More on the Spectrum of Perturbations in String Gas Cosmology

Robert H. Brandenberger \(^1\)

Sugumi Kanno \(^1\)

Jiro Soda \(^2\)

Damien A. Easson \(^3\)

Justin Khoury \(^4\)

Patrick Martinneau \(^1\)

Ali Nayeri \(^5\)

And Subodh P. Patil \(^1\)

1) Department of Physics, McGill University, Montréal, QC, H3A 2T8, Canada

2) Department of Physics, Kyoto University, Kyoto, Japan

3) Centre for Particle Theory, Department of Mathematical Sciences, Durham University, Science Laboratories, South Road, Durham, DH1 3LE, U.K.

4) Perimeter Institute, 31 Caroline St. N., Waterloo, ON, N2L 2Y5, Canada and

5) Jefferson Physical Laboratory, Harvard University, Cambridge, MA, 02138, USA

(Dated: August 28, 2006)

String gas cosmology is rewritten in the Einstein frame. In an effective theory in which a gas of closed strings is coupled to a dilaton gravity background without any potential for the dilaton, the Hagedorn phase which is quasi-static in the string frame corresponds to an expanding, non-accelerating phase from the point of view of the Einstein frame. The Einstein frame curvature singularity which appears in this toy model is related to the blowing up of the dilaton in the string frame. However, for large values of the dilaton, the toy model clearly is inapplicable. Thus, there must be a new string phase which is likely to be static with frozen dilaton. With such a phase, the horizon problem can be successfully addressed in string gas cosmology. The generation of cosmological fluctuations in the Hagedorn phase seeded by a gas of long strings in thermal equilibrium is reconsidered, both from the point of view of the string frame (in which it is easier to understand the generation of fluctuations) and the Einstein frame (in which the evolution equations are well known). It is shown that fixing the dilaton at some early stage is important in order to obtain a scale-invariant spectrum of cosmological fluctuations in string gas cosmology.

PACS numbers: 98.80.Cq

INTRODUCTION

String gas cosmology is a model of superstring cosmology which is based on coupling to a classical dilaton gravity background a gas of classical strings with a mass spectrum corresponding to one of the consistent perturbative superstring theories \(^1\) (see also \(^2\) for early work, and \(^4\) \(^5\) \(^6\) for reviews). String gas cosmology has been developed in some detail in recent years \(^7\) \(^8\) \(^9\). In particular, it was shown \(^10\) \(^11\) \(^12\) \(^13\) \(^14\) \(^15\) \(^16\) that string modes which become massless at enhanced symmetry points lead to a stabilization of the volume and shape moduli of the six extra spatial dimensions (see \(^17\) for arguments in the context of string gas cosmology on how to naturally obtain the separation between three large and six string-scale dimensions \(^19\)).

String gas cosmology is usually formulated in the string frame, the frame in which stringy matter couples canonically to the background dilaton space-time. The existence of a maximal temperature \(^21\) of a gas of weakly interacting strings in thermal equilibrium has crucial consequences for string cosmology. As discussed in \(^1\), as we follow our universe back in time through the radiation phase of standard cosmology, then when the temperature approaches its limiting value, the energy shifts from the radiative modes to the string oscillatory and winding modes. Thus, the pressure approaches zero. In the string frame, and for zero pressure, as follows from the equations derived in \(^2\) \(^21\), the universe is quasi-static. There is an attractive fixed point of the dynamics in which the scale factor of our large three dimensions is constant, but the dilaton is dynamical (we consider the branch of solutions in which the dilaton is a decreasing function of time). We call this phase the quasi-static Hagedorn phase. Since the string frame Hubble radius is extremely large in the Hagedorn phase (infinite in the limiting case that the scale factor is exactly constant), but decreases dramatically during the transition to the radiation phase of standard cosmology, all comoving scales of interest in cosmology today are sub-Hubble initially, propagate on super-Hubble scales for a long time after the transition to the radiation phase before re-entering the Hubble radius at late times. Thus, it appears in principle possible to imagine a structure formation mechanism driven by local physics.

As was recently suggested \(^22\), string thermodynamic fluctuations in the Hagedorn phase may lead to a nearly scale-invariant spectrum of cosmological fluctuations. There would be a slight red tilt for the spectrum of scalar metric perturbations. A key signature of this scenario would be a slight blue tilt for the spectrum of gravitational waves \(^24\) (see also \(^25\) \(^26\) for more detailed treatments). Since this result is surprising from the point of view of particle cosmology, and since the analyses of \(^25\) \(^24\) contained approximations, it is an interesting challenge to analyze the cosmology from the point of view of the Einstein frame, the frame in which cosmologists have a better physical intuition.

In this paper, we analyze both the background dynamics of string gas cosmology and the generation and evolu-
tion of cosmological perturbations in the Einstein frame. A naive extrapolation of the background solutions of \cite{2} into the past would yield a cosmological singularity. Such an extrapolation is clearly not justified once the dilaton reaches values for which we enter the strong coupling regime of string theory. Rather, at early times one must have a new phase of the theory in which the dynamics is consistent with the qualitative picture which emerges from string thermodynamics. This phase is meta-stable and will decay into a phase with rolling dilaton. The modified background evolution can solve the horizon and singularity problems in the context of string gas cosmology.

An improved analysis of generation of fluctuations in the string frame shows that the conclusions of \cite{23, 24} concerning the spectra of cosmological perturbations and gravitational waves (we are considering the case where our three large dimensions are toroidal) are only obtained if the dilaton velocity can be neglected.

**BACKGROUND DYNAMICS IN THE EINSTEIN FRAME**

String gas cosmology is based on T-duality symmetry and on string thermodynamics. String thermodynamics yields the existence of a maximal temperature of a gas of strings in thermal equilibrium, the Hagedorn temperature \cite{20}. If we consider \cite{1} adiabatic evolution of a gas of strings in thermal equilibrium as a function of the radius of space $R$, T-duality \cite{27} yields a temperature-radius curve (see Fig. 1) which is symmetric about the self-dual radius. Being at the self-dual radius must be a fixed point of the dynamics. The higher the entropy of the string gas is at a fixed radius, the larger is the flat region of the curve, the region where the temperature remains close to the Hagedorn temperature. This implies that the duration of the Hagedorn phase will increase the larger the energy density is.

In a regime in which it is justified to consider the dynamics in terms of a gas of strings coupled to background dilaton gravity, the string frame action for the background fields, the metric and the dilaton (we will set the antisymmetric tensor field to zero) is

$$S = -\int d^{1+N}x \sqrt{-g} e^{-2\phi} [R + 4g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi], \quad (1)$$

where $R$ is the string frame Ricci scalar, $g$ is the determinant of the string frame metric, $N$ is the number of spatial dimensions, and $\phi$ is the dilaton field. Note that we are working in units in which the dimensionful pre-factor appearing in front of the action is set to 1.

The background is sourced by a thermal gas of strings. Its action $S_m$ is given by the string gas free energy density $f$ (which depends on the string frame metric) via

$$S_m = \int d^{1+N}x \sqrt{-g} f. \quad (2)$$

The total action is the sum of $S$ and $S_m$. Note that the factor $e^{-2\phi}$ gives the value of Newton’s gravitational constant.

In the case of a spatially flat, homogeneous and isotropic background given by

$$ds^2 = dt^2 - a(t)^2 dx^2, \quad (3)$$

the three resulting equations of motion of dilaton-gravity (the generalization of the two Friedmann equations plus the equation for the dilaton) in the string frame are \cite{2} (see also \cite{21})

\begin{align*}
- N\dot{\lambda}^2 + \dot{\phi}^2 &= e^\phi E, \\
\dot{\phi} \lambda - \phi \dot{\lambda} &= \frac{1}{2} e^\phi P, \\
\ddot{\phi} - N\dot{\lambda}^2 &= \frac{1}{2} e^\phi E, \quad (4)\end{align*}

where $E$ and $P$ denote the total energy and pressure, respectively, and we have introduced the logarithm of the scale factor

$$\lambda(t) = \log(a(t)), \quad (7)$$

and the rescaled dilaton

$$\varphi = 2\phi - N\lambda. \quad (8)$$

where $N$ is the number of spatial dimensions (in the following, the case of $N = 3$ will be considered).

The Hagedorn phase is characterized by vanishing $P$ and, therefore, constant total energy $E$. Thus, combining

\begin{align*}
- N\dot{\lambda}^2 + \dot{\phi}^2 &= e^\phi E, \\
\dot{\phi} \lambda - \phi \dot{\lambda} &= \frac{1}{2} e^\phi P, \\
\ddot{\phi} - N\dot{\lambda}^2 &= \frac{1}{2} e^\phi E, \quad (4)\end{align*}

where $E$ and $P$ denote the total energy and pressure, respectively, and we have introduced the logarithm of the scale factor

$$\lambda(t) = \log(a(t)), \quad (7)$$

and the rescaled dilaton

$$\varphi = 2\phi - N\lambda. \quad (8)$$

where $N$ is the number of spatial dimensions (in the following, the case of $N = 3$ will be considered).

The Hagedorn phase is characterized by vanishing $P$ and, therefore, constant total energy $E$. Thus, combining

\begin{align*}
- N\dot{\lambda}^2 + \dot{\phi}^2 &= e^\phi E, \\
\dot{\phi} \lambda - \phi \dot{\lambda} &= \frac{1}{2} e^\phi P, \\
\ddot{\phi} - N\dot{\lambda}^2 &= \frac{1}{2} e^\phi E, \quad (4)\end{align*}

where $E$ and $P$ denote the total energy and pressure, respectively, and we have introduced the logarithm of the scale factor

$$\lambda(t) = \log(a(t)), \quad (7)$$

and the rescaled dilaton

$$\varphi = 2\phi - N\lambda. \quad (8)$$

where $N$ is the number of spatial dimensions (in the following, the case of $N = 3$ will be considered).

The Hagedorn phase is characterized by vanishing $P$ and, therefore, constant total energy $E$. Thus, combining

\begin{align*}
- N\dot{\lambda}^2 + \dot{\phi}^2 &= e^\phi E, \\
\dot{\phi} \lambda - \phi \dot{\lambda} &= \frac{1}{2} e^\phi P, \\
\ddot{\phi} - N\dot{\lambda}^2 &= \frac{1}{2} e^\phi E, \quad (4)\end{align*}

where $E$ and $P$ denote the total energy and pressure, respectively, and we have introduced the logarithm of the scale factor

$$\lambda(t) = \log(a(t)), \quad (7)$$

and the rescaled dilaton

$$\varphi = 2\phi - N\lambda. \quad (8)$$

where $N$ is the number of spatial dimensions (in the following, the case of $N = 3$ will be considered).

The Hagedorn phase is characterized by vanishing $P$ and, therefore, constant total energy $E$. Thus, combining

\begin{align*}
- N\dot{\lambda}^2 + \dot{\phi}^2 &= e^\phi E, \\
\dot{\phi} \lambda - \phi \dot{\lambda} &= \frac{1}{2} e^\phi P, \\
\ddot{\phi} - N\dot{\lambda}^2 &= \frac{1}{2} e^\phi E, \quad (4)\end{align*}

where $E$ and $P$ denote the total energy and pressure, respectively, and we have introduced the logarithm of the scale factor

$$\lambda(t) = \log(a(t)), \quad (7)$$

and the rescaled dilaton

$$\varphi = 2\phi - N\lambda. \quad (8)$$

where $N$ is the number of spatial dimensions (in the following, the case of $N = 3$ will be considered).

The Hagedorn phase is characterized by vanishing $P$ and, therefore, constant total energy $E$. Thus, combining

\begin{align*}
- N\dot{\lambda}^2 + \dot{\phi}^2 &= e^\phi E, \\
\dot{\phi} \lambda - \phi \dot{\lambda} &= \frac{1}{2} e^\phi P, \\
\ddot{\phi} - N\dot{\lambda}^2 &= \frac{1}{2} e^\phi E, \quad (4)\end{align*}

where $E$ and $P$ denote the total energy and pressure, respectively, and we have introduced the logarithm of the scale factor

$$\lambda(t) = \log(a(t)), \quad (7)$$

and the rescaled dilaton

$$\varphi = 2\phi - N\lambda. \quad (8)$$

where $N$ is the number of spatial dimensions (in the following, the case of $N = 3$ will be considered).

The Hagedorn phase is characterized by vanishing $P$ and, therefore, constant total energy $E$. Thus, combining

\begin{align*}
- N\dot{\lambda}^2 + \dot{\phi}^2 &= e^\phi E, \\
\dot{\phi} \lambda - \phi \dot{\lambda} &= \frac{1}{2} e^\phi P, \\
\ddot{\phi} - N\dot{\lambda}^2 &= \frac{1}{2} e^\phi E, \quad (4)\end{align*}

where $E$ and $P$ denote the total energy and pressure, respectively, and we have introduced the logarithm of the scale factor

$$\lambda(t) = \log(a(t)), \quad (7)$$

and the rescaled dilaton

$$\varphi = 2\phi - N\lambda. \quad (8)$$

where $N$ is the number of spatial dimensions (in the following, the case of $N = 3$ will be considered).

The Hagedorn phase is characterized by vanishing $P$ and, therefore, constant total energy $E$. Thus, combining

\begin{align*}
- N\dot{\lambda}^2 + \dot{\phi}^2 &= e^\phi E, \\
\dot{\phi} \lambda - \phi \dot{\lambda} &= \frac{1}{2} e^\phi P, \\
\ddot{\phi} - N\dot{\lambda}^2 &= \frac{1}{2} e^\phi E, \quad (4)\end{align*}

where $E$ and $P$ denote the total energy and pressure, respectively, and we have introduced the logarithm of the scale factor

$$\lambda(t) = \log(a(t)), \quad (7)$$

and the rescaled dilaton

$$\varphi = 2\phi - N\lambda. \quad (8)$$

where $N$ is the number of spatial dimensions (in the following, the case of $N = 3$ will be considered).

The Hagedorn phase is characterized by vanishing $P$ and, therefore, constant total energy $E$. Thus, combining
and (4) to eliminate the dependence on $\dot{\lambda}$, yields a second order differential equation for $\varphi$ with the solution

$$e^{-\varphi(t)} = \frac{E_0}{4}t^2 - \varphi_0 e^{-\varphi_0} t + e^{-\varphi_0}$$

subject to the initial condition constraint

$$\varphi_0^2 = e^{\varphi_0} E_0 + N\lambda_0^2$$

which follows immediately from (4). In the above, the subscripts stand for the initial values at the time $t = 0$.

In the Hagedorn phase, the second order differential equation (5) for $\lambda$ can easily be solved. If the initial conditions require non-vanishing $\lambda_0$, the solution is

$$\lambda(t) = \lambda_0 + \frac{1}{\sqrt{N}} \ln \left[ \frac{\sqrt{N} \lambda_0 - G}{\sqrt{N} \lambda_0 + G} \right] ,$$

where $G$ is an abbreviation which stands for

$$G = \frac{Et}{2} e^{\varphi_0} - \dot{\varphi}_0 .$$

For vanishing initial value of the derivative of the scale factor, the solution is simply

$$\lambda(t) = \lambda_0 ,$$

i.e. a static metric. In the static case, the result (4) simplifies to

$$e^{-\varphi(t)} = e^{-\varphi_0} \left( \frac{\varphi_0 t}{2} - 1 \right)^2 .$$

These solutions are slight generalizations of the solutions given in the appendix of (2). These solutions have also very recently been discussed in (22). We are interested in the branch of solutions with $\dot{\varphi} < 0$.

One important lesson which follows from the above solution is that, although the metric is static in the Hagedorn phase, a dilaton singularity develops at a time $t_s$ given by

$$t_s = -\frac{2}{|\varphi_0|} .$$

In fact, already at a slightly larger time $t_c$, the dilaton has reached the critical value $\phi = 0$, beyond which string perturbation theory breaks down. The times $|t_s|$ and $|t_c|$ are typically of string scale. Thus, unless the current value of the dilaton is extremely small, the duration of the phase in which the above solution is applicable will be short.

Note, however, that the solution (13) is not consistent with the qualitative picture which emerges from string thermodynamics (1) (see Fig. 1) according to which the evolution of all fields close to the Hagedorn temperature should be almost static. We know that the dilaton gravity action ceases to be justified in the region in which the theory is strongly coupled. This leads to the conclusion that the phase during which (13) is applicable must be preceded by another phase of Hagedorn density, a phase in which the dynamics reflects the qualitative picture which emerges from Fig. 1, and corresponds to fixed scale factor and fixed dilaton. We call this phase the strong coupling Hagedorn phase (50). Note that the Einstein action is not invariant under T-duality. Hence, we expect that intuition based on Einstein gravity will give very misleading conclusions when applied to the strong coupling Hagedorn phase. In particular, constant energy density should not lead to a tendency to expansion. The strong coupling phase is long-lived but meta-stable and decays into a solution in which the dilaton is free to roll, a phase described by the equations (4-6).

If the equation of state is that of radiation, namely $P = 1/3E$, then a solution with static dilaton is an attractor. For static dilaton, the equations (5) and (6) then reduce to the usual Friedmann-Robertson-Walker-Lemaitre equations.

Figure 2 shows a space-time sketch from the perspective of string frame coordinates. The Hagedorn phase lasts until the time $t_R$ (the time interval from $t_s$ to close to $t_R$ being describable by the equations of motion of dilaton gravity) when a smooth transition to the radiation phase of standard cosmology takes place. This transition is governed by the annihilation of string wind-

ing modes into oscillatory modes and is described by Boltzmann-type equations (3) (with corrections pointed out in (15, 19)). These Boltzmann equations are analogs of the equations used in the cosmic string literature (see (28) for reviews) to describe the transfer of energy between “long” (i.e. super-Hubble) strings and string loops. Note that the decay of string winding modes into radiation is the process that “heats” the universe.

Close to the time $t_s$, we reach the strong coupling Hagedorn phase, the phase responsible for generating a large horizon.

In order that the cosmological background of Figure 2 match with our present cosmological background, the radius of space at the end of the Hagedorn phase needs to be of the order of 1mm, the size that expands into our currently observed universe making use of standard cosmology evolution beginning at a temperature of about $10^{15}$GeV. This is many orders of magnitude larger than the string size. Thus, without further assumptions, there is a cosmological “horizon” and “entropy” problem, similar to the one present in Standard Big Bang (SBB) cosmology. Provided that the strong coupling Hagedorn phase is long-lived (and, based on Fig. 2 this is more likely the higher the initial energy density is chosen), string gas cosmology will be able to resolve these problems. In particular, there will be enough time to establish thermal equilibrium over the entire spatial section, a necessary condition for the structure formation scenario outlined in (23) to work.
The above issues become more manifest when the cosmological background is rewritten in the Einstein frame. It is to this subject to which we now turn.

The conformal transformation of the metric to the Einstein frame is given by

\[
\tilde{g}_{\mu\nu} = e^{-4\phi/(N-1)}g_{\mu\nu} = e^{-2\phi}g_{\mu\nu}
\]

(16)

where quantities with a tilde refer to those in the Einstein frame, and in the final expression we have set \( N = 3 \). The dilaton transforms as (for \( N = 3 \))

\[
\tilde{\phi} = 2\phi.
\]

(17)

Under this transformation, the action \( S_E \) becomes

\[
S_E = -\int d^4x \sqrt{-\tilde{g}}(\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi}) .
\]

(18)

The matter action in the Einstein frame becomes

\[
S_m = \int d^4x e^{2\phi} f(\tilde{g}, \tilde{\phi})\sqrt{\tilde{g}} ,
\]

(19)

from which we see that the factor \( e^{2\phi} \) plays the role of the gravitational constant.

We apply the conformal transformation for the metric of a homogeneous and isotropic universe. To put the resulting Einstein frame metric into the FRWL form, we have to re-scale the time coordinate, defining a new Einstein frame time \( \tilde{t} \) via

\[
d\tilde{t} = e^{-\phi}dt .
\]

(20)

The resulting scale factor \( \tilde{a} \) in the Einstein frame then is given by

\[
\tilde{a} = e^{-\phi}a .
\]

(21)

The comoving spatial coordinates are unchanged.

The Hubble parameters \( \tilde{H} \) and \( H \) in the Einstein and string frames, respectively, are related via

\[
\tilde{H} = e^\phi(H - \dot{\phi}) .
\]

(22)

It is important to note that they denote very different lengths.

Let us now calculate the evolution of the Einstein frame scale factor. Making use of the solution for the string frame dilaton \( 14 \), it follows from \( 21 \) that

\[
\tilde{a} = e^{\lambda_0}e^{-\phi_0}(1 - \dot{\phi}_0t) .
\]

(23)

By integrating \( 20 \) we find that the physical time in the Einstein frame is given by

\[
\tilde{t} = e^{-\phi_0}t(1 - \frac{1}{2}\phi_0 t) .
\]

(24)

It is important to keep in mind that \( \dot{\phi} < 0 \).

It follows that, in the Einstein frame, the evolution looks like that of a universe dominated by radiation. This is easy to verify for large times, when the factors of \(-1\) within the parentheses in \( 23 \) and \( 24 \) are negligible, and it thus follows that

\[
\tilde{a}(\tilde{t}) \sim \tilde{t}^{1/2} .
\]

(25)

The same conclusion can also be reached for times close to the singularity, by explicitly inverting \( 24 \) and inserting into \( 23 \).

We thus see that the expansion is non-accelerated and the Hubble radius is expanding linearly. The dilaton singularity in the string frame is translated into a curvature singularity in the Einstein frame, a singularity which occurs at the Einstein frame time

\[
\tilde{t}_s = \frac{exp(-\phi_0)}{2}\phi_0^{-1} .
\]

(26)
From the constraint equation (10) it follows that this of the order of the string scale.

As stressed earlier, these solutions cannot be applied when the value of the dilaton is larger than 0 (when the string theory enters the strong coupling regime). Instead, we will have a strong coupling Hagedorn phase characterized by constant dilaton, and hence also almost constant Einstein frame scale factor.

The space-time sketch of our cosmology in the Einstein frame is sketched in Figure 3. In the absence of the strong coupling Hagedorn phase, the horizon (forward light cone beginning at $t_s$) would follow the Einstein frame Hubble radius (up to an irrelevant factor of order unity), thus yielding a horizon problem. However, during the strong coupling Hagedorn phase and, in particular, during the transition between the strong coupling Hagedorn phase and the phase described by the rolling dilaton, the horizon expands to lengths far greater than the Einstein frame Hubble radius. This phase can, in particular, establish thermal equilibrium on scales which are super-Hubble in the rolling dilaton phase. In the last section of this paper we will come back to a discussion of how to model the strong coupling Hagedorn phase.

We will close this section with some general comments about the relationship between the string and the Einstein frames. Obviously, since the causal structure is unchanged by the conformal transformation, the comoving horizon is frame-independent. The Hubble radius, on the other hand, is a concept which depends on the frame. The string frame and the Einstein frame Hubble radii are two very different length scales. Both can be calculated in any frame, but they have different meanings.

**COSMOLOGICAL PERTURBATIONS IN THE STRING FRAME**

From the point of view of the string frame, the scale of cosmological fluctuations is sub-Hubble during the Hagedorn phase. In [23] it was assumed that thermal equilibrium in the Hagedorn phase exists on scales of the order of 1mm. It was proposed to follow the string thermodynamical matter fluctuations on a scale $k$ until that scale exits the string frame Hubble radius at the end of the Hagedorn phase, to determine the induced metric fluctuations at that time, and to follow the latter until the present.

To study cosmological perturbations, we make use of a particular gauge choice, longitudinal gauge (see [20] for an in depth review article on the theory of cosmological perturbations and [21] for a pedagogical introduction), in which the metric takes the form

$$ds^2 = e^{2\Lambda(t)} \left( (1 + 2\Phi) dt^2 - (1 - 2\Psi) \delta_{ij} dx^i dx^j \right), \tag{27}$$

where $\eta$ is conformal time, and where $\Psi$ and $\Phi$ are the fluctuation variables which depend on space and time. In Einstein gravity, and for matter without anisotropic stress, the two potentials $\Phi$ and $\Psi$ coincide, and $\Phi$ is the relativistic generalization of the Newtonian gravitational potential. In the case of dilaton gravity, the two potentials are related via the fluctuation of the dilaton field.

According to the proposal of [23], the cosmological perturbations are sourced in the Hagedorn phase by string thermodynamical fluctuations, in analogy to how in inflationary cosmology the quantum matter fluctuations source metric inhomogeneities [51]. On sub-Hubble scales, the matter fluctuations dominate. Hence it was suggested in [23] to track the matter fluctuations on a fixed comoving scale $k$ until the wavelength exits the Hubble radius at time $t_s(k)$ (see Fig. 2). At that time, the induced metric fluctuations are calculated making use of the Poisson equation

$$\nabla^2 \Psi = 4\pi Ga^2 \delta T_{0}^{0}. \tag{28}$$

The key feature of string thermodynamics used in [23] (and discussed in more detail in [24, 21]) is the fact that the specific heat $C_V$ scales as $R^2$, where $R$ is the size of the region in which we are calculating the fluctuations. This result was derived in [31] and holds in the case of three large dimensions with topology of a torus. The
specific heat, in turn, determines the fluctuations in the energy density. The scaling

\[ C_V \sim R^2 \]  

leads to a Poisson spectrum

\[ P_{\varphi\varphi}(k) \sim k^4 \]  

for the energy density fluctuations, and a similar spectrum for the pressure fluctuations \[ [21, 22] \]. Making use of the Poisson equation \[ [25] \), this leads to a scale-invariant spectrum for the gravitational potential \( \Phi \). \[ [52] \)

However, in the context of a relativistic theory of gravity, what should be used to relate the matter and metric fluctuations is the time-time component of the perturbed Einstein equation, or - more specifically - its generalization to dilaton gravity, and not simply the Poisson equation. There are correction terms compared to \[ [28] \) coming from the expansion of the cosmological background and from the dynamics of the dilaton. The analysis of \[ [23, 24] \) was done in the string frame. In this frame, during the Hagedorn phase the correction terms coming from the expansion of space are not present, but the dilaton velocity is important. The important concern is whether the resulting correction terms will change the conclusions of the previous work.

The perturbation equations in the string frame were discussed in \[ [32] \. Denoting the fluctuation of the dilaton field by \( \chi \), the equations read

\[ \nabla^2 \Phi = 3\mathcal{H}\Phi' - 3\mathcal{H}^2\Phi \]

\[ = \frac{1}{2}\epsilon^{2\phi+2\lambda}\left(2\chi T_0^0 + \delta T_0^0\right) - 6\mathcal{H}\Phi' \]

\[ - 3\Psi' \Phi' - \nabla^2 \chi + 3\mathcal{H}\chi' + 2\Phi\Phi' - 2\chi'\Phi', \]

\[ \partial_t\Psi' + \mathcal{H}\partial_t\Phi = \frac{1}{2}\epsilon^{2\phi+2\lambda}\delta T_{0i}^i + \partial_i\Phi\phi' - \partial_i\chi' + \mathcal{H}\partial_i\chi \]

\[ \partial_i\partial_j\left(\Phi - \Psi - 2\chi\right) = 0 \quad i \neq j, \]

\[ - 2\Psi'' - 4\mathcal{H}\Psi' - 2\mathcal{H}^2\Phi - 4\mathcal{H}'\Phi - 2\Phi'\mathcal{H} \]

and

\[ - 2\Phi\phi'' + 2\phi'\chi' + \Phi\phi'' + \frac{1}{2}\Phi\phi' + 2\mathcal{H}\Phi' \]

\[ + \frac{3}{2}\Psi' \phi' - \frac{1}{2}\chi'' + \frac{1}{2}\nabla^2 \chi - \mathcal{H}\chi' = \frac{1}{4}\epsilon^{2\phi+2\lambda}\left(2\chi T + \delta T\right), \]

where \( T \equiv T_{\mu}^{\mu} \) is the trace. The first equation is the time-time equation, the second the space-time equation, the next two the off-diagonal and diagonal space-space equations, respectively, and the last one is the matter equation.

The times \( t_{\mu}(k) \) when the metric perturbations were computed are in the transition period between the Hagedorn phase and the radiation phase of standard cosmology. Space is beginning to expand. In this case, neither the terms containing the Hubble expansion rate nor those containing the dilaton velocity vanish.

We will first consider the case when the dilaton velocity is negligible (we will come back to a discussion of when this is a reasonable approximation), and then the case when the dilaton velocity is important.

If the dilaton velocity is negligible, then the equations simplify dramatically. We first note that at the time \( t_{\mu}(k) \), the comoving Hubble constant \( \mathcal{H} \) is of the same order of magnitude as \( k \). Consider now, specifically, the time-time equation of motion \[ [31] \). The terms containing \( \mathcal{H} \) and its derivative on the left-hand side of this equation are of the same order of magnitude as the first term. We will, therefore, neglect all terms containing \( \mathcal{H} \). Hence, the equation simplifies to

\[ \nabla^2 \Psi = \frac{1}{2}\epsilon^{2\phi+2\lambda}\left(2\chi T_0^0 + \delta T_0^0\right) - \nabla^2 \chi. \]

Similarly, the perturbed dilaton equation simplifies to

\[ - \frac{1}{2}\chi'' + \frac{1}{4}\nabla^2 \chi = \frac{1}{2}\epsilon^{2\phi+2\lambda}\left(2\chi T + \delta T\right), \]

From the latter equation, it follows that the Poisson spectrum of \( \delta T \) induces a Poisson spectrum of the dilaton fluctuation \( \chi \). Subtracting \[ [37] \) from \[ [36] \) and keeping in mind that the background pressure vanishes in the Hagedorn phase yields

\[ \nabla^2 \Psi = \frac{1}{2}\epsilon^{2\phi+2\lambda}\left(\delta T_0^0 - \delta T\right) - \chi'' \]

from which it follows that the induced spectrum of \( \Psi \) will be scale-invariant

\[ P_{\Psi} \sim k^0. \]
It is easy to check that the other equations of motion are consistent with this scaling. If the Hagedorn phase is modeled by the equations (41) and (42), then the dilaton velocity is not negligible, since it is related to the energy density via the constraint equation (31):

$$\dot{\phi}^2 = \frac{1}{4} e^{2\phi} E.$$

(40)

Thus, the terms containing the dilaton velocity are as important as the other terms in the Hagedorn phase. Now, the prescription of (23, 24) was to use the constraint equation at the time $t_i(k)$ when the scale $k$ exits the Hubble radius. This time is towards the end of the Hagedorn phase. However, in the context of our action, Eq. (41) always holds. The right-hand side of this equation must be large since it gives the (square of the) Hubble expansion rate at the beginning of the radiation phase. At the time $t_i(k)$, then, for scales which are large compared to the Hubble radius at the beginning of the radiation phase, the value of the terms containing $\dot{\lambda}$ in (41) are negligible, and hence the dilaton velocity term is non-negligible.

Let us now compute the induced metric fluctuations in the Hagedorn phase taking into account the terms depending on the dilaton velocity. The time-dependence of the dilaton introduces a critical length scale into the problem, namely the inverse time scale of the variation of the dilaton. Translated to the Einstein frame, this length is the Einstein frame Hubble radius (this radius is, up to a numerical constant, identical to the forward light cone computed beginning at the time of the dilaton singularity). On smaller scales, the dilaton-dependent terms in the time-time Einstein constraint equation (31) are negligible, Eq. (31) reduces to the Poisson equation (25) and we conclude that the Poisson spectrum of the stringy matter induces a flat spectrum for the metric potential $\Psi$.

On larger scales, however, it is the dilaton-dependent terms in (31) which dominate. If we insist on the view that it is the string gas matter fluctuations which seed all metric fluctuations, then we must take all terms independent of the string sources to the left-hand side of the equations of motion. If we do this and keep the terms on the left-hand side of the equations which dominate in a gradient expansion, then the time-time equation becomes

$$3\Psi' \dot{\phi} - 2\Phi \ddot{\phi} + 2\chi' \dot{\phi} = \frac{1}{2} e^{2\phi + 2\lambda} \left( 2\chi T_0^0 + \delta T_0^0 \right),$$

(41)

and the analogous approximation scheme applied to the dilaton equation yields

$$-2\Phi \ddot{\phi} + 2\dot{\phi} \chi' + \Phi \ddot{\chi} + \frac{1}{2} \Phi' \dot{\phi}' + \frac{3}{2} \Psi' \phi' - \frac{1}{2} \chi''$$

$$= \frac{1}{4} e^{2\phi + 2\lambda} \left( 2\chi T + \delta T \right).$$

(42)

Inspection of (12) shows that a Poisson spectrum of $\delta T$ will induce a Poisson spectrum of $\chi$. Subtracting two times (12) from (11) shows that, given a Poisson spectrum of $\chi$, the resulting equation is no longer consistent with a scale-invariant spectrum for $\Psi$ and $\Phi$, since terms which have a scale-invariant spectrum would remain on the left-hand side of the equation. Hence, we conclude that the induced spectrum of $\Phi$ and $\Psi$ will also be Poisson:

$$P_{\Phi} \sim k^4.$$  

(43)

The above conclusion is consistent with the Traschen Integral constraints (32) which state that in the absence of initial curvature fluctuations, motion of matter cannot produce perturbations with a spectrum which is less red than Poisson on scales larger than the horizon. This view is consistent with the fact that on small scales, the spectrum is scale-invariant: on small scales it is possible to move around matter by thermal fluctuations to produce new curvature perturbations.

As we have stressed earlier, however, the equations (41) and (42) are definitely not applicable early in the Hagedorn phase, namely in the strong coupling Hagedorn phase. In that phase, the dilaton is fixed, and thus the arguments of (23) imply the presence of scale-invariant metric fluctuations seeded by the string gas perturbations (32). These fluctuations will persist in the phase in which the dilaton is rolling (the fluctuations cannot suddenly decrease in magnitude). Hence, we believe that the conclusions of (23) are robust.

**COSMOLOGICAL PERTURBATIONS IN THE EINSTEIN FRAME**

From the point of view of the Einstein frame, the scales are super-Hubble during the phase of dilaton rolling. How is this consistent with their sub-Hubble nature from the point of view of the string frame? The answer is that, whereas the causal structure of space-time (and thus concepts like horizons) are frame-independent, the Hubble radius depends on the frame. The physical meaning of the Hubble radius is that it separates scales on which matter oscillates (sub-Hubble) from scales where the matter oscillations are frozen in (super-Hubble). Matter which is coupled minimally to gravity in the string frame feels the string frame Hubble radius (54), matter which is minimally coupled to gravity in the Einstein frame feels the Einstein frame Hubble radius. Strings couple minimally to gravity in the string frame and hence feel the string frame Hubble radius.

If we take into account the presence of the strong coupling Hagedorn phase, then it becomes possible, also in the Einstein frame, to study the generation of fluctuations. During the strong coupling Hagedorn phase, the dilaton is fixed and hence the fluctuation equations are
those of Einstein gravity. Since scales of cosmological interest today are sub-Hubble during this phase, a scale-invariant spectrum of metric fluctuations is induced by the string gas fluctuations, as discussed in the previous section. If the strong coupling Hagedorn phase is long in duration, then a scale-invariant spectrum can be induced consistent with the Traschen integral constraints. Note that the a long duration of the strong coupling phase is required in order to justify the assumption of thermal equilibrium on the scales we are interested in.

We can also obtain the Einstein frame initial conditions by conformally transforming the initial conditions obtained in the string frame. The transformation of the perturbation variables in straightforward:

\[ \dot{\Psi} = (\Psi + \chi) , \]
\[ \dot{\Phi} = (\Phi - \chi) , \]
\[ \dot{\chi} = 2\chi , \]

where, as before, tilde signs indicate quantities in the Einstein frame.

Note, in passing, that the string frame off-diagonal spatial equation of motion (35) immediately implies that

\[ \dot{\Psi} = \Phi , \]

which is the well-known result for Einstein frame fluctuations in the absence of matter with anisotropic stress.

Given the transformation properties (44 - 46) of the fluctuation variables, it is obvious that the conclusions about the initial power spectra of the fluctuations variables are the same as in the string frame: if the dilaton velocity can be neglected, the spectrum of \( \Phi \) and \( \Psi \) is scale-invariant, if the dilaton velocity is important, the spectra of these variables are Poisson.

Whereas setting the initial conditions for the fluctuations may look more ad hoc in the Einstein frame, the evolution of the perturbations is easier to analyze since we can use all of the intuition and results developed in the context of fluctuations in general relativity. In particular, we can use the Deruelle-Mukhanov \[ \text{[34]} \] matching conditions to determine the fluctuations in the post-Hagedorn radiation phase of standard cosmology from those at the end of the Hagedorn phase. The Deruelle-Mukhanov conditions are generalization to space-like hyper-surfaces of the Israel matching conditions \[ \text{[35]} \] which state that the induced metric and the extrinsic curvature need to be the same on both sides of the matching surface. Applied to the case of cosmological perturbations, the result \[ \text{[34]} \] is that (in terms of longitudinal gauge variables) both \( \Phi \) and \( \zeta \) need to be continuous \[ \text{[33]} \] where \( \zeta \) is defined as

\[ \zeta \equiv \tilde{\Phi} + \frac{\mathcal{H}}{\dot{\mathcal{H}}} (\dot{\Phi}' + \mathcal{H} \tilde{\Phi}') , \]

where \( \mathcal{H} \) the Einstein frame Hubble expansion rate with respect to conformal time.

In the Einstein frame, the universe is radiation-dominated both before and after the transition. Hence, in both phases the dominant mode of the equation of motion for \( \tilde{\Phi} \) is a constant. The constant mode in the phase \( \dot{t} < \dot{t}_R \) couples dominantly to the constant mode in the phase \( \dot{t} > \dot{t}_R \). The initial value of the spectrum of \( \tilde{\Phi} \) will seed both the constant and the decaying mode of \( \tilde{\Phi} \) with comparable strengths and with the same spectrum. Hence, the late-time value of \( \tilde{\Phi} \) is given, up to a factor of order unity, by the initial value of \( \tilde{\Phi} \) at the time \( \tilde{t}_i(k) \)

\[ P_{\tilde{\Phi}}(k, \dot{t}) \sim P_{\tilde{\Phi}}(k, \tilde{t}_i(k)) \sim k^\theta \dot{t} >> \tilde{t}_R . \]

**DISCUSSION AND CONCLUSIONS**

In this paper we have recast string gas cosmology in the Einstein frame rather than in the string frame in which the analysis usually takes place. Our analysis sheds new light on several important cosmological issues.

At the level of the background evolution, it becomes clear that solutions of the dilaton gravity equations \[ \text{[4]} \] - \[ \text{[9]} \] with decreasing dilaton contain an initial singularity. From the point of view of the Einstein frame, there is an initial curvature singularity which follows from the presence of a singularity in the dilaton field in the string frame. Obviously, however, these solutions are not applicable at very early times since they correspond to times when the string theory is strongly coupled. Hence, there must be, prior to the phase of rolling dilaton, a strong coupling Hagedorn phase in which both the size of space and the dilaton must be quasi-static. Provided that this phase lasts sufficiently long, thermal equilibrium over all scales relevant to current observations can be established.

Note that during the period when the solutions of \[ \text{[4]} \] - \[ \text{[9]} \] have a rolling dilaton, then in the Einstein frame the expansion of space never accelerates. The evolution corresponds to that of a radiation-dominated universe. The Einstein frame Hubble radius increases linearly in time throughout. However, the presence of the strong coupling Hagedorn phase can solve the horizon problem in the sense of making the comoving horizon larger than the comoving scale corresponding to our currently observed universe.

How to model the strong coupling Hagedorn phase now becomes a crucial question for string gas cosmology \[ \text{[50]} \]. Since the singularity of string gas cosmology is (from the point of view of the string frame) associated with the dilaton becoming large, and thus with string theory entering a strongly coupled phase, it is interesting to conjecture that a process like tachyon condensation \[ \text{[53]} \] will occur and resolve the singularity (like in the work of \[ \text{[14]} \]). If this phase lasts for a long time, it will produce a large space in thermal equilibrium.

There are other possible scenarios in which there is a precursor phase of the rolling dilaton period which es-
establishes thermal equilibrium on large scales. One such possibility was discussed in [12] and makes use of a pre-Hagedorn phase in which the extra spatial dimensions initially expand, driven by a gas of bulk branes. The resulting increase in the energy stored in the branes leads to the increase in size and entropy which solves both the horizon and entropy problems. Once the extra spatial dimensions have contracted again to the string scale, the size of our three spatial dimensions can be macroscopic while the temperature of matter is of string scale.

Another possibility to obtain a solution to the horizon problem and to justify thermal equilibrium over large scales is to invoke a bouncing cosmology such as obtained in the context of higher derivative gravity models in [14] (see also [16] for an earlier construction). The phase of contraction could produce the high densities required to form a string gas with the necessary requirements.

We have also studied the mechanism for the generation of fluctuations proposed in [22, 27] in more detail, both from the point of view of the string frame and the Einstein frame. We have shown that a small value of the dilaton velocity in the Hagedorn phase is required in order that the string thermodynamic fluctuations are able to generate a scale-invariant spectrum of cosmological fluctuations. If dilaton velocity terms are important, then a Poisson spectrum is negligible, and if thermal equilibrium on large scales is produced. If the dilaton velocity in the Hagedorn phase is required in order that the string thermodynamic fluctuations are able to generate a scale-invariant spectrum of cosmological fluctuations. If dilaton velocity terms are important, then a Poisson spectrum is negligible, and if thermal equilibrium on large scales is produced.

Acknowledgements

Two of us (R.B. and A.N.) are grateful to Lev Kofman, Andrei Linde and Slava Mukhanov for detailed discussions, and for emphasizing the need to analyze the dynamics of string gas cosmology completely in the Einstein frame. R.B. thanks Misao Sasaki and the Yukawa Institute for Theoretical Physics for hospitality during the time when this project was initiated. We wish to thank Misao Sasaki, Scott Watson and in particular Cumrun Vafa for fruitful conversations. J.S. would like to thank the Department of Physics, McGill University for kind hospitality during the course of this work. The research of R.B. and S.P. is supported by an NSERC Discovery Grant, by the Canada Research Chairs program, and by an FQRNT Team Grant. The work of D.E. is supported in part by PPARC. S.K. is supported by a JSPS Postdoctoral Fellowship for Research Abroad. The research of J.K. at Perimeter Institute is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MEDT. The work of A.N. is supported in part by NSF grant PHY-0244821 and DMS-0244464. The work of J.S. is supported by the Japan-U.K. Research Cooperative Program, the Japan-France Research Cooperative Program, and the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science and Culture of Japan No.18540262 and No.17340075.

References

A. Chatrabhuti, “Target space duality and moduli stabilization in string gas cosmology,” arXiv:hep-th/0602031


See, however, [18, 19] for some caveats.

Another argument supporting the assumption that in the strong coupling Hagedorn phase the dilaton is fixed can be given making use of S-duality. Under S-duality, the dilaton $\phi$ is mapped to $-\phi$. It is reasonable to assume that close to the maximal temperature, the system is in a configuration which is self S-dual, and in which the dilaton is hence fixed - we thank C. Vafa for stressing this point. Note that we are assuming that the existence of the maximal temperature remains true at strong coupling.

In spite of this analogy, there is a key difference, a difference which is in fact more important: In inflationary cosmology, the fluctuations are quantum vacuum perturbations, whereas in our scenario they are classical thermal fluctuations.

Note that thermal particle fluctuations would not give rise to a scale-invariant spectrum - see 15 for an interesting study of the role of thermal particle fluctuations in cosmological structure formation.

However, in this phase one must reconsider the computation of the string thermodynamic fluctuations, since our analysis implicitly assumes weak string coupling.

Note, however, that the dilaton coupling to stringy matter can produce friction effects which are similar to Hubble friction.

Note that the application of Israel matching conditions is, in our case, well justified. The concerns raised in 36 regarding the application of the matching conditions in the base of the Pre-Big-Bang 37 and Ekpyrotic/Cyclic 38 scenarios do not apply since in our case the matching conditions are satisfied at the level of the background solution.

See also 18 for an interesting discussion of these issues.