CM-P00061867

Ref.TH.2250-CERN

Archives

TOPOLOGICAL FLUCTUATIONS AND BREAKING OF CHIRAL SYMMETRY
IN GAUGE THEORIES INVOLVING MASSLESS FERMIONS

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ABSTRACT

Using the method of Euclidean functional integrals, we show the occurrence of a certain type of chiral symmetry breaking as a result of local fluctuations in the winding number. We mainly restrict our discussion to QED_2 in order to have an independent check of our methods from the known solution of this model. We do not use instantons (pseudoparticles), and we also avoid functional integration over fields with global winding (Pontryagin) number different from zero.

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1. INTRODUCTION

Recently several authors ¹⁾⁻³⁾ have discussed a new type of non-Goldstonian symmetry breaking in non-Abelian gauge theories. The important aspect of this new mechanism is the topological "winding" property ⁴⁾ of gauge fields in the Euclidean functional integrals.

In this paper we attempt to obtain a better understanding of the topological origin of these new ideas. The physical manifestation of the topological mechanism is the breakdown of the cluster decomposition property, at least if zero-mass fermions are present. It is well known that general local quantum field theory relates the failure of this property with the degeneracy of the vacuum. A more economical description, in terms of a unique vacuum and an explicit symmetry breaking, can then be constructed ^{*)}. In this respect the analysis resembles discussions of the BCS model of superconductivity or the Bose gas in which the less redundant description, in terms of a unique vacuum (characterized by the Josephson angle), exhibits an explicit breaking of a charge superselection rule.

In the models of QED₂ and QCD₄ with massless fermions, the breaking is seen in certain chirality carrying operators. Every gauge invariant chirality carrying operator $\mathcal{O}$ in our terminology factorizes into a dynamical and a "spurion" part:

$$\mathcal{O} = \mathcal{O}_{dyn.} \cdot \mathcal{O}_{spur} \quad (1.1)$$

The spurion part carries no energy momentum but only chirality, whereas the dynamical part describes all the complicated physics. In QED₂, such a spurion picture holds also for the colour. In the Landau gauge the "label colour" of the massless fermion fields condenses via spurions into the vacuum ⁶⁾, whereas the "Gaussian colour" (i.e., in the sense of span integrals over local densities) vanishes; a well-known prerequisite for confinement.

In our discussion of chiral breaking we deviate in two aspects from the existing literature. First, we do not make use of solutions of classical gauge field equations. We find it very satisfying that the understanding of "symmetry breaking due to topological fluctuations" does not depend on one's ability to solve classical Euclidean Yang-Mills equations ⁷⁾. 't Hooft's observation ¹⁾ that a massless Dirac field in an instanton

^{*)} We refer to the Appendix of Ref. 5) where this construction has been carried out and applied to the BCS Hamiltonian of superconductivity.

potential produces a zero energy normalizable eigenstate is, in fact, a special case of the more general statement that a winding field implies always a vanishing functional fermion determinant. The second aspect is the functional integration. Here we take a completely conventional point of view. We always integrate over A_μ field configurations with total winding number zero. Only the concept of non-trivial "local winding" enters our functional methods. For example, in order to evaluate the two-point function of chirality carrying operators, we find it useful to decompose our A_μ according to ^{*}):

$$A_\mu = A_\mu^{(1)} + A_\mu^{(-1)} + A_\mu^{2n} \quad (1.2)$$

where $A^{(\pm 1)}$ are special representations of ± 1 winding number centered at an appropriate position (these positions will be coupled to the co-ordinates of the correlation functions).

In this way we avoid the unsolved, and perhaps even ill-defined, problem of functionally integrating over different "principle bundles" ⁸⁾, which in a more conventional language is related to the problem of relative normalizations between functional integrals over different winding numbers.

For reasons of simplicity we discuss our approach first in QED₂, i.e., the Schwinger model ⁹⁾. In Section 2 and in the Appendix, we review briefly this model in the language of Euclidean functional integration, and indicate the Lowenstein-Swieca ⁶⁾ "operator fit" of the resulting correlation functions. This operator formalism allows us to expose the chiral vacuum degeneracy and the accompanying chiral spurions in a simple fashion.

In Section 3 we rearrange the functional integrand after the fermion integration, but before the A_μ integration, in such a way that the violation of cluster properties (as an example we consider the two-point function of $\bar{\Psi}\Psi$) and the resulting chiral symmetry breaking is exposed on the level of the functional integrand. This is done with the help of a classical background field of the form as described in formula (1.2). We obtain the qualitative picture that Green's functions in appropriately chosen background fields (with total winding zero, but non-vanishing local winding) increase for large separations. On the other hand, the fermion determinant in such fields leads to a decrease in such a way that certain products of Green's functions with the determinant stay asymptotically finite. The exact

^{*}) For QCD₄ the composition of local winding fields is more involved.

two-point correlation function in the asymptotic long-distance limit converges to a constant which does not contain quasiclassical e^{-1/e^2} factors; the violation of the cluster property is equally strong for all $e \neq 0$, and the behaviour at $e=0$ is both non-analytic and non-smooth.

Section 4 contains some qualitative remarks on the QCD_4 model with massless fermions, but a systematic study of this model along our lines will hopefully be treated for a future publication.

Finally, we would like to stress one point which in the existing literature has been somewhat obscure. This concerns the necessity of having massless fermions in order to obtain any interesting physical manifestation of topological vacuum fluctuations. Consider, for example, the massive Schwinger model. The existing literature emphasizes the role of the $F_{\mu\nu}$ behaviour at spatial infinity¹⁰⁾ and concludes that this effect is described by a new parameter θ which was not there in the original massive QED_2 Lagrangian. This effect for $\theta \neq 0$ is then interpreted as some kind of spontaneous parity breaking due to boundary effects. We believe, however, that this way of phrasing the results is somewhat misleading because the resulting theory is undistinguishable from the explicit symmetry breaking model:

$$\mathcal{L} = \mathcal{L}_{\text{Schwinger}} - m \cos \theta \bar{\psi} \psi - m \sin \theta \bar{\psi} \gamma_5 \psi \quad (1.3)$$

The discussion of Callan, Dashen and Gross³⁾ suggests that non-Abelian gauge theories without massless fermions (i.e., the tunnelling situation) are also equivalent to a new Lagrangian theory with an additional parameter θ ¹¹⁾.

The reader may wonder how our topological picture is related to Kogut's and Susskind's¹²⁾ idea of a "seized vacuum". Their picture applies to a Hamiltonian formulation of gauge theories in which the Hilbert space is entirely physical, i.e., all ghosts have been removed at the expense of obtaining a manifestly non-local interaction which does not meet the prerequisites for the applicability of Goldstone's theorem¹³⁾. For non-Abelian four-dimensional gauge theories such a description has, however, not been proposed; the most favourable axial gauge propagators, which seemed to be free of Faddeev-Popov¹⁴⁾ ghosts, can still not be represented in a physical Hilbert space. Even if such a description turns out to be possible in the future, we expect it to be so complicated and unmanageable that we think the conventional description leading to chirality (and perhaps colour) carrying spurions is more pragmatic.

2. THE SCHWINGER MODEL

In this Section we recapitulate briefly the Schwinger model ⁹⁾, using functional methods in a two-dimensional Euclidean space and comparing the outcome with the "operator fit" of Lowenstein and Swieca ⁶⁾. The Appendix contains material completing our discussion.

The generating functional of Green's functions is

$$Z[J, \eta, \bar{\eta}] = \int [dA][d\psi][d\bar{\psi}] \exp \left\{ -S[A, \psi, \bar{\psi}] + \int d^2x \left(A_\mu(x) J_\mu(x) + \bar{\eta}(x) \psi(x) + \bar{\psi}(x) \eta(x) \right) \right\} \quad (2.1)$$

where we have introduced sources J_μ , η and $\bar{\eta}$ of the photon and electron fields A_μ , $\bar{\psi}$ and ψ , respectively. The action is:

$$S[A, \psi, \bar{\psi}] = -i \bar{\psi} \gamma_\mu (\partial_\mu - ie A_\mu) \psi + S[A] \quad (2.2)$$

with

$$S[A] = \frac{1}{4} F_{\mu\nu}^2 + \text{gauge terms} \quad (2.3)$$

where $F_{\mu\nu}$ is the field strength.

The circumstance which allows the model to be solved is the fact that the electron propagator in an arbitrary external field can be found exactly ⁹⁾, i.e., the equation:

$$i \gamma_\mu (\partial_\mu - ie A_\mu(x)) G(x, y; A) = \delta(x-y) \quad (2.4)$$

has the solution:

$$G(x, y; A) = \exp \{ ie [\phi(x) - \phi(y)] \} \cdot G(x-y) \quad (2.5)$$

where

$$\phi(x) = - \int d^2 z D(x-z) \left(\partial_\mu A_\mu(z) - \frac{i}{2} \gamma_5 \varepsilon_{\mu\nu} F_{\mu\nu} \right) \quad (2.6)$$

Here we have introduced the free massless Green's functions $G(x)$ and $D(x)$ in two-dimensional Euclidean space:

$$D(x) = - \frac{1}{4\pi} \ln \mu^2 x^2 \quad (2.7)$$

$$G(x) = -i \gamma_\mu \partial_\mu D(x) \quad (2.8)$$

where μ is an infra-red cut-off. Our γ matrices are chosen such that γ_5 is diagonal:

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.9)$$

The integration over electron fields in Eq. (2.1) can for $\eta = \bar{\eta} = 0$ be carried out immediately:

$$Z[J] = \int [dA] \text{Det} \{ i \gamma (\partial - i e A) \} \exp \{ -S[A] + \int d^2 x A_\mu J_\mu \} \quad (2.10)$$

The value of the functional determinant can be found from Eqs. (2.4)-(2.6):

$$\begin{aligned} \text{Det} \{ i \gamma (\partial - i e A) \} &= \exp \left\{ -\frac{e^2}{2\pi} \int d^2 x A_\mu \left(\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) A_\nu \right\} \\ &\equiv \exp \{ -\Gamma^{[1]}[A] \} \end{aligned} \quad (2.11)$$

where a divergent A independent factor has been discarded. $\Gamma^{[1]}(A)$ is the one-loop generating function of photon vertex functions.

Equations (2.10) and (2.11) show that the photon Green's functions are those of a free particle with mass $e/\sqrt{\pi}$ (the Schwinger mechanism).

The $2n$ point electron Green's functions are found from (2.1) by differentiation after the spinor sources η and $\bar{\eta}$ and integration over the electron fields:

$$\begin{aligned} & \langle \psi(x_1) \dots \psi(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_n) \rangle = \\ & Z^{-1}(0) \int [dA] \exp \{ -S[A] - \Gamma^{(n)}[A] \} \times \\ & \times \langle \psi(x_1) \dots \psi(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_n) \rangle |_A \end{aligned} \quad (2.12)$$

where we have introduced the $2n$ point function in an external field, known from (2.5):

$$\begin{aligned} & \langle \psi(x_1) \dots \psi(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_n) \rangle |_A = \\ & \langle \psi(x_1) \dots \psi(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_n) \rangle_0 \times \\ & \times \exp \left\{ ie \int d^2z A_\mu(z) J_\mu(z; x_1, \dots, x_n, y_1, \dots, y_n) \right\} \end{aligned} \quad (2.13)$$

with

$$\begin{aligned} J_\mu(z; x_1, \dots, x_n, y_1, \dots, y_n) &= \sum_i \left(\frac{\partial}{\partial z_\mu} - i \gamma_{5x_i} \varepsilon_{\lambda\mu} \frac{\partial}{\partial z_\lambda} \right) D(x_i - z) \\ &= \sum_i \left(\frac{\partial}{\partial z_\mu} + i \gamma_{5y_i} \varepsilon_{\lambda\mu} \frac{\partial}{\partial z_\lambda} \right) D(y_i - z) \end{aligned} \quad (2.14)$$

Green's functions containing both electron and photon fields can be generated by adding the source of the photon field to (2.14).

The integration over A in Eq. (2.12) can now be carried out with the result:

$$\begin{aligned}
 & \langle \psi(x_1) \dots \psi(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_n) \rangle = \\
 & \langle \psi(x_1) \dots \psi(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_n) \rangle_0 \times \\
 & \times \exp \left\{ \frac{-1}{2} e^2 \iint d^2z d^2z' J_\mu(z; x_1, \dots, x_n, y_1, \dots, y_n) \times \right. \\
 & \times D_{\mu\nu}(z-z') J_\nu(z'; x_1, \dots, x_n, y_1, \dots, y_n) \left. \right\}
 \end{aligned}
 \tag{2.15}$$

where $D_{\mu\nu}(z-z')$ is the exact Euclidean photon propagator which in the transverse (Landau) gauge used originally by Schwinger has the form:

$$D_{\mu\nu}(z-z') = \delta_{\mu\nu} \Delta(z-z'; \frac{e^2}{\pi}) - \frac{\pi}{e^2} \frac{\partial}{\partial z_\mu} \frac{\partial}{\partial z'_\nu} \left(D(z-z') - \Delta(z-z'; \frac{e^2}{\pi}) \right)
 \tag{2.16}$$

where $\Delta[z_i(e^2/\pi)]$ is the Green's function of a free scalar massive particle:

$$(\square - \frac{e^2}{\pi}) \Delta(z; \frac{e^2}{\pi}) = -\delta(z)
 \tag{2.17}$$

Equation (2.15) can now be reduced to the form:

$$\begin{aligned}
 & \langle \psi(x_1) \dots \psi(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_n) \rangle = \\
 & \langle \psi(x_1) \dots \psi(x_n) \bar{\psi}(y_1) \dots \bar{\psi}(y_n) \rangle_0 \times \\
 & \times \exp \left\{ \frac{\pi}{2} \int d^2z d^2z' \left[\sum_i \gamma_{sx_i} \delta(x_i - z) + \sum_i \gamma_{sy_i} \delta(y_i - z) \right] \times \right. \\
 & \times \left. \left[\sum_i \gamma_{sx_i} \delta(x_i - z') + \sum_i \gamma_{sy_i} \delta(y_i - z') \right] \left(D(z-z') - \Delta(z-z'; \frac{e^2}{\pi}) \right) \right\}
 \end{aligned}
 \tag{2.18}$$

The Euclidean correlation functions (2.18) can now be continued into the Minkowski region where Δ and D go over into the Wightman function $i\Delta^{(+)}$, $iD^{(+)}$. The contributions $i\Delta^{(+)}(0)$ cancel identically with those of $iD^{(+)}(0)$ if we choose the normalization mass μ in (2.7) equal to the physical mass $m = e^2/\pi$.

It was observed by Lowenstein and Swieca¹⁰⁾ that Eqs. (2.16) and (2.18) can be fitted by the operator identifications

$$A_\mu(x) = -\frac{1}{m^2} (\varepsilon_{\mu\nu} \partial_\nu \Sigma(x) + \partial_\mu \eta(x)) \quad (2.19)$$

$$\psi(x) = : \exp i\sqrt{\pi} \gamma_5 (\Sigma(x) + \tilde{\eta}(x)) : \psi_0(x) \quad (2.20)$$

and

$$\varepsilon^{\mu\nu} \partial_\nu \tilde{\eta} = \partial^\mu \eta$$

where $\Sigma(x)$ is a free scalar field with mass $e/\sqrt{\pi}$, $\psi_0(x)$ is a free massless spinor field, and $\eta(x)$ is a free massless scalar field quantized with indefinite metric and identified as an unphysical gauge excitation field.

Green's functions of gauge invariant operators are also obtainable from the functional method. Consider thus the two-point function of the gauge-invariant operators:

$$J(x) = N \left[\bar{\psi} \frac{1+\gamma_5}{2} \psi \right] (x) \quad (2.21)$$

and its Hermitean adjoint. Corresponding to (2.12) we have:

$$\begin{aligned} \langle J(x) J^\dagger(y) \rangle &= \bar{Z}'(0) \int [dA] \exp \{ -S[A] - \Gamma''[A] \} \times \\ &\times \text{tr} G(x,y;A) \frac{1-\gamma_5}{2} G(y,x;A) \frac{1+\gamma_5}{2} \end{aligned} \quad (2.22)$$

Inserting (2.5) and (2.6), carrying out some γ algebra, and performing the functional integration, one ends up with:

$$\langle J(x) J^\dagger(y) \rangle = t_r G(x-y) G(y-x) \frac{1+\epsilon}{2} \times \\ \times \exp \left\{ 4\pi \left[D(0) - \Delta(0; e_\pi^2) - D(x-y) + \Delta(x-y; e_\pi^2) \right] \right\} \quad (2.23)$$

Note that the exponential is finite, even though the separate terms are not.

The reason why the operator J is of interest for our purposes can be seen directly from (2.23); this equation implies that the linked cluster property of the theory is violated and the vacuum of the theory consequently degenerates. This occurs through the factor:

$$\exp \{ -4\pi D(x-y) \} = m^2 (x-y)^2 \quad (2.24)$$

which for large values of $(x-y)^2$ grows sufficiently fast to make the whole expression remain constant. This means that the operator J creates a state out of the original vacuum $|0\rangle$ with a translation invariant component $|1\rangle$ which is degenerate with $|0\rangle$. This degeneracy of the vacuum can also be explicitly seen in the Lowenstein-Swieca operator formulation. Consider the behaviour of the operator in Minkowski space:

$$J^\dagger(x) = N[\psi_2^\dagger \psi_1](x) = : e^{-2i\sqrt{\pi}(\Sigma + \tilde{\eta})} : : \psi_{20}^\dagger \psi_{10} : \quad (2.25)$$

Sandwiched between physical states ^{*)}, the Σ part drops out in the large-distance limit (weak limit on gauge invariant subspace):

$$U = \lim_{x \rightarrow \infty} J^\dagger(x) = : e^{-2i\sqrt{\pi}\tilde{\eta}} : : \psi_{20}^\dagger \psi_{10} : \quad (2.26)$$

If the vacuum would be non-degenerate, the weak limit of a density would converge towards a multiple of the identity. It is, however, easy to see that U is a non-trivial constant operator. The non-triviality follows from the fact that $\tilde{\eta}$ and ψ_0 are independent fields and the independence of x can easily be seen by computing the two-point function:

*) Physical states are those which may be obtained by applying gauge invariant operators as $F_{\mu\nu}$ and gauge invariant bilocals containing line integrals over A_μ on an arbitrary number of times to the vacuum.

$$\begin{aligned} \langle U^\dagger(x) U(y) \rangle &= \exp. -4\pi D^{(+)}(x-y) \times \\ &\times \langle \psi_{20}(x) \psi_{20}^\dagger(y) \rangle \langle \psi_{10}^\dagger(x) \psi_{10}(y) \rangle \end{aligned} \quad (2.27)$$

The cancellation of the co-ordinate dependence is precisely the one already observed after (2.23). U is a pure spurion operator which only carries chirality. Its repeated application to our reference vacuum $|0\rangle$ creates a chain of states $|n\rangle$ which are degenerate with $|0\rangle$:

$$|n\rangle = U^n |0\rangle$$

It is easy to see that every gauge invariant chirality carrying operator $\Theta(x)$, which carries $2n$ units of chirality, factorizes in the physical subspace:

$$\Theta(x) = \Theta_{dyn}(x) \cdot U^n \quad (2.28)$$

The chirality decouples completely from the dynamics and "condensates" into the vacuum analogous to the Bose-Einstein condensation of a Bose gas.

A convincing argument that the chiral spurions U are related to the topology of gauge transformations comes from a recent paper of Rothe and Swieca¹⁵⁾. They construct explicit gauge transformations T in the Schwinger (Landau) gauge:

$$T(\Lambda) A_\mu(x) T^\dagger(\Lambda) = A_\mu(x) + \frac{1}{e} \partial_\mu \Lambda(x) \quad (2.29a)$$

$$T(\Lambda) \psi(x) T^\dagger(\Lambda) = e^{i\Lambda(x)} \psi(x) \quad (2.29b)$$

with

$$\square \Lambda(x) = 0$$

For gauge transformation T satisfying the usual boundary conditions^{*)}

*) The boundary conditions are stable under time translation.

$$\Lambda(0, x) \xrightarrow{x \rightarrow \pm\infty} 0 \quad (2.30)$$

they find the expected statement

$$\langle \phi | T(\Lambda) - 1 | \psi \rangle = 0 \quad (2.31)$$

where $|\phi\rangle$ and $|\psi\rangle$ are states in the gauge invariant subspace. For gauge transformation with "winding", i.e.,

$$\Lambda^{(1)}(0, x) \xrightarrow{x \rightarrow \pm\infty} \begin{cases} 0 \\ 2\pi \end{cases} \quad (2.32)$$

they establish the following relation:

$$\langle \phi | T(\Lambda^{(1)}) - U | \psi \rangle = 0 \quad (2.33)$$

This clearly shows that in relativistic gauges the spurion operator may be interpreted in terms of non-trivial winding gauge transformations in the physical Minkowski space.

3. REARRANGED FUNCTIONAL INTEGRATION AND TOPOLOGICAL VACUUM FLUCTUATIONS

We shall now demonstrate how the breaking of cluster properties reviewed in Section 2 can be understood in terms of the concept of topological vacuum fluctuations. We do this by means of a rearrangement of the functional integral (2.22). A background field is introduced carrying a local winding number (1.2) in the sense that it is the sum of a vortex line field¹⁶⁾ and anti-vortex line, at large separation. This makes it possible to exhibit the breaking of cluster properties already before the A integration in (2.22) is carried out.

After insertion of (2.5) into (2.22), it can be written:

$$\begin{aligned} \langle J(x) J^\dagger(y) \rangle &= \bar{Z}(0) \int [dA] \exp \{ -S[A] - \Gamma^{(1)}[A] \} \times \\ &\times \text{tr} G(x-y) G(y-x) \frac{1+\gamma_5}{2} \exp \left\{ 2e\gamma_5 \varepsilon_{\lambda\mu} \int d^2z A_\mu(z) \frac{\partial}{\partial z_\lambda} (D(x-z) - D(y-z)) \right\} \end{aligned} \quad (3.1)$$

In this formula we change the functional integration variable:

$$A_\mu \longrightarrow A_\mu + A_\mu^{\text{cl}} \quad (3.2)$$

where the background field A_μ^{cl} has been chosen, for example, as:

$$A_\mu^{\text{cl}}(z) = -\frac{1}{e} \varepsilon_{\lambda\mu} \left\{ \frac{(z-x')_\lambda}{1+(z-x')^2} - \frac{(z-y')_\lambda}{1+(z-y')^2} \right\} \quad (3.3)$$

This field is not a solution of the classical field equation, but it is the sum of two constituents with winding numbers 1 and -1, respectively, where the winding number is defined by:

$$\eta = \frac{1}{4\pi} \int d^2z \varepsilon_{\lambda\mu} F_{\lambda\mu}(z) \quad (3.4)$$

The co-ordinates x' and y' are sites of an "anti-instanton" and an "instanton", respectively, which here can be represented by means of vortex lines. We envisage a situation with x near x' and y near y' , and for simplicity we can take $x=x'$ and $y=y'$.

The A dependent factors in the integrand of (3.1) must now be estimated, with A replaced by A^{cl} , for large values of

$$\xi = x - y \quad (3.5)$$

Consider first:

$$\begin{aligned} \Gamma^{(1)}[A^{\text{cl}}] &= \frac{e^2}{2\pi} \int d^2z A_\mu^{\text{cl}}(z) = \\ &= -\frac{1}{\pi} \int d^2z \left[\frac{z(z-\xi)}{(1+z^2)(1+(z-\xi)^2)} - \frac{z^2}{(1+z^2)^2} \right] \end{aligned} \quad (3.6)$$

Here we scale $\xi \rightarrow \xi/\lambda$ and substitute $z \rightarrow z/\lambda$:

$$\Gamma^{(1)}[A^{\text{cl}}] = -\frac{1}{\pi} \int d^2z \left[\frac{z(z-\xi)}{(\lambda^2+z^2)(\lambda^2+(z-\xi)^2)} - \frac{z^2}{(\lambda^2+z^2)^2} \right] \quad (3.7)$$

The two terms in the integrand each yield a logarithmically divergent integral from the upper limit, but these divergences cancel. When we want the part to become singular for $\lambda \rightarrow 0$, corresponding to $x-y$ large, we need only keep a small circle C_ϵ around the origin of the z plane, and there is only a contribution from the last term in the integrand:

$$\Gamma^{(1)}[A^c] \approx \int_{C_\epsilon} d^2 z \frac{z^2}{(\lambda^2 + z^2)^2} \approx -\ln\left(\frac{\xi}{\lambda}\right)^2 \quad (3.8)$$

The contribution from $S[A]$ is:

$$S[A^c] = -\frac{1}{2} \int d^2 z A_\mu^c(z) \square A_\mu^c(z) \quad (3.9)$$

which is $O(1/e^2)$, but remains finite for $\xi \rightarrow \infty$. This leaves us with

$$\begin{aligned} & 2e\gamma_5 \epsilon_{\lambda\mu} \int d^2 z A_\mu^c(z) \frac{\partial}{\partial z_\lambda} (D(x-z) - D(y-z)) \\ &= -\gamma_5 \frac{2}{\pi} \int d^2 z \left[\frac{1}{(1+z^2)} - \frac{z(z-\xi)}{z^2(1+(z-\xi)^2)} \right] \end{aligned} \quad (3.10)$$

which, under the scale transformation already carried out upon (3.6), behaves like:

$$-2\gamma_5 \ln\left(\frac{\xi}{\lambda}\right)^2 \quad (3.11)$$

We can now go back to (3.1) and collect the factors coming from the background field:

$$\begin{aligned} & \exp \left\{ -S[A] - \Gamma^{(1)}[A] + 2e\gamma_5 \epsilon_{\lambda\mu} \int d^2 z A_\mu^c(z) \frac{\partial}{\partial z_\lambda} \right. \\ & \quad \left. \times (D(x-z) - D(y-z)) \right\} \\ & \approx \begin{pmatrix} \left[\left(\frac{\xi}{\lambda}\right)^2\right]^{-3} & 0 \\ 0 & \left(\frac{\xi}{\lambda}\right)^2 \end{pmatrix} \end{aligned} \quad (3.12)$$

The matrix $1 + \gamma_5/2$ picks out the matrix element in the lower corner, which has the correct rate of growth to compensate for the decrease of $G(x-y)G(y-x)$, so we have, indeed, obtained the wanted effect.

It is clearly essential for this argument that the sites of J and the anti-instanton coincide, as well as those of J^+ and the instanton. The effect would vanish if we made an interchange.

It is easy to see that the integration over the remaining fluctuating variable (this is also a Gaussian integral) does not undo the violation of cluster properties (3.12) from the background. For $\xi \rightarrow \infty$ it just contributes numerical (ξ independent) factors. It is also easy to see that this integration (which does not involve a zero energy problem) together with the factor (3.12) reconstitutes the exact result (2.23) as it should be. The advantage of "tuning" the background field to the x and y of the correlation function by setting $x' = x$, $y' = y$ is that the exact result (2.23) is split into a classical and a fluctuation factor, each of which does not contain additional space-time variables (which would only cancel in the product), i.e., the covariance property is retained in the factorization.

By a judicious choice of the background field, one may even attempt to see something about the spectrum properties of (2.23) in the classical factor. For example, one could imagine a rapid approach of the classical factor at infinity (corresponding to a gap in the spectrum), or at least a decrease which is sufficient to rule out any intermediate Goldstone boson. It is, in particular, with respect to those problems that our approach looks promising. The approach of 't Hooft (which in our terminology means going directly to a description with a unique vacuum) aims directly at $\langle \bar{\Psi}\Psi \rangle$ in winding number 1 and does not seem to allow the discussion of spectral gaps and absence of Goldstone particles as coming from the same origin as the vacuum degeneracy.

4. FUNCTIONAL INTEGRALS IN GLOBALLY WINDING FIELDS

In this Section we would like to convince the reader that for those correlation functions we have been interested in, there is actually no difference between the results of a suitably extended 't Hooft's approach and ours. The main reason for this is that the functional integral representation for a correlation function of a product of fields of total chirality zero vanishes as soon as the integration over A_μ has a non-vanishing winding number:

$$\langle \mathcal{O} \rangle_n = \int [dA] e^{-S[A]} \text{Det } i\mathcal{D} \cdot \mathcal{O}_A = 0 \quad (4.1)$$

$\mathcal{O}$ = product of fields with total chirality zero;

$\mathcal{O}_A$ = product of A's contained in $\mathcal{O}$ and A field-dependent spinor Green functions.

The basis for this claim is a generalization of 't Hooft's ¹⁾ observation that instantons lead to vanishing eigenvalues in the massless Dirac equation.

Statement:

The Euclidean massless Dirac equation:

$$i\mathcal{D}\psi = E\psi, \quad \mathcal{D}_\mu = \partial_\mu - ig\mathbf{A}_\mu \quad (4.2)$$

in an $\mathbf{A}_\mu$ field with a non-vanishing winding number has a normalizable state ψ_0 with $E_0 = 0$.

Here we will not prove this statement in full generality. We will only demonstrate a mild generalization of 't Hooft's result, enough in order to see that pseudoparticles (as solutions of Yang-Mills equations) are not important for obtaining vanishing eigenvalues.

Consider a class of fields with a winding number, one which contains, as a special case, the "pseudoparticle" solution:

$$\mathbf{A}_\mu(x) = f(x^2) \mathbf{\hat{g}}^i(x) \partial_\mu \mathbf{g}(x) \quad (4.3)$$

with

$$\mathbf{g}(x) = \frac{x_4 + i\vec{\sigma}\vec{x}}{r_{x^2}} \quad (4.4)$$

$$f(x^2) \xrightarrow{x^2 \rightarrow \infty} \begin{cases} 1 \\ 0 \end{cases} \quad (4.5)$$

Apart from the two boundary conditions which express the absence of non-integrable singularities at zero and the presence of a unit winding number, there are no restrictions on f .

We write ψ_0 as:

$$\psi_0(x) = F(x^2) u \quad (4.6)$$

and obtain for u the equation (4.2) with $E=0$:

$$\gamma_\mu \left[x_\mu - i \gamma_{\mu\nu} \sigma_\nu x_\nu + \frac{f(x^2) \bar{F}(x^2)}{2x^2 F'(x^2)} \right] u = 0$$

which is identical to 't Hooft's equation ¹⁾ for the constant spinor, if we choose:

$$\bar{F}(x^2) = \exp -\frac{3}{2} \int_a^{x^2} \frac{dt}{t} f(t) \quad (4.7)$$

Under the assumption (4.5) the function F comes out to be integrable.

After this brief illustration of the statement, we go back to (4.1). Consider first Θ 's which do not contain spinors. In that case the presence of a discrete zero eigenvalue in the determinant leads to the vanishing of the integrand and hence (4.1). If Θ contains spinors, the presence of zero-energy eigenstates of the Dirac equation would seem to lead to trouble, since:

$$G(x, y; A) = \sum_E \frac{\psi_E(x) \bar{\psi}_E(y)}{E} \quad (4.8)$$

contains a contribution $1/0$.

However, the u spinor structure of $\psi_0 \bar{\psi}_0$ is such that the chirality conserving part of (4.8) does not contain this term. Even the chirality violating part is only different from zero if the violation matches the winding number [for simplicity, think of a $U(1)$ flavour model]. For example, if $\psi_0^r(x) \bar{\psi}_0^\ell(y) \neq 0$ (r and ℓ are the left- or right-handed polarized spinors), then $\psi_0^\ell(x) \bar{\psi}_0^r(y)$ vanishes.

On the other hand, if we are computing a functional integral corresponding to a chirality changing expectation value (whose chirality takes the winding number), then our formula (4.1) would lead to an undeterminate integrand: $0 \cdot \infty$, the zero coming from the vanishing factor in the determinant and the infinity from the ψ_0 contribution with the external field

dependent Green's function (4.8). In that case, the bookkeeping has to be changed. As explained in the Appendix the necessary change emerges naturally from the basic rules of functional integration over fermion fields.

Arguments of this kind, based on the previous statement, allow one to specify those Green's functions which, for a functional integral of a given winding number, are different from zero. One has to work considerably harder if one wants to compute their value. For the reasons mentioned in the Introduction and at the end of Section 3, we have investigated only those Green's functions which are zero precisely if we integrate over a global zero winding number.

We should not hide from the reader that there is one aspect in QCD_2 which does not fit into the previous discussion. Namely, for this case the Euclidean Dirac equation in winding fields does not have a normalizable zero energy eigenvalue. Instead there is a zero energy eigenfunction with a norm which only diverges logarithmically, whereas the non-zero energy eigenstates have the quadratically divergent norm as expected in the continuum. It can be seen by stereographic mapping onto a three-dimensional sphere that this suppression of the divergence suffices ¹⁶⁾ to make $\text{Det } i \not{D}$ vanish.

In order to apply our method to QCD_4 we would have to construct convenient representations for a local unit winding number field centred around x and an opposite winding field around y . This and other related matters will hopefully be dealt with in a future publication.

ACKNOWLEDGEMENTS

We thank R. Crewther for discussions. One of the authors (B.S.) is indebted to J.A. Swieca for a very fruitful exchange of correspondence.

APPENDIX

In the following we compile some formulae which are useful for the evaluation of fermionic functional determinants in external (winding) fields. From the basic rule of fermion integration ¹⁷⁾

$$\int d\bar{\eta} d\eta \begin{Bmatrix} 1 \\ \eta \\ \bar{\eta} \\ \eta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \quad (\text{A.1})$$

as well as $\eta^2 = 0 = \bar{\eta}^2$, we deduce

$$\int [d\bar{\eta}] [d\eta] \bar{\eta} \Gamma \eta e^{\bar{\eta} K \eta} = \sum_j \psi_j \Gamma \bar{\psi}_j \frac{\pi}{j} K_j \quad (\text{A.2})$$

where K_j and ψ_j are the eigenvalues and eigenstates of the kernel K .

In the case where $K = i\cancel{D}$ is a continuous kernel, we obtain

$$\sum_j \frac{1}{K_j} \psi_j \Gamma \bar{\psi}_j \det K \longrightarrow \langle \psi(x) \Gamma \bar{\psi}(y) \rangle_A \text{Det}' i\cancel{D} \quad (\text{A.3})$$

This formula only holds if the external field has zero winding. Because of the theorem mentioned in the last Section, an external field A_μ with winding number 1 leads, for $\Gamma = 1$, to an undetermined expression $\infty \cdot 0$. This factor arises from the $1/E_0$ with $E_0 = 0$ in the Green's function, whereas $i\cancel{D}$ contains this vanishing eigenvalue as a factor. Going back to (A.2) one finds, in this case,

$$\psi_0 \Gamma \bar{\psi}_0 \frac{\pi}{j \neq 0} K_j \longrightarrow \psi_0(x) \Gamma \bar{\psi}_0(y) \Big|_A \text{Det}' i\cancel{D} \quad (\text{A.4})$$

where the prime indicates the omission of the discrete zero eigenvalue.

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