{A}B}$$

Furthermore, the shape of the diffraction peaks must be the same.

With some technical complications which I shall not describe here ¹⁰⁾ the argument can be extended to shrinking or expanding diffraction peaks. In general we must have

$$\lim_{s \rightarrow \infty} \frac{T_W^{AB \rightarrow AB}}{T_W^{\bar{A}B \rightarrow \bar{A}B}} = 1$$

At the time when this theorem was established by Cornille and myself, the experimental evidence on pp and $p\bar{p}$ scattering was not in agreement with this prediction : it looked as if the $p\bar{p}$ peak was expanding and the pp peak was shrinking. Recent measurements at higher energies presented at the Palermo Conference and reported here by Professor Zichichi ²⁸⁾ indicate that the pp and $p\bar{p}$ slopes become approximately equal above 200 GeV. The same happens to K^+p and K^-p scattering (the slopes for π^+p and π^-p are already equal at lower energies). Experiment agrees almost too well with the asymptotic predictions ! Why is it so ? Because the phase of the amplitude is very close to $\pi/2$, much smaller than what is allowed by inequality (21).

5. "EXISTENCE" OF THE DIFFRACTION PEAK ²⁹⁾

I already said, at the beginning of these lectures, that the presence of a diffraction peak in elastic reactions at high energy was so familiar that it was not considered any more as a problem. First of all why is the differential cross-section maximum in the forward direction ? If one accepts the fact that the amplitude is purely imaginary at high energies, then there is no problem. The imaginary part of the amplitude is maximum in the forward direction, because it is a sum of Legendre polynomials with positive coefficients. Here, however, we want to take into

account the effect of the real part. Cornille and myself have succeeded in showing the existence of a peak if one assumes that the odd signature amplitude is negligible, i.e., that the dominant part of the amplitude is symmetric in the exchange of s and u .

Let us first indicate the information we shall use on the absorptive part. We want to study the slope of the diffraction peak, which, for a purely imaginary amplitude, reduces to

$$\eta_{abs} = \frac{\frac{dA}{dt}(s, t=0)}{A(s, t=0)} \quad (37)$$

(A , the absorptive part, coincides with $\text{Im } F$ for t real). On dA/dt we have the double inequality, valid for $A \gg 1$:

$$c \frac{|A(s, t=0)|^2}{s} < \left. \frac{d}{dt} A(s, t) \right|_{t=0} < c' A(s, t=0) (\log s)^2 \quad (38)$$

The right-hand side of (38) comes from the existence of an effective cut-off $L_{\max} \sim \sqrt{s} \log s$ in the partial wave distribution ^{6), 7)}; the left-hand side of (38) is a direct consequence of unitarity ³⁰⁾.

The problem is how to use analyticity in s , and $s - u$ symmetry, in conjunction with (38) to get inequalities on

$$\begin{aligned} \eta &= \frac{1}{2} \frac{d}{dt} \log \left(\frac{d\sigma}{ds} \right) = \frac{R \frac{dR}{dt} + A \frac{dA}{dt}}{R^2 + A^2} \\ &= \text{Re} \left(\frac{dF}{dt} / F \right), \end{aligned} \quad (39)$$

where $F = R + iA$

First we shall present a heuristic argument in which A and dA/dt have power or logarithmic behaviour. Let

$$\begin{aligned} A &\sim s^\alpha & \alpha \leq 1 \\ \frac{dA}{dt} &\sim s^\beta \end{aligned} \quad (40)$$

Inequality (38) gives

$$2\alpha - 1 < \beta < \alpha \quad (41)$$

Then it is very easy to estimate the corresponding real parts by using Prágmen-Lindelöf arguments or, if you prefer, the recipes of Regge theory :

$$R \frac{dR}{dt} \sim \cotan \frac{\pi\alpha}{2} \cotan \frac{\pi\beta}{2} s^{\alpha+\beta} \quad (42)$$

If $1 \geq \alpha > \frac{1}{2}$, i.e., if $\sigma_t > s^{-\frac{1}{2}}$

$$\beta > 0$$

$$\text{and } R \frac{dR}{dt} > 0.$$

Hence η is positive. Furthermore, if $\alpha > \frac{1}{2} + \epsilon$

$$\eta/\eta_{\text{abs}} < \text{const} \quad \eta_{\text{abs}} < (\log s)^2$$

and therefore

$$\eta < (\log s)^2$$

In the case $\alpha = 1$ we can make a slightly more refined treatment, including logarithmic factors. Let

$$\begin{aligned} A &\sim s (\log s)^\delta \\ \frac{dA}{dt} &\sim s (\log s)^\delta \end{aligned} \quad (43)$$

Condition (38) gives

$$2\gamma < \delta < \gamma + 2 \quad (44)$$

We easily find F and dF/dt by using s - u crossing :

$$\begin{aligned} F &\simeq i s \left(\log s - i \frac{\pi}{2} \right)^\gamma \\ \frac{dF}{dt} &\simeq i s \left(\log s - i \frac{\pi}{2} \right)^\delta \end{aligned} \quad (45)$$

Hence

$$\frac{R \frac{dR}{dt}}{A \frac{dA}{dt}} = \frac{\gamma \delta}{(\log s)^2} \quad (46)$$

Equation (46) can be negative in the case $-2 < \gamma < 0$ but $R(dR/dt)$ remains negligible as compared to $A(dA/dt)$.

This is about as far as we can go using simple arguments. We have already discovered that the existence of the peak is not linked to the dominance of the imaginary part ; indeed in the case $\frac{1}{2} < \alpha < 1$ the real and imaginary parts are of the same order of magnitude and there is nevertheless a peak. We also know that there is no hope to prove the existence of a peak for total cross-section decreasing like $s^{-\frac{1}{2}}$ or faster ; however, a rigorous treatment is needed because we have been assuming that the total cross-section is smooth, which is not necessarily the case. Among possible interpretations of present pp experimental data, some authors propose to believe that we see the beginning of an oscillatory behaviour³¹⁾. This shows the need for a more rigorous proof.

The procedure is as follows. We try to compute the quantity

$$\frac{dF}{dt}(s, t=0) / F(s, t=0)$$

for $s \rightarrow i\infty$, by writing separately dispersion relations for dF/dt and F ; we find

$$F(i|s|, 0) \sim -\frac{s^2}{\pi} \int \frac{A(s', 0) ds'}{s'(s'^2 + s^2)} \quad (47)$$

$$\frac{dF}{dt}(i|s|, 0) \sim -\frac{s^2}{\pi} \int \frac{\frac{dA(s', 0)}{dt} ds'}{s'(s'^2 + s^2)} \quad (48)$$

and combining (47) and (48) with inequalities (38) we get, after some work,

$$|s| \int \frac{A(s', 0) ds'}{s'(s'^2 + s^2)} < \frac{\frac{dF}{dt}(i|s|, 0)}{F(i|s|, 0)} < C (\log |s|)^2, \quad (49)$$

$s \rightarrow i\infty.$

We now write an inverse dispersion relation for $(dF/dt)/F$ which gives

$$\begin{aligned} \frac{1}{|s|} \frac{\frac{dF}{dt}(is, 0)}{F(is, 0)} &\sim \frac{1}{\pi} \int \frac{\operatorname{Re} \left[\frac{dF}{dt} / F \right] ds'}{s'^2 + s^2} \\ &= \frac{1}{\pi} \int \frac{\eta(s') ds'}{s'^2 + s^2} \end{aligned} \quad (50)$$

Consistency of (50) with inequality (49) requires

$$(\log s)^2 > \eta(s) > c s^{-1/2} \quad (51)$$

provided $s^{1/2} \sigma_{\text{tot}} \rightarrow \infty$, at least for a sequence of energies. We find essentially the same result as in the heuristic argument but we have eliminated the smoothness assumption. Of course, our result holds only for a sequence of energies approaching infinity. This is unavoidable in all problems of that kind, when one tries to obtain results on the behaviour of an analytic function along its cut.

Furthermore, if $\eta_{\text{abs}}(s)$ is slowly varying [like, for instance $(\log s)^\alpha$] we succeeded also in proving that η/η_{abs} has finite, non-zero upper and lower bounds.

6. CAN ONE PROVE GEOMETRIC SCALING OF THE DIFFRACTION PEAK ?

I had not intended to include this topic in my lectures but since the recent data on pp scattering at ISR²⁸⁾ indicate that : i) the small t peak shrinks with energy ; ii) the diffraction minimum also moves towards small t , it may be worth while to try to remember the theoretical situation.

We know that the total cross-section satisfies the bound^{6),7),32)}

$$\sigma_t < \frac{\pi}{m_\pi^2} (\log s)^2. \quad (52)$$

If

$$\sigma_t \sim c (\log s)^2 \quad (53)$$

where the constant c may be a very small fraction of π/m_π^2 , we are in an extremal situation, with various interesting consequences which have been discussed some years ago by Auberson, Kinoshita and myself³³⁾, at a time where there was no indication that σ_t was rising. First of all,

$$\sigma_{el} \sim (\log s)^2 \quad (54)$$

Therefore

$$(\sigma_t)^2 / \sigma_{el} \sim (\log s)^2 \quad (55)$$

The most spectacular consequence is that the reduced scattering amplitude

$$F(s, t) / F(s, 0)$$

exhibits geometric scaling

$$\frac{F(s, t)}{F(s, 0)} \sim f\left(t \frac{(\sigma_t)^2}{\sigma_{el}}\right) \quad (56)$$

Naturally

$$t \frac{(\sigma_t)^2}{\sigma_{el}} \sim t (\log s)^2$$

but this form fixes the scale and has the advantage of making unitarity manifest in the sense that if we neglect $\text{Re } F$,

$$\begin{aligned}
 \sigma_{el} &\simeq C \int_{-s}^0 \left| \frac{F(s,t)}{s} \right|^2 dt \geq \\
 &C' \left| \frac{F(s,0)}{s} \right|^2 \int_{-\infty}^0 \frac{\sigma_{el}}{(\sigma_t)^2} |f(z)|^2 dz \\
 &= C'' \sigma_{el} \int_{-\infty}^0 |f(z)|^2 dz .
 \end{aligned} \tag{57}$$

The function $f(z)$ is proved to be an entire function (i.e., analytic for all values of z) of order $\frac{1}{2}$ (i.e., bounded by $\exp\sqrt{|z|}$ for $|z| \rightarrow \infty$) which is not trivial (i.e., not identically equal to 1). Incidentally such a function has many zeros and one of them may be close to the real axis and produce a diffraction minimum.

Present experimental data are compatible with a $(\log s)^2$ rise of σ_t ; they also indicate a rise of σ_{el} , and it is, of course, very tempting to believe this because the geometrical scaling follows automatically. Let me indicate that this is what happens with field theory models which saturate the Froissart bound.

However, we may have a rise slower than $(\log s)^2$. Then (56) is not really necessary but may be postulated as a phenomenological fit ³⁴⁾ which seems rather good.

Note added after the lectures

Stimulated by the interest in geometrical scaling at the Erice School, Cornille and myself have tried to find a more general basis for (56). To date, our results are as follows : neglect the real part of the amplitude. Define the width of the diffraction peak by $T(s)$ such that

$$\frac{d\sigma}{dt}(s, -T(s)) / \frac{d\sigma}{dt}(s, 0) = 1/2$$

then if

$$\frac{\frac{dA}{dt}(s, t=0)}{A(s, t=0)} \times T(s) < \text{const}, \quad (58)$$

for $s \rightarrow \infty$,

$$A(s, t)/A(s, 0) \rightarrow f(t/T(s))$$

for some sequence of energies approaching infinity. $f(z)$ is now a continuous function, not identically equal to unity, but it is only defined for real $z \leq 0$. It is not necessarily analytic.

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