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DYNAMICAL THEORY OF QUARK CONFINEMENT AND HADRON SPECTRUM

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A B S T R A C T

A dynamical approach to quark confinement and, being on equal footing, the hadron spectrum is presented on field-theoretical grounds. The basic ingredient will be the two-body Bethe-Salpeter kernel which governs both the Schwinger-Dyson equation for the quark propagator and the quark-antiquark bound state spectrum which we will restrict to. The Bethe-Salpeter kernel is inferred from the β function directly in momentum space. For the β function we only assume that it has no infra-red stable zero. The quark propagator turns out to be a constant in momentum space (i.e., a δ function in configuration space) consistent with quark confinement and absence of free quark-like states. The bound states are found to lie on linear Regge trajectories.

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1. - INTRODUCTION

A satisfactory dynamical model of hadrons and their interactions should, among others, exhibit the following three features :

- i) the fundamental constituents of hadrons are quarks, and hadrons are classified as in the naive quark model ;
- ii) the hadron spectrum generates rising Regge trajectories (preferably linear) ;
- iii) quarks are confined.

It is the aim of this paper to incorporate these features into the framework of (relativistic) quantum field theory.

Over the past couple of years various quark constituent models have been proposed where quarks are bound in a long-range effective potential ¹⁾⁻⁷⁾. The R^2 potential (harmonic oscillator) is preferably chosen as to give linear Regge trajectories. In all these models quarks are treated as quasi-free particles which is highly inconsistent. As is well known, the quark propagator, entering in the bound state equations, cannot be chosen independently of the interaction but is given by the Schwinger-Dyson equation which involves the interaction kernel. In Ref. 8) it has been shown that the quark propagator is drastically changed under such long-range forces. This casts many doubts upon what has been gained. In particular, it implies that the non-relativistic limit is not justifiable.

None of these models is consistent with quark confinement. Though the hadron wave functions do not have quark-mass singularities due to the long-range forces which guarantees that freely moving quarks are absent from, e.g., the diagrams shown in Fig. 1, higher order diagrams such as, e.g., shown in Fig. 2, will - however - acquire quark-mass singularities, in contradiction to quark confinement.

Under quark confinement we shall understand, more precisely, the situation where the physical Hilbert space does not contain any state with quark quantum numbers. This might not be the only possible definition, but conceptually it is the most straightforward one. A corollary of this definition is that the quark propagator should (in momentum space) have no discontinuity in the mass squared variable ^{9),10)}.

Despite the fact that, fundamentally, little has been achieved, we believe the field theoretical approach to be particularly suited to tackle (i)-(iii). Once the Bethe-Salpeter (BS) kernel, say, of the quark-antiquark system is determined, the quark propagator as well as the static and electromagnetic properties of the quark-antiquark system such as the meson spectrum, form factors, deep inelastic structure functions, etc., can be calculated in a straightforward way. In this respect, e.g., the MIT bag ¹¹⁾ meets with great difficulties. As to the link to any specific field theoretical model, the BS kernel can be inferred from the properties of the β function directly in momentum space. This, once more, puts us in a favourable position as we do not have to speculate about the Higgs field.

The massive quark model ¹²⁾ combines (i)-(iii) into a model of hadrons in an unconventional way. As we are going along, we shall find our work to support some of the assumptions going into this approach.

The paper is organized as follows. In Section 2 we shall present the basic ingredients of our calculation including a brief recital of the BS framework. The main part of this section is devoted to specifying the BS kernel. In Section 3 we calculate the quark propagator, being the critical quantity as to quark confinement. Finally, in Section 4 we discuss the bound state spectrum.

2. - FORMULATION OF THE MODEL

In this paper we shall restrict ourselves to spin-zero quarks. Quark quantum numbers will also be suppressed. We believe that everything what is said also holds for the more realistic case of spin $\frac{1}{2}$ quarks. Furthermore, we shall only consider mesonic bound states, i.e., the quark-antiquark sector.

The main ingredient will be the quark-antiquark BS kernel K which stands for the sum over all two-particle irreducible diagrams. In terms of K , the quark propagator

$$\Delta_F(q) = \int d^4x e^{iqx} \langle 0 | T(q(x) q^\dagger(0)) | 0 \rangle \quad (2.1)$$

is determined by the integral equation (Fig. 3)

$$\Delta_F^{-1}(q) = Z \Delta_0^{-1}(q) - i \int d^4k K(q, k, P=0) \Delta_F(k),$$

$$\Delta_0(q) = \frac{1}{q^2 - m_q^2}, \quad (2.2)$$

where m_q is the bare mass of the quark and Z denotes the wave function renormalization constant. Knowing $\Delta_F(k)$, the mesonic bound states, being described by the BS amplitudes

$$\phi_P(q) = (2\pi)^{\frac{3}{2}} \int d^4x e^{iqx} \langle 0 | T(q(\frac{x}{2}) q^\dagger(-\frac{x}{2})) | \phi_P \rangle, \quad (2.3)$$

can then be calculated from the BS equation^{*)} (Fig. 4)

$$\phi_P(q) = \Delta_F(\frac{1}{2}P+q) \Delta_F(\frac{1}{2}P-q) i \int d^4k K(q, k, P) \phi_P(k). \quad (2.4)$$

In our aim to determine the BS kernel, perturbation theory will not be of any help^{**) .} Any successful BS kernel must encounter long-range forces which cannot be traced back to a few diagrams. So a reasonable thing to do is to orientate at the renormalization group equations.

In order to have long-range forces, the β function must have no infra-red stable zero. It is a wide-spread hope that this is going to hold in non-Abelian gauge theories^{14),15)}. In view of this we shall phrase the quark interaction in terms of an effective vector gluon exchange. The β function is conjectured to be (typically) of the form shown in Fig. 5. This has been impossible to prove because of the irrelevance of perturbation theory, though there are a couple of arguments supporting this conjecture^{16),17)}.

*) See also the discussion of the inhomogeneous BS equation at the beginning of Section 4.

**) See, e.g., Ref. 13) which shows that even non-Abelian gauge theories are infra-red finite in low-order perturbation theory.

We shall assume the β function having the form sketched in Fig. 5. This will probably be at the bottom of any successful field theory incorporating (i)-(iii). By further specification of β one may obtain for the effective quark-gluon coupling constant (in agreement with asymptotic freedom)

$$g(t) \underset{t \rightarrow 0_-}{\sim} t^{-\frac{\kappa}{2}} + O(t^{-\frac{\kappa}{2}+1}) , \quad \beta(g) \sim g , \quad (2.5)$$

t being the momentum transfer. The basic (effective) one-gluon exchange diagram (Fig. 6), as being given by (2.5), then reads

$$(P+q+k)_\mu g^{\mu\nu} D(t) (P-q-k)_\nu , \quad t = (q-k)^2 , \quad (2.6)$$

where, for the gluon propagator (e.g.) being constant near $t \approx 0$,

$$D(t) \underset{t \approx 0}{\simeq} c_\kappa t^{-\kappa} + c_{\kappa-1} t^{-\kappa+1} + \dots \quad (2.7)$$

In the following we shall take $\kappa = 3$, mainly for definiteness. Qualitatively, our results will not depend on the precise value of κ as we shall see. We shall comment on the general situation at given times. Furthermore, for our purpose it is sufficient to retain only the first two terms in the expansion (2.7), i.e., the long-range part. As it stands, (2.7) is a generalized function having an algebraic singularity (in κ for integer $\kappa \geq 2$). This is regularized in the canonical fashion¹⁸⁾ [note that the coefficients in (2.7) are undetermined] giving

$$D(t) = -i\beta \square_{(q-k)} \delta(q-k) + i\alpha \delta(q-k) . \quad (2.8)$$

Here, α and β are both taken to be positive. In Euclidean configuration space we obtain from (2.8) :

$$D(R^2) = \beta R^2 - \alpha , \quad R^2 = \vec{x}^2 + x_4^2 . \quad (2.9)$$

The corresponding (effective) one-gluon exchange potential has the form shown in Fig. 7, which is a typical confinement potential. This justifies our choice of constants α and β . At short distances our approximation of what is believed to be the true potential (Fig. 7) is somewhat poor. But we shall need to know the potential only in the shaded region.

Generally, the BS kernel will acquire higher-order contributions beyond the (effective) one-gluon exchange, though the latter is probably dominant at larger distances. Since it is the shaded area in Fig. 7 which will be of major importance for the spectrum, medium range forces are expected to also play a rôle.

The crossed two-gluon exchange will not give any new contribution. The next contribution one can envisage is shown in Fig. 8. When summed over all gluons this is (a posteriori) equivalent to the sum over all two-meson intermediate states. Clearly, this forms a regular contribution to the BS kernel. Topologically, diagram 8 belongs to the Pomeron exchange.

The Pomeron is the leading regular contribution to the BS kernel ^{*)} with respect to P^2 and also (most likely) with respect to the range of interaction. The latter follows from its sharp forward peaking in momentum space, being a consequence of the infra-red singularities encountered, e.g., in diagram 8. Note also that the Pomeron does not mix quarks of different flavours which should be kept in mind as we come to the spectrum. The Pomeron contribution can, for small momentum transfer (i.e., medium/long distance) and high P^2 , be written

$$i g^2 ((q-k)^2) P^2. \quad (2.10)$$

This is just the region where it will play a rôle in our following considerations (cf. earlier this section). The coupling g^2 will be sharply peaked at $(q-k)^2 = 0$. This means that its Fourier transform is expected to be smooth. The exact form of g^2 will be irrelevant for our aims.

^{*)} For the distinct rôle of the Pomeron within the context of confined quarks, see also Ref. 19).

Altogether, the BS kernel then has the form

$$K(q, k, P) = -i\beta \square_{(q-k)} \delta(q-k) + i\alpha \delta(q-k) + i g^2 ((q-k)^2) P^2 + \dots \quad (2.11)$$

The dots stand for the non-leading regular contribution.

3. - THE QUARK PROPAGATOR

In Ref. 8), the Schwinger-Dyson equation (2.2) has been carefully studied for the BS kernel $K(q, k, P=0) = -i\beta \square_{(q-k)} \delta(q-k)$ ($= \beta R^2$, i.e., a harmonic oscillator in Euclidean configuration space), giving

$$\Delta_F^{-1}(q) = Z(q^2 - m_q^2) - 2\beta \frac{d^2}{d(q^2)^2} q^2 \Delta_F(q) \quad (3.1)$$

This is a non-linear differential equation with movable singularities [i.e., the zeroes of $\Delta_F(q)$]. To be consistent with quark confinement (see Section 1), $\Delta_F(q)$ should be an entire function (i.e., have no cut in q^2). The only entire solution Eq. (3.1) admits is $\Delta_F(q) = -(Zm_q^2)^{-1}$ with $Z \rightarrow 0$, $m_q^2 \rightarrow \infty$ but $Zm_q^2 \rightarrow \text{constant}$.

In case of the slightly more complicated kernel (2.11) the Schwinger-Dyson equation (2.2) reads

$$\Delta_F^{-1}(q) = Z(q^2 - m_q^2) + 2\beta \left[g_{\mu\nu} \frac{\partial}{\partial q_\mu} q^2 \frac{\partial}{\partial q_\nu} - \frac{\alpha}{\beta} q^2 + 2 \right] \Delta_F(q) \quad (3.2)$$

Note that the Pomeron contribution (2.10) does not contribute ($P \equiv 0$). The regular part has been left out since it cannot be relevant for confinement (see later). Equation (3.2) has the constant solution

$$\Delta_F(q) = \frac{Z}{2\alpha}, \quad Z^2 = \frac{\alpha}{\frac{\beta}{\alpha} - \frac{m_q^2}{2}} \quad (3.3)$$

In contrast to $Z=0$ in Eq. (3.1) we have (fortunately), however, $Z \neq 0$ which tells us via the vertex equations ²⁰⁾ that the photon couples point-like to the quarks. We did not examine Eq. (3.2) as carefully as Eq. (3.1), but very likely the constant propagator will remain the only entire solution. In fact, the singularity structure of Eq. (3.2) is mainly determined by the non-linear term [i.e., $\Delta_F^{-1}(q)$].

In configuration space, Eq. (3.3) becomes

$$\Delta_F(x-y) = \frac{Z}{2\alpha} \delta(x-y) \quad (3.4)$$

which means that quarks do not propagate. Such a quark propagator has also been obtained in Wilson's lattice gauge theory in the strong coupling limit ²¹⁾. Moreover, it has also gone into the massive quark model ⁹⁾.

As to the regular contribution omitted in Eq. (3.2), it can be shown that the constant propagator is consistent with adding a (regular) convolution type spin-zero and spin-one exchange to the BS kernel. In particular, this holds for elementary spin-zero and spin-one exchange with asymptotic freedom-type high energy behaviour.

Coming back to the more general situation of κ being different from 3 (but $\kappa > 2$) as, e.g., one finds in case of a linear confinement potential, it can be inferred from Ref. 18) that for $4 > \kappa \geq 2$, Eq. (3.2) remains roughly the same ^{*}). For $\kappa \geq 4$ even higher derivatives add to Eq. (3.2) so that the constant propagator comes out quite generally.

The constant propagator can be thought as coming from a Lehmann representation with one subtraction and the quark mass m_q going to infinity, i.e.,

$$\Delta_F(q) = \Delta_F(0) + q^2 \int_{m_q^2}^{\infty} d\mu^2 \frac{\sigma(\mu^2)}{(q^2 - \mu^2)\mu^2} \quad (3.5)$$

^{*}) The main difference lies in the regularization of the algebraic singularity ¹⁸⁾ (see Section 2) which, however, has no effect on the form of Eq. (3.2). Note that $\beta=0$ for $3 > \kappa \geq 2$.

4. - BETHE-SALPETER EQUATION AND SPECTRUM

In the classical paper of Gell-Mann and Low ²²⁾ it is shown that the BS amplitude

$$\phi_p(x_1, x_2) = \langle 0 | T(\psi(x_1) \bar{\psi}(x_2)) | \phi_p \rangle \quad (4.1)$$

satisfies the inhomogeneous integral equation

$$\begin{aligned} \phi_p(x_1, x_2) G_p = & \lim_{t \rightarrow -\infty} \int d^3\vec{x}_3 d^3\vec{x}_4 \Delta_F(x_1 - (t, \vec{x}_3)) \Delta_F(x_2 - (t, \vec{x}_4)) \phi_p((t, \vec{x}_3), (t, \vec{x}_4)) \\ & - G_p \int d^4x_5 d^4x_6 d^4x_7 d^4x_8 \Delta_F(x_1 - x_5) \Delta_F(x_2 - x_6) K(x_5, x_6, x_7, x_8) \phi_p(x_7, x_8) \end{aligned} \quad (4.2)$$

where

$$G_p = \int d^3\vec{x} \bar{\phi}_p((0, \vec{x})) \phi_p((0, \vec{x})), \quad \phi_p(x) = \langle 0 | T(\psi(\frac{x}{2}) \bar{\psi}(-\frac{x}{2})) | \phi_p \rangle$$

and the L limit being defined by

$$\lim_{t \rightarrow -\infty} f(t) = \lim_{t \rightarrow -\infty} \frac{1}{|t|} \int_{2t}^t d\tau f(\tau). \quad (4.3)$$

In a standard theory such as that considered in Ref. 22), Eq. (4.2) describes the scattering states. As for the bound states defined by $P^2 < 4m_q^2$ (here, m_q being the physical mass of the constituents), the inhomogeneous term vanishes because of the mass condition. In our case the mass of the constituents is not defined so that we cannot use this argument. Inserting (3.4) into Eq. (4.2), we obtain

$$\begin{aligned} & \lim_{t \rightarrow -\infty} \int d^3\vec{x}_3 d^3\vec{x}_4 \Delta_F(x_1 - (t, \vec{x}_3)) \Delta_F(x_2 - (t, \vec{x}_4)) \phi_p((t, \vec{x}_3), (t, \vec{x}_4)) \\ & = \left(\frac{Z}{2\alpha}\right)^2 \delta(x_1^0 - x_2^0) \lim_{t \rightarrow -\infty} \frac{1}{|t|} \int_{2t}^t d\tau \delta(x_1^0 - \tau) \phi_p((\tau, \vec{x}_1), (\tau, \vec{x}_2)) = 0, \end{aligned} \quad (4.4)$$

i.e., the inhomogeneous term in Eq. (4.2) drops out for all P^2 (and also for $x_1^0 = x_2^0$ on test functions). This means that the scattering states (i.e., the continuum of free quark states) are totally removed from the theory as it is expected from the quark propagator (3.3).

Let us now consider the quark-antiquark bound states, i.e., what remains left in the $q\bar{q}^+$ sector. This means to solve the BS equation (2.4) on the accessory field theoretical normalization condition ²³⁾

$$(\phi_P, \frac{\partial}{\partial P^2} (\Delta_F^{(1)-1} \Delta_F^{(2)-1} - K) \phi_P) = 1. \quad (4.5)$$

From Eqs. (2.4), (2.11) and (3.3), we derive the bound state wave equation

$$\begin{aligned} \phi_P(q) = & \left(\frac{Z}{2\alpha}\right)^2 \int d^4k \{ (-\beta \square_{(q-k)} + \alpha) (P^2 - (q+k)^2) \delta(q-k) \\ & + g^2 ((q-k)^2) P^2 \} \phi_P(k). \end{aligned} \quad (4.6)$$

Here, the Pomeron now comes up for part of the kinetic energy of the quark-antiquark system replacing the quark propagators. We assume that we can perform the Wick rotation/continuation. In Euclidean momentum space we then obtain

$$\begin{aligned} \left[\sum_{i=1}^4 \frac{\partial}{\partial q_i} \left(\frac{P^2 + q^2}{4} \right) \frac{\partial}{\partial q_i} + \frac{\alpha}{\beta} \left(\frac{P^2 + q^2}{4} \right) + \frac{\alpha^2}{\beta Z^2} + i \right] \phi_P(q) \\ + \frac{1}{4\beta} P^2 \int d^4k g^2 ((q-k)^2) \phi_P(k) = 0. \end{aligned} \quad (4.7)$$

It will be profitable to solve Eq. (4.7) in configuration space. There it turns out to be an eigenvalue equation in P^2 . Furthermore, it will make transparent that the short-distance behaviour and the precise form of the long-range forces are also irrelevant for the global structure of the bound state spectrum. After performing the Fourier transformation, Eq. (4.7) becomes

$$\begin{aligned} \left[\sum_{i=1}^4 \frac{\partial}{\partial x_i} \left(\frac{\alpha}{\beta} - R^2 \right) \frac{\partial}{\partial x_i} - \frac{1}{4} P^2 \left(\frac{\alpha}{\beta} - R^2 \right) - \frac{1}{4\beta} P^2 g^2(R^2) \right. \\ \left. - \frac{\alpha^2}{\beta Z^2} - i \right] \phi_P(x) = 0, \end{aligned} \quad (4.8)$$

where $R^2 = \vec{x}^2 + x_4^2$ and $\bar{\phi}_P(x)$ and $g^2(R^2)$ are the Fourier transforms of $\bar{\phi}_P(q)$ and $g^2((q-k)^2)$, respectively. The normalization condition (4.5) reads (in the Euclidean)

$$\int d^4q d^4k \bar{\phi}_P(q) \{ (\beta \square_{(q-k)} + \alpha) \delta(q-k) + g^2((q-k)^2) \} \phi_P(k) \\ = \int d^4x \bar{\phi}_P(x) (\alpha - \beta R^2 + g^2(R^2)) \phi_P(x) = 1. \quad (4.9)$$

Substituting

$$\chi_P(x) = \sqrt{\frac{\alpha}{\beta} - R^2} \phi_P(x), \quad (4.10)$$

we obtain the "wave equation"

$$[\square_x - \frac{1}{4} P^2 - \mathcal{U}(R)] \chi_P(x) = 0 \quad (4.11)$$

with the energy dependent "potential" (beware of the negation sign of $\frac{1}{4}P^2$)

$$\mathcal{U}(R) = \frac{\frac{\alpha^2}{\beta R^2} - 2 + \frac{1}{4\beta} P^2 g^2(R^2)}{\frac{\alpha}{\beta} - R^2} - \frac{R^2}{(\frac{\alpha}{\beta} - R^2)^2}. \quad (4.12)$$

In Eq. (4.11), $\chi_P(x)$ can be expanded in terms of hyperspherical harmonics $Y_{nlm}(\psi, \theta, \phi)$:

$$R \chi_P(x) = \sum_{nlm} c_{nlm} g_n(R) Y_{nlm}(\psi, \theta, \phi). \quad (4.13)$$

Furthermore, the d'Alembertian can be expressed as follows ²⁴⁾:

$$\square_x = \frac{1}{R^3} [(R \frac{d}{dR})^2 - (N^2 + 1)] R = 4 \frac{d^2}{d(R^2)^2} R^2 - \frac{1}{R^2} N^2, \quad (4.14)$$

where N^2 is the $O(4)$ Casimir operator, diagonal on the Y_{nlm} , i.e.,

$$N^2 Y_{nlm} = n(n+2) Y_{nlm}. \quad (4.15)$$

Writing $r = \sqrt{\frac{\beta}{\alpha}} R$ and

$$y_n(r) = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{4}} \sqrt{r} f_n(R), \quad (4.16)$$

we finally obtain the Schrödinger-like equation

$$y_n''(r) - \left[E + \frac{\frac{\alpha^2}{\beta^2} - 2 + \frac{1}{\alpha} E g^2\left(\frac{\alpha}{\beta} r^2\right)}{1 - r^2} - \frac{r^2}{(1 - r^2)^2} + \frac{(n+1)^2 - \frac{1}{4}}{r^2} \right] y_n(r) = 0, \quad (4.17)$$

where $E = \frac{\alpha}{4\beta}$, together with the normalization condition

$$\int_0^1 dr \bar{y}_n(r) \left[1 + \frac{1}{\alpha} \frac{g^2\left(\frac{\alpha}{\beta} r^2\right)}{1 - r^2} \right] y_n(r) = O(1). \quad (4.18)$$

Note that $y_n(1) = 0$.

The energy-dependent "potential" entering in Eq. (4.17) is qualitatively drawn in Fig. 9 for a smooth g^2 . This indicates that the situation is very much like an infinite well (note again the negative sign of E). At short distances the potential will for large n be governed by the centrifugal barrier. In the framework of asymptotically free gauge theories, this will be true for all n ($n=0,1,\dots$). Here, we shall mainly be interested in the variety of the spectrum (Regge trajectories, large n) so that, in any case, the short-distance behaviour will not be of importance. Figure 9 also indicates (finally) that we need not know the precise large-distance behaviour of the BS kernel. Any other κ ($\kappa > 2$, see Section 2) will qualitatively lead to the same type of "potential" (Fig. 9). We notice that it is the shaded region of Fig. 7, i.e., the zero of $D(R^2)$ [Eq. (2.9)] which transforms into the infinite well. Moreover, as can be deduced from Fig. 9, the quarks will be bound to the infinite well. This means, the relative quark separation will centre around $r \approx 1$. Hence, g^2 can be well approximated by

$$g^2\left(\frac{\alpha}{\beta} r^2\right) \approx g^2\left(\frac{\alpha}{\beta}\right) \equiv g_0^2, \quad (4.19)$$

i.e., retaining the residual part of the potential only. Again, it occurs that the precise form of g^2 is unimportant (as long as it is smooth).

Equation (4.17) can be solved for large E . The solution is discussed in some detail in Appendix A. The results are as follows :

$$y_{n,k}(r) \underset{E \text{ large}}{=} r^{\frac{3n+3}{2}} \sqrt{1-r^2} e^{(n + \sqrt{n^2 + E(1 + \frac{g_0^2}{4\alpha}) + \frac{n}{2}})(1-r)} \\ \times L_k \left(-2 \sqrt{n^2 + E(1 + \frac{g_0^2}{4\alpha}) + \frac{n}{2}} (1-r) \right), \quad (4.20)$$

where L_k are Laguerre polynomials and $k=0,1,\dots$ is the hyper-radial quantum number. The corresponding mass spectrum is

$$E \underset{n \text{ large}}{=} \frac{2\alpha}{g_0^2} (1+k) n, \quad (4.21)$$

i.e., a linear Regge trajectory. The more general form of the trajectory for large E but finite n (i.e., large k) can be found in Appendix A. The characteristics of the Regge spectrum are sketched in Fig. 10. There is a degeneracy with respect to the angular momentum ℓ . Hyper-radially excited states ($k=1,2,\dots$) are not lying on parallel daughters as one is used to from the harmonic oscillator model but have a smaller slope which decreases as k is increased. Altogether, the eigenvalue spectrum forms a complete set of states.

As to the normalization condition, there is no problem with negative "relative-time parity" states which in the conventional BS framework lead to negative norm²⁶⁾. In our case, the wave function $\phi_P(q)$ has no cut due to quark confinement which implies $\bar{\phi}_P(q) = \phi_P^*(q)$. Moreover, there is no danger that the norm (4.18) becomes negative since $y_n(r)$ is exponentially damped for $r > 1$.

In the general case, Eq. (4.17) [with (4.19)] cannot be solved, but a lot of information on the solution can be derived from the behaviour at the singular points. An account of the singularity structure of Eq. (4.17) will be given in Appendix A.

The BS wave functions have no direct physical interpretation like the Schrödinger wave functions. But they define a set of hadronic states $\{|\Phi_p\rangle\}$ which span up the physical Hilbert space. Since there are no scattering states and the bound state spectrum is complete, our physical Hilbert space (strictly speaking the qq^+ sector) is made up of quark-antiquark bound states [of quantum numbers n, ℓ, k and mass (A.8)].

In a forthcoming paper ²⁸⁾ we shall continue calculating the form factor and deep-inelastic structure function of mesonic bound states.

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APPENDIX A

Let us write $u = 1-r$ and

$$y_n(r) = r^\lambda \sqrt{1-r^2} \, g(u), \quad \lambda = \frac{2n+3}{2}. \quad (\text{A.1})$$

Equation (4.17) then becomes

$$g'' - \left[\frac{2\lambda}{1-u} - \frac{2(1-u)}{u(2-u)} \right] g' - \left[E + \frac{\frac{\alpha^2}{\beta^2 Z^2} - 1 + \frac{1}{\alpha} E g_0^2 + 2\lambda}{u(2-u)} \right] g = 0. \quad (\text{A.2})$$

Making the change of variables $t = \Theta u$ where $\Theta \rightarrow \infty$ as $E \rightarrow \infty$ (e.g., $\Theta = \sqrt{E}$) and writing $\eta(t) = g(u)$, Eq. (A.2) reduces to the confluent hypergeometric differential equation

$$\eta'' + \left[-\frac{2n}{\Theta} + \frac{1}{t} \right] \eta' - \left[\frac{E(1 + \frac{1}{\alpha} g_0^2) + \frac{\alpha^2}{\beta^2 Z^2} - 2}{\Theta^2} + \frac{\frac{1}{\alpha} E g_0^2 + \frac{\alpha^2}{\beta^2 Z^2} - 2}{\Theta} \frac{2}{t} \right] \eta = 0 \quad (\text{A.3})$$

for large E . Equation (A.3) has the general solution ²⁵⁾

$$\eta(t) = e^{(n + \sqrt{n^2 + E(1 + \frac{g_0^2}{4\alpha}) + \frac{n}{2}}) \frac{t}{\Theta}} \left[C_1 M(\delta, 1, \tau) + C_2 U(\delta, 1, \tau) \right], \quad (\text{A.4})$$

where

$$\delta = \frac{-\frac{1}{2} \left(\frac{\alpha^2}{\beta^2 Z^2} - 2 + \frac{1}{\alpha} E g_0^2 \right) + \sqrt{n^2 + E(1 + \frac{g_0^2}{4\alpha}) + \frac{n}{2}}}{2 \sqrt{n^2 + E(1 + \frac{g_0^2}{4\alpha}) + \frac{n}{2}}} \quad (\text{A.5})$$

and

$$\tau = - \frac{2t}{\Theta} \sqrt{n^2 + E \left(1 + \frac{g_0^2}{4\alpha}\right) + \frac{n}{2}}. \quad (\text{A.6})$$

The function M is regular at $\tau=0$ ($r=1$) whereas U has a logarithmic singularity at $\tau=0$.

We must now impose that $\eta(t)$ be finite at $t=-\infty$ ($r=+\infty$). The asymptotic behaviour of (A.4) is

$$\begin{aligned} \eta(t) \underset{t \rightarrow -\infty}{\sim} e^{(n + \sqrt{n^2 + E \left(1 + \frac{g_0^2}{4\alpha}\right) + \frac{n}{2}}) \frac{t}{\Theta}} & \left\{ \frac{c_1}{\Gamma(\delta)} e^{\tau} \tau^{\delta-1} \right. \\ & + \left[\frac{c_1}{\Gamma(1-\delta)} + c_2 (-1)^{-\delta} \right] (-\tau)^{-\delta} \Big\} \\ \underset{(r \rightarrow +\infty)}{=} \frac{c_1}{\Gamma(\delta)} e^{(n - \sqrt{n^2 + E \left(1 + \frac{g_0^2}{4\alpha}\right) + \frac{n}{2}}) (1-r)} & \left[-2 \sqrt{n^2 + E \left(1 + \frac{g_0^2}{4\alpha}\right) + \frac{n}{2}} (1-r) \right]^{\delta-1} \\ & + \left[\frac{c_1}{\Gamma(1-\delta)} + c_2 (-1)^{-\delta} \right] e^{(n + \sqrt{n^2 + E \left(1 + \frac{g_0^2}{4\alpha}\right) + \frac{n}{2}}) (1-r)} \\ & \times \left[2 \sqrt{n^2 + E \left(1 + \frac{g_0^2}{4\alpha}\right) + \frac{n}{2}} (1-r) \right]^{-\delta}. \end{aligned} \quad (\text{A.7})$$

For $t \rightarrow -\infty$ ($r \rightarrow +\infty$) the first term in Eq. (A.7) goes to infinity unless $-\delta = k = 0, 1, \dots$. The latter condition leads to the desired discretization of E :

$$\begin{aligned}
 E &= F + \sqrt{F^2 + G}, \\
 F &= \frac{4\alpha^2(2k+1)^2(1 + \frac{g_0^2}{4\alpha}) - 2\alpha(\frac{\alpha^2}{\beta Z^2} - 2)g_0^2}{g_0^2}, \\
 G &= \frac{4\alpha^2(2k+1)^2(n^2 + \frac{n}{2}) - \alpha^2(\frac{\alpha^2}{\beta Z^2} - 2)^2}{g_0^4}. \tag{A.8}
 \end{aligned}$$

For large n this reduces to Eq. (4.21). The discretization removes the logarithmic singularity of U , and M and U become linearly dependent. Altogether this gives the solution stated in Eq. (4.20).

We could have also satisfied the boundary condition at $t = -\infty$ ($r = +\infty$) by imposing $C_1 = 0$ in Eq. (A.7). This would leave us with a logarithmic singularity at $\tau = 0$ ($r = 1$), which is unphysical, and no discretization. Let us, however, note that in passing from Eq. (A.2) to Eq. (A.3) we have lost the singularity at $t = 0$ ($r = 0$). In the real case both solutions, the limiting functions of which are M and U , must therefore combine to a physical solution being regular at $t = 0$ ($r = 0$). This obviously will constrain C_1 to $C_1 \neq 0$. Note that in passing from Eq. (A.2) to Eq. (A.3) the singularity at $t = -\infty$ ($r = +\infty$) persists. For further details of the singularity structure of Eq. (4.17), see Appendix B.

APPENDIX B

Equation (4.17) has three singular points ²⁷⁾ for $r \geq 0$ at $r=0$, $r=1$ and $r=\infty$. The behaviour of its solutions at the singular points are as follows :

(i) $r=0$ is a Fuchsian singularity. There are two solutions $y^{(1)} = r^{n+\frac{3}{2}} Z^{(1)}$ and $y^{(2)} = r^{-n-\frac{1}{2}} Z^{(2)}$ where $Z^{(1)}$ and $Z^{(2)}$ are regular at the origin. Only the first solution is acceptable.

(ii) $r=1$ is a Fuchsian singularity. We have two solutions $y^{(1)} = \sqrt{1-r} W^{(1)}$ and $y^{(2)} = \sqrt{1-r} \log(1-r) W^{(2)}$ where $W^{(1)}$ and $W^{(2)}$ are regular at $r=1$. Both solutions are consistent with the normalization condition (4.18).

(iii) $r=\infty$ is not Fuchsian, but the behaviour of the solution can easily be deduced from Eq. (4.17). The general solution is $y \simeq C_1 e^{-\sqrt{E}r} + C_2 e^{\sqrt{E}r}$. Only if $C_2=0$ ($E > 0$), y is integrable.

The logarithmic singularity at $r=1$ is certainly an unpleasant feature of our wave function. However, Appendix A shows that for large E the wave functions are actually regular at $r=1$. This makes us hope that the logarithmic singularity will, after the discretization, also disappear from the general solution.

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FIGURE CAPTIONS

- Figure 1 Lowest order (non-relativistic) quark diagrams of the hadronic
 (a) three-point function, and (b) four-point function.
- Figure 2 Handbag diagram.
- Figure 3 The Schwinger-Dyson equation for the quark propagator.
- Figure 4 The Bethe-Salpeter equation for the wave function.
- Figure 5 The Callan-Symanzik function vs. the coupling constant.
- Figure 6 One-gluon exchange diagram.
- Figure 7 Confinement potential. Solid curve is our approximation.
 Dashed curve indicates true short-distance behaviour.
- Figure 8 Two- and higher-gluon exchange diagram having Pomeron-type
 topology.
- Figure 9 Effective "potential" entering into the Schrödinger-type
 equation (4.17).
- Figure 10 Regge spectrum.

(a)

(b)

FIG.1

FIG.2

FIG.3

FIG.4

FIG.5

FIG.6

FIG.7

FIG.8

FIG.9

FIG.10