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ONE - DIMENSIONAL QUANTUM THEORY OF THE POMERON

V. Alessandrini ^{*)}
ICTP - Trieste

D. Amati
CERN - Geneva

R. Jengo
Istituto di Fisica, Università di Trieste

ABSTRACT

The scattering amplitude, originated by the sum over all multi-Pomeron exchanges, is evaluated in the zero slope limit by solving a one-dimensional quantum field theory. In the classical limit, the theory gives back the tree diagram results obtained in a preceding paper. The constant cross-sections obtained in that case for a Pomeron with intercept larger than one, are modified only at ultra-asymptotic energies by quantum effects. In the one-dimensional theory, this modification gives rise to cross-sections behaving as $s^{-\Delta E}$ where ΔE is a small quantity generated by a tunnel effect.

^{*)} Permanent address: Departamento de Fisica,
Universidad de la Plata, Argentina.

1. INTRODUCTION

In a preceding paper ¹⁾ we have attempted the summation over all Pomeron trees exchanged between particles which act as sources for the Pomeron field. The tree exchange is equivalent to the classical solution described by the interacting Pomeron Lagrangian

$$\mathcal{L}_0 = p \partial_y q + \alpha' \vec{\partial}_b p \cdot \vec{\partial}_b q + \mu p q - \lambda p q (p+q) \quad (1)$$

plus the source terms. The Pomeron fields $p(\vec{b}, y)$ and $q(\vec{b}, y)$ are related to the Gribov fields ²⁾ by $p = i\psi^+$, $q = i\psi$, and $\mu = \alpha'_P(0) - 1$. This change of variables is made in order to work in a theory with a real triple Pomeron coupling λ . The field variables depend, as usual, on the rapidity y and the impact parameter $\vec{b}$. The rapidity y plays, in this theory, the role of a fictitious time (we shall often call it time), and the conventions we have chosen in writing the Lagrangian correspond to the case in which this time variable has already been rotated to imaginary values.

Due to the smallness of α' , it was argued in Ref. 1) that over a large region of rapidity, the α' term in Eq. (1) could safely be neglected in the solution of the classical equations. This brought the investigation of the one-dimensional problem described by the Lagrangian

$$\mathcal{L} = p \partial_y q + p q (\mu - \lambda(p+q)) + g_1(\vec{b}) \delta(y-y_1) q + g_2(\vec{b}) \delta(y-y_2) p \quad (2)$$

which corresponds to two sources with impact parameter distribution $g_1(\vec{b})$ and $g_2(\vec{b})$ acting at times y_1 and y_2 , respectively. The time difference $Y = y_2 - y_1$ is the total rapidity of the colliding system, and the impact parameter distribution of each source is, in principle, related to the internal structure of the colliding objects. The only b dependence of the solution comes indeed from the dependence on b of the sources $g(\vec{b})$.

The solution of the one-dimensional classical problem ¹⁾ showed an interesting feature. While for $\mu < 0$ the asymptotic S matrix approached one (i.e., vanishing cross-sections), for $\mu > 0$ the scattering amplitude approaches a non-unit value (constant cross-sections). This was due to a phenomenon closely related to a spontaneous symmetry breaking thanks to

which the classical solution $p(y) \rightarrow q(y)$, started at one of two fixed points of the problem for $y = -\infty$, described an orbit in the pq plane which depended on g_1, g_2 and $Y = y_2 - y_1$, and ended into the other fixed point. Neither of these two fixed points were the origin: the origin being instead the departure ($x = -\infty$) and the arrival ($y = +\infty$) of the classical solution for the $\mu < 0$ case. In Appendix A we re-state some of the results of Ref. 1) in a way better suited for the quantum extension attempted here.

In this paper our goal is the investigation of what can be the effect of the inclusion of Pomeron loops, i.e., of quantum effects. We will attempt the solution of the quantum theory of the one-dimensional problem of Eq. (2), thanks to the well-known fact that in this case the field theory can be reduced to the solution of a Schrödinger equation.

We are aware of the fact that setting $\alpha' = 0$ in $\mathcal{L}_0$ and reducing the dimensionality of the problem can be doubtful at the quantum level. Really speaking $\sqrt{\alpha'}$ controls the scale on which b is measured. What we shall do is introduce a lattice in impact parameter keeping the spacing (Δb) larger than the correlation length which should be of the order of $(\alpha')^{\frac{1}{2}}$.

This procedure will be introduced in the next section, and at the end we will comment on the results that can be more crucially dependent on having treated the problem as unidimensional.

2. REDUCTION TO A ONE-DIMENSIONAL PROBLEM

As discussed in Ref. 1), the scattering amplitude is given by

$$S(g_1, g_2) = \frac{Z(g_1, g_2)}{Z(g_1, 0) Z(0, g_2)} \quad (3)$$

where Z is the partition function of the theory which, in the path integral formalism, is given by

$$Z(g_1, g_2) = N \int \mathcal{D}p \mathcal{D}q \exp - \int \alpha b^2 dy [\mathcal{L}_0 + g_1(b^2) \delta(y-y_1) q + g_2(b^2) \delta(y-y_2) p] \quad (4)$$

$\mathcal{L}_0$ is the interacting Pomeron Lagrangian given by Eq. (1), and N is defined by the condition $Z(0,0) = 1$. Our expression is obtained under the

assumption that the Pomerons are emitted or absorbed by the particles (sources) in an uncorrelated way ¹⁾. By dividing by $Z(g_1, 0) \cdot Z(0, g_2)$ we cancel any possible contribution of disconnected diagrams where a number of Pomerons are emitted by a source, and make a transition to the vacuum without reaching the other source.

As mentioned before, let us introduce a lattice in impact parameter space, thereby replacing the functional integral on functions of three variables by a product of functional integrals over functions of one variable. Taking a one-dimensional impact parameter space to simplify the notation, we can write

$$Z(g_1, g_2) = \prod_i \int \mathcal{D}p_i \mathcal{D}q_i \exp -\Delta b \int_{-x}^{+x} dy \left[\mathcal{L}_0(p_i, q_i) + g_1^i q_i \delta(y - y_1) + g_2^i p_i \delta(y - y_2) \right] \quad (5)$$

where Δb is the lattice spacing in impact parameter space.

If Δb is chosen larger than the correlation length - which is supposedly small due to the smallness of α' - the gradient terms in $\mathcal{L}$ in the Gribov Lagrangian will contribute negligibly to the action, so that $\mathcal{L}$ can be replaced by the one-dimensional Lagrangian of Eq. (2). Then the impact parameter label i in the fields $p_i(y), q_i(y)$ is totally irrelevant because the fields are integration variables, and the one-dimensional path integral in Eq. (5) gives the partition function of a one-dimensional field theory as a function of the sources g_1^i, g_2^i and the lattice spacing Δb . Therefore, we can write the full partition function of our problem in terms of the one-dimensional partition function $Z(g_1^i, g_2^i, \Delta b)$ as

$$Z(g_1, g_2) = \prod_i Z(g_1^i, g_2^i, \Delta b) = \exp \sum_i Z(g_1^i, g_2^i, \Delta b) \quad (6)$$

with a similar form for $Z[g_1, 0]$ and $Z[0, g_2]$.

This result, however, is valid only as long as the lattice space Δb is kept different from zero and larger than the correlation length in b . To have neglected the b gradient terms in $\mathcal{L}_0$, not only ceases to be justifiable for $\Delta b \sim 0$, but formally generates divergences in Z that

render Eq. (5) meaningless. The origin of these divergences is easily understood. Suppose we take the limit $\alpha' = 0$ after quantizing the theory and expanding in Feynman diagrams. Each internal line involves an integration over an energy E - conjugate to y - and a two-dimensional transverse momentum $\vec{k}$, conjugate to $\vec{b}$. When $\alpha' = 0$, all the dependence on $\vec{k}^2$ of the Pomeron propagator is lost, and we are left with a one-dimensional propagator integrated in a three-dimensional phase space. An ultra-violet cut-off Λ is needed to render the integral over d^2k finite, and loops will in general contribute over-all powers of the factor

$$\int_0^\Lambda d^2k \sim \frac{1}{(\Delta b)^2} \quad (7)$$

because Λ is inversely proportional to our lattice spacing in $\vec{b}$. The fact that at the level of the tree approximation there are no divergences follows at once from the fact that in this case Z is simply the exponential of the minimal action, and therefore $\log Z$ is proportional to Δb , giving the correct measure that transforms the sum over the lattice into an integral over $\vec{b}$ in Eq. (5). For loop diagrams, $\log Z$ depends on Δb in a more complicated way, which cannot be reabsorbed into the definition of the b integration over a one-dimensional action.

From now on we shall restrict ourselves to the one-dimensional theory and will use the notation $Z(g_1, g_2, Y)$ for the one-dimensional partition function.

3. THE ONE-DIMENSIONAL QUANTUM PROBLEM

It is a well-known fact that a one-dimensional field theory is just the quantum mechanics of a system, which in our case is defined by the Lagrangian of Eq. (2) where p and q are functions of the time y . The Hamiltonian is therefore given by ^{*)}

$$H = \mu p q - \lambda p q (p + q) \quad (8)$$

and the evolution operator is formally given by

*) This Hamiltonian H has the opposite sign convention from the one used in Ref. 1). This sign difference appears clearly in the classical limit of the theory (cf. Appendix B).

$$U(y_2, y_1) = \iint \mathcal{D}p \mathcal{D}q \exp - \int_{y_1}^{y_2} (p\dot{q} + H) dy \quad (9)$$

where the path integral is understood as a Hamiltonian path integral with the appropriate boundary conditions. For example, if we want to compute the Feynman propagator in q space, we must first perform the integral over p to transform it into the more usual Lagrangian path integral, and then integrate over all trajectories such that $q(y_1) = q_1$, $q(y_2) = q_2$. Another possibility is to perform both integrals at the same time with the given boundary conditions in $q(y)$, but no boundary conditions in $p(y)$ as discussed at length in Ref. 3).

Thanks to the fact that the source terms in Eq. (2) act only at time y_1 and y_2 , we can formally perform the integration over all $p(y)$ as well as all $q(y)$, but not $q(y_1)$ and $q(y_2)$. The partition function can then be written as

$$Z(g_1, g_2, \nu) = \frac{\iint dq(y_1) dq(y_2) U(\infty, y_2) e^{-g_2 P(y_2)} U(y_2, y_1) e^{-g_1 q(y_1)} U(y_1, -\infty)}{U(\infty, -\infty)} \quad (10)$$

where $p(y_2)$ must now be considered as the translation operator of $q(y_2)$. We realize the equivalence of (10) with the classical description (cf. Appendix A). Now q and p cannot now be followed simultaneously, and therefore we must choose a specific representation [configuration space q in Eq. (10)]. $U(y_1, -\infty)$ implies that the system evolves in the ground state (classically $E=0$ trajectories) between $-\infty$ and y_1 , where q acquires the value $q(y_1)$. At that moment the operator $e^{-g_1 q(y_1)}$ acts (reminiscent of a translation of g_1 in p !), then it evolves between y_1 and y_2 (classically, orbits of the problem). At y_2 , $q(y_2)$ is translated by g_2 and then the system evolves in the ground state up to $y = +\infty$.

We will attempt the evaluation of the evolution operator by its well-known relation to eigenfunctions and eigenvalues of a Schrödinger equation. In order to do this we shall transform variables so as to bring H to the

standard $p^2 + V(q)$ form as done, classically, in Appendix A. But before doing this, we are faced with two problems which cannot be avoided at the quantum level. These are:

- (a) the integration contours in p and q in the path integrals;
- (b) the ordering problem.

Let us discuss first (a). This is a relevant problem since, due to the typical cubic interaction, integrals as those of Eq. (9) are ill-defined and formally divergent. With our choice of integrating first over p , the convergence of the integral is controlled by the $\lambda p^2 q$ term which a priori has not a definite sign. We must choose integration contours for which $p^2 q$ is definite negative. This leaves us with two choices: p real and $q < 0$, or p imaginary and $q > 0$.

In addition to having a convergent path integral, we want to be able to derive a Schrödinger equation for our quantum mechanical system, that is to say, we want the time derivative of the wave function to be given by the action of a local operator. In the Lagrangian form of path integrals, the Schrödinger equation can be derived whenever the kinetic term $p\dot{q}$ in the Lagrangian provides a damping, such that the major contribution to the path integral comes from trajectories where $q(y+\epsilon)$ differs very little from $q(y)$. The same requirement can be put in a slightly different form using the Hamiltonian form of path integrals. In this case, the Feynman kernel $K(q_2, q_1, \epsilon)$ which in ordinary quantum mechanics propagates the wave function from time t to time $t + \epsilon$

$$\Psi(q_2, t+\epsilon) = \int_{-\infty}^{+\infty} dq_1 K(q_2, q_1, \epsilon) \Psi(q_1, t) \quad (11)$$

can be written as

$$K(q_2, q_1, \epsilon) = \int_{-\infty}^{+\infty} dp e^{ip(q_2 - q_1) - iH(p, \frac{q_1 + q_2}{2})\epsilon} = \int_{-\infty}^{+\infty} dp e^{ip(q_2 - q_1)} (1 - i\epsilon H) \quad (12)$$

where the ordinary integral over p is the only surviving trace of the original path integral: the integral over $p[t + (\epsilon/2)]$. Now one sees immediately that powers of p in H will give derivatives of $\delta(q_1 - q_2)$

which generate, in Eq. (11), differential operators acting on the wave function at time t .

The previous discussion shows that, in order to have a Schrödinger equation, it is crucial to have an imaginary term $ip\dot{q}$ in the exponential of the action ^{*}). Since in our case, given by Eq. (9), all the terms in the exponential are, in principle, real, we choose an integration contour for $p(y)$ along the imaginary axis. This means that the function $p(y)$ takes all possible imaginary values. This contour in p , together with the requirement $q > 0$, also renders the path integral convergent. This last requirement will be implemented by making a field transformation $q = x^2$ and defining the theory in terms of the field $x(y)$. This is equivalent to selecting an integration contour in q that runs above and below the positive real axis. The p, q contours of integration are exhibited in Fig. 1.

At the classical level, the theory we are considering is invariant under the interchange of p and q , provided y is changed into $(-y)$. With our choice of integration contours, the p, q symmetry is no longer explicit, since both variables are treated in a non-symmetrized way. It is possible to construct formal operations which accomplish the interchange of the p and q integration contours, and to show that the p, q symmetry is actually preserved ⁵⁾. We shall take this symmetry for granted and keep track of it at various stages of our calculations. We will also be careful to solve the ambiguities of our quantization procedure with prescriptions which preserve that symmetry.

With our choice of integration contours, the path integral (9) becomes the Hamiltonian path integral version of the Lagrangian path integrals encountered in statistical mechanics. Reggeon field theory leads therefore naturally - after the rotation to imaginary times and the choice of the integration contours - to a problem in the statistical mechanics of a one-dimensional system.

The integration over p with our choice of integration contours is realized, in practice, by performing the canonical transformation

$$q = x^2 \quad p = -\frac{1}{2x} \frac{\partial}{\partial x} = -\frac{\partial}{\partial x^2} \quad (13)$$

^{*}) The integration contours chosen by Cardy and Sugar ⁴⁾ do not satisfy this condition. They do not give rise, therefore, to a local equation for the time evolution (Schrödinger equation in one dimension).

which guarantees that q is positive, and of course $p = -i\partial_q$ because it is integrated along the imaginary axis. This canonical transformation must be supplemented with a selection rule; namely, we will accept in our Hilbert space only even functions of x . Clearly, the operator p^n is singular at $x=0$, unless it acts on functions of x^2 , in which case it is perfectly regular. But we can also argue that the selection rule is really necessary in order to make sure that we have the same Hilbert space we started from. In the original Reggeon theory, q and p are creation and annihilation operators, respectively, and the Hilbert space is spanned by powers of q acting on the ground state. Indeed, when the triple-Pomeron coupling λ is set equal to zero, the Hamiltonian of Eq. (8) becomes the Hamiltonian of a one-dimensional harmonic oscillation of fundamental frequency $\omega = \mu$. However, powers of q correspond to powers of x^2 in our case, that is to say, to all even functions of x . Of course, if we were computing the path integrals of the interacting theory exactly with our integration contours, this selection rule should come out automatically. In one particular case that can be discussed exactly (the free theory when $\lambda \rightarrow 0$), we will see later that this is indeed the case.

Let us now turn to the ordering problem. Different orderings of the operators in the interaction part of the Hamiltonian differ from each other by terms linear in p and q . The only guiding principle we have available here is the $(p \rightleftharpoons q)$, $(y \rightleftharpoons -y)$ symmetry of our dynamical problem, which at the quantum level means that an operator like qp^2 is replaced by q^2p and not pq^2 , as one would have naively expected. This statement can be proved by using the path integral formulation to compute the differential operator corresponding to a classical expression, in complete analogy with the derivation of the Schrödinger equation. The expression qp^2 becomes, under the interchange of p and q , equal to pq^2 . But the further change of sign in y has the effect, in the derivation of the Schrödinger equation, that it gives differential operators acting on the left instead of the right. Then the operator p does not differentiate the q^2 standing beside it, and the term pq^2 becomes the operator q^2p acting on the right.

Consistently with p, q symmetry, we will write the Hamiltonian as

$$H = \mu pq - \frac{\lambda}{2} (q^2p + qp^2 + qpq + pqp) \quad (14)$$

where a definite ordering has been chosen for reasons that will be discussed later. This ordering is the obvious one, $\lambda(q^2 p + q p^2)$, plus a linear term of the form $(\lambda/2)(p+q)$, which in perturbation theory gives rise to tadpole graphs. If we perform the canonical transformation given by Eqs. (13) on this Hamiltonian, we obtain

$$H = -\frac{\lambda}{4} \frac{\partial^2}{\partial x^2} + \left(\frac{\lambda}{2} x^3 - \frac{\mu}{2} x \right) \frac{\partial}{\partial x} + \frac{\lambda}{2} x^2 \quad (15)$$

which is clearly a non-Hermitean operator, due to the presence of the first derivative terms. However, this operator can be transformed into another Hamiltonian $\tilde{H}$ without linear derivatives by means of the transformation

$$\begin{aligned} \hat{H} &= e^{-F(x)} H e^{F(x)} = \\ &= -\frac{\lambda}{4} \left(F' + \frac{\partial}{\partial x} \right)^2 + \left(\frac{\lambda}{2} x^3 - \frac{\mu}{2} x \right) \left(F' + \frac{\partial}{\partial x} \right) + \frac{\lambda}{2} x^2 \end{aligned} \quad (16)$$

Therefore, the condition that the linear derivative terms disappear determines $F'(x)$ to be

$$F'(x) = x^3 - \frac{\mu}{\lambda} x \quad (17)$$

i.e.,

$$F(x) = \frac{1}{4} \left(x^4 - \frac{\mu}{\lambda} x^2 \right) \quad (18)$$

and the Hamiltonian $\tilde{H}$ becomes

$$\hat{H} = -\frac{\lambda}{4} \frac{\partial^2}{\partial x^2} + V(x) \quad (19)$$

$$V(x) = \frac{1}{4} F'^2 - \frac{1}{4} x^2 + \frac{\mu}{4} = \frac{1}{4} \left[x^2 \left(x^2 - \frac{\mu}{\lambda} \right)^2 + \frac{\mu}{\lambda} - x^2 \right] \quad (20)$$

The transformation (16) (18) is the operator analogue of the classical shift in p [cf. Eq. (A.1)]. The eigenvalues of H and $\tilde{H}$ are, of course, the same. Let us define φ_n as the eigenfunctions of $\tilde{H}$, i.e.,

$$\tilde{H} \varphi_n = E_n \varphi_n \quad (21)$$

We will use φ_n as our complete set. This means that all operators O of the original theory should be replaced by $\tilde{O} = e^{-F} O e^F$.

We can, at this point, provide a justification of our choice (14) for the ordering. Had we chosen differently, we would have found an additional term in $\tilde{H}$ of the form $\lambda A x^{-2}$ [$A = 3/16$ for the ordering $\lambda(p^2 q + q^2 p)$]. This implies a behaviour of the wave function at the origin which violates [except for $A = \ell(\ell+1)$, $\ell = 0, 2, \dots$] our selection rule that only even functions of x are admitted in our Hilbert space. The ordering of Eq. (14) seems to be the only one giving a consistent theory at the quantum level.

The potential $V(x)$ of Eq. (20) is shown in Fig. 2. Apart from the sign, it differs from the classical potential (Fig. 6 of Appendix A) by the additional $\mu/\lambda - x^2$ term. This term is originated by the non-commutativity of p and q , and is therefore a typical quantum effect. If track had been kept of $\hbar$, the $\mu/\lambda - x^2$ term would contain an extra $\hbar$ with respect to the rest.

The sign difference between the classical and the quantum potential (at lowest order in $\hbar$) is a well-known effect. In the Appendix B it is shown explicitly how the sign arrives by deriving the classical motion from the quantum theory.

Let us formally obtain the scattering amplitude of Eq. (3) through the solutions of the Schrödinger equation (19), (21). This is done by inserting in Eq. (10), the expression

$$\mathcal{U}(x_1, x_2) = \varphi_n^*(x_2) \varphi_n(x_1) e^{-E_n \tau} \quad (22)$$

Then

$$S(g_1, g_2; Y) = \frac{\sum_n \langle \varphi_0 | e^{-F} O_1 e^F | \varphi_n \rangle \langle \varphi_n | O_2 | \varphi_0 \rangle e^{-(E_n - E_0)Y}}{\langle \varphi_0 | e^{-F} O_2 e^F | \varphi_0 \rangle \langle \varphi_0 | O_1 | \varphi_0 \rangle} \quad (23)$$

where φ_0 is the ground state wave function, $O_1 = e^{-g_1 x^2}$, $O_2 = e^{+g_2 x^2}$ translates x^2 by g_2 , the function $F(x)$ is given by Eq. (18), and $\langle \varphi_0 | O | \varphi_n \rangle$ simply means $\int_{-\infty}^{+\infty} dx \varphi_0(x) O \varphi_n(x)$.

We see that the high-energy behaviour (large Y) of the scattering amplitude is controlled by the structure of the lower lying levels.

As a check of our quantization procedure, we show in Appendix C how the free theory ($\lambda = 0$) results are recovered.

4. SOLUTION OF THE SCHRÖDINGER EQUATION FOR SMALL λ/μ

We have been unable to solve explicitly the Schrödinger equation for the lower lying levels. We shall attempt the analysis of the level structure in the limit $\lambda/\mu \rightarrow 0$. This limit is of some interest because it can throw some light on the magnitude and origin of quantum effects in the situation in which the classical results (sum over trees) seem to be relevant, i.e., $\lambda \rightarrow 0$, λg_1 fixed [cf. Ref. 1)]. For small λ/μ the barriers separating the three minima of the potential become very high. In the limit $\lambda/\mu \rightarrow 0$, the potential represents three harmonic oscillators separated by infinitely high barriers. Let us develop $\tilde{H}$ around the different minima. For the central well (call it C), introducing $\xi = x/\sqrt{\lambda}$

$$\tilde{H} = -\frac{1}{4} \frac{\partial^2}{\partial \xi^2} + \frac{\mu^2}{4} \xi^2 + \frac{\mu}{4} + O\left(\frac{1}{\mu}\right) \quad (24)$$

Therefore, the energy levels are given by

$$E_n^C = \frac{\mu}{4} + \left(m + \frac{1}{2}\right) \frac{\mu}{2} \quad m = 0, 1, \dots \quad (25)$$

the ground state wave function being

$$\varphi_0^C = \exp -\frac{\mu \xi^2}{2} \quad (26)$$

For the side wells (call them L and R), let us introduce $\xi = (x/\sqrt{\lambda}) \pm (\sqrt{\lambda}/\lambda)$ so that for small ϵ

$$\tilde{H} = -\frac{1}{4} \frac{\partial^2}{\partial \xi^2} + \mu^2 \xi^2 + O\left(\frac{\lambda}{\mu}\right) \quad (27)$$

so that the eigenvalues are now

$$E_n^{L,R} = \left(n + \frac{1}{2}\right) \mu \quad n = 0, 1, \dots \quad (28)$$

and

$$\varphi_0^{L,R} = \exp -\mu \xi^2 \quad (29)$$

The ground state energies of the three oscillators are therefore equal. If we now remember the rule that in the Hilbert space we are considering only even wave functions which appear, it is clear that we have to retain the wave functions which are even in $\xi \rightarrow -\xi$ for the central oscillator, i.e., even m in Eq. (25), whereas for the side oscillators we have to take the sum of the same wave function on the right and left sides. In the limit $\lambda/\mu \rightarrow 0$, the energy spectrum of the central allowed excitations and of the left-right excitations is therefore the same. The level spacing is μ .

In the $\lambda/\mu \rightarrow 0$ limit, we find therefore for every eigenvalue $E_n = (n + \frac{1}{2})\mu$, two degenerate eigenfunctions which we can choose to be the symmetric and antisymmetric combination of φ_{2n}^C and $\varphi_n^{LR} = (\varphi_n^L + \varphi_n^R)/(\sqrt{2})$, i.e.,

$$\varphi_n^S = \frac{\varphi_{2n}^C + \varphi_n^{LR}}{\sqrt{2}}, \quad \varphi_n^A = \frac{\varphi_{2n}^C - \varphi_n^{LR}}{\sqrt{2}} \quad (30)$$

These levels and, in particular the ground state, are however not truly degenerate. This can be due to two possible reasons: corrections of the order $\lambda (\lambda/\mu)^n$, and the tunnel effect which allows the three oscillators to know each other. Therefore there will be another energy scale in our theory, namely the splitting between the two ground states, which is smaller than the normal energy scale of the excitations within a single oscillator, in the sense that it goes to zero for $\lambda/\mu \rightarrow 0$.

In the diagonalization of the 2×2 energy matrix, the first possibility arises from different diagonal elements. This gives level splittings going to zero as powers $[\lambda (\lambda/\mu)]$ for different first orders in the diagonal elements]. The pure tunnel effect arises when the splitting is entirely given by the non-diagonal elements. It gives level splittings which go to zero exponentially, i.e., like $e^{-\mu^2/\lambda^2}$.

At present we do not have a precise answer to this alternative, because we have not solved exactly the Schrödinger equation, and moreover because we are not completely sure that we did not miss some terms in our potential due to the order ambiguities we already discussed. The pq symmetry must, however, guarantee equal diagonal elements, so the splitting should be dominated by the tunnel effect. If this were not the case (first alternative), one would find the asymmetric situation in which the ground state wave function would be concentrated either in the central well or in the lateral ones, and therefore [cf. Eqs. (32)] the expectation values of p and q would be very different. If the diagonalization is entirely due to the non-diagonal elements (tunnel effect), the ground state would be ϕ_0^S . The splitting ΔE between it and the first excited level ϕ_0^A would be given by the tunnel effect, i.e.,

$$\Delta E \sim \exp - \int \sqrt{\frac{4}{\lambda} V(x)} dx \sim \exp - \frac{\mu^2}{\lambda^2} \quad (31)$$

α being a number of the order of one ($\alpha = 1/4$ in the W.K.B. approximation).

Let us show that ϕ_0^S treats indeed the three oscillators on the same footing, preserving therefore the pq symmetry. The ground state being the true vacuum of our theory, this is a basic feature for the preservation of the pq symmetry of the whole theory.

For $\lambda/\mu \rightarrow 0$, it is easy to verify

$$\begin{aligned} \langle \psi_0^C | e^{-F} p e^F | \psi_0^C \rangle &= \frac{\mu}{\lambda}, \quad \langle \psi_0^L | e^{-F} p e^F | \psi_0^L \rangle = \langle \psi_0^R | e^{-F} p e^F | \psi_0^R \rangle = 0 \\ \langle \psi_0^C | q | \psi_0^C \rangle &= 0, \quad \langle \psi_0^L | q | \psi_0^L \rangle = \langle \psi_0^R | q | \psi_0^R \rangle = \mu/\lambda \end{aligned} \quad (32)$$

while non-diagonal expectation values are all zero as $e^{-\alpha\mu^2/\lambda^2}$.

Therefore φ_0^C and $\varphi_0^L = (\varphi_0^L + \varphi_0^R)/\sqrt{2}$ represent, in the classical limit, the wave functions corresponding to the fixed points $(\mu/\lambda, 0)$ and $(0, \mu/\lambda)$, respectively, which are the relevant fixed points for the $\mu > 0$ case we are now discussing (cf. Appendix A).

5. EVALUATION OF THE SCATTERING AMPLITUDE

We can summarize our results on the spectrum for the $\mu > 0$ case, by stating that we find two energy scales. One is μ and separates successive pairs of level which are themselves split by a much smaller energy ΔE which vanishes for $\lambda/\mu \rightarrow 0$. The scale ΔE controls the asymptotic behaviour of cross-sections. Indeed from Eq. (25), $S - 1 \propto e^{-\Delta E Y} = S^{-\Delta E}$.

At the classical level, $\Delta E = 0$. In the one-dimensional quantum mechanics, the tunnel effect unavoidably gives rise to a non-vanishing ΔE of the order given by Eq. (31). How much relevance we must assign to this ΔE is another matter. The approximation made here to consider the problem as a one-dimensional one, can play a basic role for what regards ΔE . On one side, the non-zero value of α' can build up scales which are larger than ΔE . But independently of this, a tunnel effect is absent in a quantum problem with more than one dimension. It can therefore happen that the consideration of the complete three-dimensional problem will bring back ΔE to its zero classical value, and therefore cross-sections to be constants. Not having solved the three-dimensional problem, these are only speculations. What we can say is that if ΔE is small, as we find for the tunnel effect situation of Eq. (31), there is a long range of rapidity for which we recover the classical results of Ref. 1) for small λ/μ . Indeed for $1/\Delta E \gg Y \gg 1/\mu$, the S matrix is almost constant. Let us now find the value of such constants as function of g_1 and g_2 .

For sufficiently small g_1 (i.e., $g_1, g_2 < \mu/\lambda$), we can retain the harmonic approximations (32), in calculating the matrix elements appearing in Eq. (23), i.e.,

$$\begin{aligned} \langle \psi_0^C | e^{-F} O_2 e^F | \psi_0^C \rangle &\sim e^{-g_2 \mu / \lambda} \\ \langle \psi_0^{CR} | O_1 | \psi_0^{CR} \rangle &\sim e^{-g_1 \mu / \lambda} \end{aligned} \quad (33)$$

where on the right-hand side we neglect polynomials in λ/μ which, being typical quantum effects, will not appear in a classical approximation. Moreover,

$$\begin{aligned} \langle \psi_0^C | O_1 | \psi_0^C \rangle &\sim 1 \\ \langle \psi_0^{CR} | e^{-F} O_2 e^F | \psi_0^{CR} \rangle &\sim 1 \end{aligned} \quad (33)$$

again neglecting polynomials of λ/μ . Introducing (30) and (33) in Eq. (23), we obtain for Y in the interval mentioned above

$$\begin{aligned} S &\sim \frac{\langle \psi_0^S | e^{-F} O_2 e^F | \psi_0^C \rangle \langle \psi_0^C | O_1 | \psi_0^S \rangle + \langle \psi_0^S | e^{-F} O_2 e^F | \psi_0^{CR} \rangle \langle \psi_0^{CR} | O_1 | \psi_0^S \rangle}{\langle \psi_0^S | e^{-F} O_2 e^F | \psi_0^S \rangle \langle \psi_0^S | O_1 | \psi_0^S \rangle} \sim \\ &\sim e^{-g_1 \mu / \lambda} + e^{-g_2 \mu / \lambda} \end{aligned} \quad (34)$$

which is the classical result. Of course, classically for the least action principle, one discards the higher of the two terms. Similarly, we have neglected $\langle \psi_0^C | e^F O_2 e^{-F} | \psi_0^{LR} \rangle$ and $\langle \psi_0^C | O_1 | \psi_0^{LR} \rangle$ which are of the order $e^{-\alpha(\mu^2/\lambda^2)}$.

For $g_1, g_2 > \mu/\lambda$

$$\begin{aligned} \langle \psi_0^C | e^{-F} O_2 e^F | \psi_0^{CR} \rangle &\sim e^{-\alpha \mu^2 / \lambda^2} \\ \langle \psi_0^{CR} | O_1 | \psi_0^C \rangle &\sim e^{-\alpha \mu^2 / \lambda^2} \end{aligned} \quad (35)$$

where now we neglect $e^{-g_1(\mu/\lambda)}$ and $e^{-g_2(\mu/\lambda)}$. In this case we get

$$S \sim e^{-2\alpha \mu^2 / \lambda^2} \quad (36)$$

which is the classical result if we recall that $\alpha = 1/4$ in the W.K.B. approximation.

6. DISCUSSION

The one-dimensional quantum theory we have developed here gives, for $\alpha_P(0) - 1 \equiv \mu < 0$, the expected result that cross-sections drop as s^μ . For the $\mu > 0$ case, the relevant result is the appearance of a spectrum controlled by two energy scales μ and ΔE , the second being much smaller than the first and vanishes for $\lambda/\mu \rightarrow 0$. The first controls the separation of pairs of levels which are themselves separated by energies of the order of the second. Both effects have a basically non-perturbative nature which is interesting to discuss.

The first effect is a typical result of a situation in which there is a spontaneous symmetry breaking. The Pomeron that was originally a tachyon ($\mu > 0$) becomes a normal "particle" in the sense that, if it should be for the aforementioned splitting, cross-sections would drop to zero as $s^{-\mu}$ and all successive cuts would be spaced as expected for a "Pomeron" shifted at $\alpha_P(0) = 1 - \mu$. The origin of the other effects is due to the presence of two minima in the Schrödinger potential which classically gives rise to two degenerate ground states (constant cross-section), while in the one-dimensional quantum mechanics the tunnel effect gives unavoidably a splitting ΔE as in Eq. (31) and, therefore, cross-sections behaving as $s^{-\Delta E}$. We have, however, already discussed the fact that the actual value of ΔE (in particular, $\Delta E \neq 0$) can well be related to the approximation made in considering just the one-dimensional quantum theory.

We see therefore that the approximate constancy of cross-sections can well be obtained if $\mu > 0$ and ΔE very small ^{*)}. In the one-dimensional theory discussed here ($\alpha' \sim 0$), this can be achieved for small values of λ/μ and, therefore, for large μ . As discussed in Ref. 1), this generates cross-sections that rise with the rapidity Y for small Y -- perturbative Pomeron with $\mu > 0$ -- and then tend towards constants (or slightly decreasing cross-sections if ΔE is small but different from zero) for large rapidities.

*) In Ref. 6) a different conclusion was reached, i.e., that only $\mu = 0$ (a critical Pomeron) could be compatible with non-vanishing cross-sections. The approach of Ref. 6) is basically different from ours: the pq symmetry is broken to start with, so that the vacuum is asymmetric. In our language the ground state wave function would be concentrated in one of the minima of the potential. They then develop the theory by coupling the field developed around that minima (i.e., the shifted field) to the external sources. The excitation energy in this one minimum situation is controlled by μ and this unavoidably gives rise to vanishing cross-sections if $\mu \neq 0$.

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A P P E N D I X A

In this Appendix, we shall briefly restate some of the results of Ref. 1) in language better suited to the quantum extension attempted here.

We shall first rediscuss the classical paths $q(y)$, $p(y)$ by following them from "time" $y = -\infty$ to $y = +\infty$. The external sources act at times y_1 and y_2 ($Y = y_2 - y_1$ being large and positive) by inducing at y_1 a jump in p of g_1 and at y_2 in q of $-g_2$. In the three intervals $-\infty < y < y_1$, $y_1 < y < y_2$ and $y_2 < y < \infty$ the paths will follow some of the orbits (solutions of the sourceless problem) characterized each by some energy E .

Having an infinite time to spend in the first interval, in order to obtain a finite action (whose geometrical interpretation is the area subtended by the path in the pq plane) the classical path must start, at $y = -\infty$, in one of the fixed points of the problem. And, with a similar argument at $y = +\infty$ it must land in a fixed point. We have three candidates for departure and landing, i.e., $pq = (0,0)$, $(\frac{\mu}{\lambda}, 0)$, $(0, \frac{\mu}{\lambda})$. The other fixed point $(\frac{\mu}{3\lambda}, \frac{\mu}{3\lambda})$ is stable and therefore there is no possibility to leave it when there.

The fixed points in question, being on trajectories with $E=0$, determine the value of E for the first and the last of the intervals discussed before.

For the $\mu > 0$ case, we want to discuss first, the time evolution on the classical trajectories with $E=0$ are represented by the arrows on the straight lines passing through the fixed points in Fig. 3. Now, at time y_1 the jump in p will take the configuration away from our $E=0$ trajectory. At time $y = y_2$ the jump in q must take the point back into our $E=0$ trajectory. It is easy to realize that this can happen only by leaving at $y = -\infty$ from the fixed point $(0, \frac{\mu}{\lambda})$ along the abscissa axis towards the origin or by leaving from $(\frac{\mu}{\lambda}, 0)$ following the line $p+q = \frac{\mu}{\lambda}$ towards the fixed point $(0, \frac{\mu}{\lambda})$. Similarly, there is only one way to land in each of the fixed points $(\frac{\mu}{\lambda}, 0)$ and $(0, \frac{\mu}{\lambda})$. In particular, the origin cannot be a departure or an arrival point.

All four combinations of the two departure ($y = -\infty$) and two arrival ($y = +\infty$) situations are possible. The minimal action is, however, obtained when the departure point is $(0, \frac{\mu}{\lambda})$ and the arrival one $(\frac{\mu}{\lambda}, 0)$ as shown in Fig. 4 for the situation in which $g_1, g_2 > \frac{\mu}{\lambda}$. In this combination the $E=0$ sector of the path, first and third intervals, follows the abscissa and ordinate axes, respectively.

The discussion of the $\mu < 0$ in similar terms is straightforward. The fixed points appear now for negative values of p and/or q (apart from the origin) and arrows on the $E = 0$ orbits (straight lines) are reversed. The sequence is the same as before, i.e., jump in p of g_1 at $y=y_1$ and jump in q of $-g_2$ at $y=y_2$. It is easy to see that now the origin can be a departure point (at $y=-\infty$) as well as an arrival one (at $y=+\infty$). The other departure point is $(0, \frac{\mu}{\lambda})$ - leaving along the hypotenuse of the triangle - and the other arrival one is $(\frac{\mu}{\lambda}, 0)$ - arriving along the same hypotenuse. The minimal action is given by departing and arriving at the origin, the path is described also in Fig. 5. The asymptotic action is obviously zero, and the approach is given by the perturbative result ¹⁾ $g_1 g_2 s^{\mu}$ which gives the correct scattering amplitude (cf., Appendix B).

We want, finally, to give the co-ordinate transformation which brings for $\mu > 0$ the Hamiltonian $H = pq[\lambda(p+q)-\mu]$ ^{*} to the standard form $\frac{1}{2}p^2 + V(x)$ which allows the definition of a potential. Introducing new variables η and x given by

$$q = x^2, \quad p = \frac{\eta}{2x} + \frac{\mu}{2\lambda} - \frac{\xi^2}{2} \quad (\text{A.1})$$

we find

$$H = \frac{1}{4}\eta^2 + V(x) \quad (\text{A.2})$$

$$V(x) = -\frac{1}{4}\left(x^2 - \frac{\mu}{\lambda}\right)^2 \quad (\text{A.3})$$

The potential V is depicted in Fig. 6. The stable fixed points $x = \pm \sqrt{\frac{\mu}{3\lambda}}$ correspond to the $(\frac{\mu}{3\lambda}, \frac{\mu}{3\lambda})$ in the pq plane. The maxima at $x = \pm \sqrt{\frac{\mu}{\lambda}}$ correspond to the $(0, \frac{\mu}{\lambda})$ while the $x=0$ one corresponds to both the $(0, 0)$ and the $(\frac{\mu}{\lambda}, 0)$ one. It is easy to see, by studying the equations of motion, that $p = \frac{\mu}{\lambda}$ if $dx^2/dy > 0$ (i.e., the point leaves the unstable fixed point) while $p = 0$ if $dx^2/dy < 0$ (i.e., it reaches the unstable fixed point).

Given the potential (A.3), the orbits of Ref. 1) appear clearly. For $E=0$ we find the "soliton-like solutions" which, after spending an infinite time in one of the equipotential maxima, make a transition to another one, where they stay up to $+\infty$. For $-\mu^3/27 < E < 0$ the solution can be trapped in one of the holes where they oscillate around the minimum [equivalent to the $(\frac{\mu}{3\lambda}, \frac{\mu}{3\lambda})$ fixed point of Ref. 1)].

^{*}) We use here the sign convention of Ref. 1). This implies an opposite sign of H with respect to the expression of Eq. (8). Cf., footnote on p. 4.

A P P E N D I X B

Let us compute the classical limit of our theory by reintroducing $\hbar$ (set equal to 1 at the beginning) and letting $\hbar \rightarrow 0$ in the evaluation of the evolution operator

$$\mathcal{U}(x_2, x_1; Y) = \sum_n \varphi_n^*(x_2) \varphi_n(x_1) e^{-E_n Y} \quad (\text{B.1})$$

In the limit we are considering, we can use a WKB approximation for the eigenfunctions $\varphi_n(x)$

$$\varphi_n = N_n e^{\frac{2i}{\hbar} \int_{x_1}^x \sqrt{E_n - V(x')} dx'} \quad (\text{B.2})$$

where N_n is a normalization factor, and $V(x)$ is the quantum mechanical potential. Then the evolution operator becomes,

$$\begin{aligned} \mathcal{U}(x_2, x_1; Y) &= \sum_n \exp \frac{2}{\hbar} \int_{y_1}^{y_2} (i\hbar \sqrt{E_n - V(x)} - E_n) dy = \\ &= \int dE \, \delta(E) \exp \frac{2}{\hbar} \int_{y_1}^{y_2} (i\hbar \sqrt{V(x) - E} - E) dy \end{aligned} \quad (\text{B.3})$$

where $x(y_1) = x_1$, and $x(y_2) = x_2$. We can see that the introduction of the i inside the square root has changed the sign of the potential. For $\hbar \rightarrow 0$ the only contribution to the integral of Eq. (B.3) comes from the stationary value of the exponential. This fixes E to be given by

$$y = \int_{x_1}^{x_2} \frac{dx'}{\sqrt{\lambda(V(x') - E)}} = \frac{2}{\lambda} \int_{x_1}^{x_2} \frac{dx'}{\sqrt{(x'^2 - \frac{\mu^2}{\lambda})^2 - \frac{4E}{\lambda x'^2}}} = \int_{x_1}^{x_2} \frac{dx}{\lambda x \sqrt{(x - \frac{\mu}{\lambda})^2 - \frac{4E}{\lambda x}}} \quad (\text{B.4})$$

which is exactly the classical trajectory [Eq. (9) of Ref. 1)], with energy $-E$. The sign difference in E is simply due to the opposite sign convention of the Hamiltonian of the quantum problem (cf. footnote on page 4).

A P P E N D I X C

It is interesting to see how the results of the free-field theory ($\lambda=0$) emerge in our case. This is a rather non-trivial check on the consistency of our quantization procedure, because we have used part of the interaction term of H - the term $\lambda q p^2$ - to build up the kinetic energy term of $\tilde{H}$. In the usual scheme, with q and p treated as creation and annihilation operators, the Hamiltonian of Eq. (8) becomes, for $\lambda=0$, the Hamiltonian of a harmonic oscillator with fundamental frequency $\omega = -\mu$. Consider the case $\mu < 0$, which is the only one in which the free theory makes sense (Pomeron intercept below 1). Then the S matrix is given by

$$S(g_1, g_2; Y) = \frac{\sum_n \langle 0 | e^{-g_1 P} | n \rangle \langle n | e^{-g_2 Q} | 0 \rangle}{\langle 0 | e^{-g_1 P} | 0 \rangle \langle 0 | e^{-g_2 Q} | 0 \rangle} \quad (C.1)$$

$$= \sum_n \frac{g_1 g_2^n}{n!} e^{n\mu Y} = \exp g_1 g_2 e^{\mu Y}$$

which is the eikonalization of the one-Pomeron exchange diagram, whose contribution is precisely $g_1 g_2 e^{\mu Y}$.

In our case, when $\lambda \rightarrow 0$ for negative μ the dominant term in the potential of Eq. (20) is the quadratic term, so we can work with the harmonic oscillator potential ($\beta \equiv -\mu$)

$$\tilde{H} = -\frac{\lambda}{4} \frac{\partial^2}{\partial x^2} + \frac{\beta^2}{4\lambda} x^2 \quad (C.2)$$

which corresponds to a harmonic oscillator with mass $m = 2/\lambda$ and frequency $\omega = \alpha/2$. Moreover, when λ is very small

$$\exp F(x) = \exp -\frac{\mu x^2}{2\lambda} = \exp \frac{\beta}{2\lambda} x^2 \quad (C.3)$$

For the harmonic oscillator, the evolution operator $U(y_2, y_1)$ is known explicitly. We can therefore compute directly the partition function of Eq. (10). Working in the x representation, as in Eq. (24),

$$Z(g_1, g_2, Y) = \left\{ \sqrt{\frac{\beta}{2\pi\lambda}} \iint_{-\infty}^{+\infty} dx_1 dx_2 e^{-\frac{\beta}{2\lambda} x_2^2} e^{-F(x_2)} O_2 e^{F(x_2)}, \sqrt{\frac{\beta}{2\pi\lambda \sinh wY}} \right. \\ \left. e^{-\frac{\beta}{2\lambda \sinh wY} [(x_1^2 + x_2^2) \cosh wY - 2x_1 x_2]} e^{-g_1 x_1^2} e^{-\frac{\beta}{2\lambda} x_1^2} \right\} \frac{1}{Z(0, 0, Y)} \quad (C.4)$$

We have included all the intermediate states because the odd parity levels are automatically decoupled in the matrix element $\langle 0 | e^{-F} O_2 e^F | n \rangle$. First notice that the operator e^F does not change the parity of the wave function. If n is even, we consider the matrix element of O_2 between two even wave functions. Since O_2 is a translation operator in $q = x^2$, the matrix element is

$$\int_{-\infty}^{+\infty} dx \varphi_0(x) O_2 \varphi_n(x) = \int_{-\infty}^{+\infty} dx \varphi_0(x) \varphi_n(\sqrt{x^2 + g_2}) \quad (C.5)$$

If, however, n is odd, φ_n is an odd function of x . In this case O_2 shifts x to $\pm\sqrt{x^2 + g_2}$, where the positive or negative values of the square root are taken in agreement with the sign of x . Then, using the parity properties of

$$\varphi_n \int_{-\infty}^{+\infty} dx \varphi_0(x) O_2 \varphi_n(x) = \int_{-\infty}^{+\infty} dx \varphi_0(x) \varphi_n(-\sqrt{x^2 + g_2}) + \int_0^{+\infty} dx \varphi_0(x) \varphi_n(\sqrt{x^2 + g_2}) = 0 \quad (C.6)$$

Using the explicit form of Eq. (C.3), we obtain from (C.4)

$$Z(g_1, g_2, Y) = \left\{ \frac{\beta}{2\pi\lambda} (\sinh wY)^{-1/2} \iint_0^{+\infty} dx_1 dx_2 e^{-\frac{\beta}{\lambda} x_2^2} \right. \\ \left. \cdot \exp - \frac{\beta}{2\lambda \sinh wY} \left(\sqrt{x_2^2 + g_2} e^{-\frac{wY}{2}} - x_1 e^{\frac{wY}{2}} \right)^2 \cdot e^{-g_1 x_1^2} \right\} \frac{1}{Z(0, 0, Y)} \quad (C.7)$$

In the limit $\lambda \rightarrow 0$ both integrals can be done explicitly by using the fact that we have in the integrated two Gaussians with the correct normalization factors to become δ functions. Then, the final result is

$$Z(g, g, Y) = \exp -g, g_2 e^{-2wY} \quad (C.8)$$

Recalling that now $w = -\mu/2$, this result gives an S matrix [cf. Eq. (3)] which coincides with the result of Eq. (C.1).

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FIGURE CAPTIONS

- Fig. 1 : The integration paths for p and q in the complex plane.
- Fig. 2 : The quantum potential as given by Eq. (21) (the curve is
computed for $\lambda/\mu = 1/4$).
- Fig. 3 : y evolution on $E=0$ classical orbits.
- Fig. 4 : Classical trajectory for $\mu > 0$ (case $g_1, g_2 < \mu/\lambda$).
- Fig. 5 : Classical trajectory for $\mu < 0$.
- Fig. 6 : The classical potential.

Fig. 1

Fig. 2

FIG. 3

FIG. 4

FIG. 5

FIG. 6