Coherence and Entanglement in a Stern-Gerlach experiment

T. R. Oliveira and A. O. Caldeira
Departamento de Física da Matéria Condensada,
Instituto de Física Gleb Wataghin, Universidade Estadual de Campinas
Caixa Postal 6165, Campinas, SP, CEP 13083-970, Brazil

We give a simple example of the tight connection between entanglement and coherence for pure bipartite systems showing the double role played by entanglement; it allows for the creation of superpositions of macroscopic objects but at the same time makes subsystems lose their quantum mechanical coherence. For this we study the time evolution of the spin coherence in the Stern-Gerlach (SG) experiment. We also show that, contrary to the naive intuition, the spin coherence is lost before the two beams become separated in the spatial coordinates.

I. INTRODUCTION

The SG experiment is viewed as the standard evidence of the quantum nature of the spin of a particle. When a beam of spin 1/2 particles, in the eigenstate \(|+\rangle\) of \(S_z\), goes through a variable magnetic field in the \(\hat{z}\) direction and the emerging particles are detected on a screen, one observes the presence of two distinct peaks corresponding to the spins in the positive and negative \(\hat{z}\) direction.

Besides furnishing the evidence of the spin quantization, this experiment is considered the paradigm of the measurement process being the simplest example of coherence and entanglement; two features of quantum mechanics of which one finds no analogue in the classical world.

In the SG experiment the particle enters the magnet in a pure state, for example, the eigenstate \(|+\rangle\rangle_x\) of \(S_x\), and for the effect of measurement of the \(S_z\) spin component it is described as a coherent superposition of the eigenstates of \(S_z\). Later on, as an effect of the interaction with the non-uniform part of the magnetic field, these spin degrees of freedom entangle with the spatial coordinates generating the state \(|\psi(t)\rangle = \alpha |+\rangle \varphi_+ (t) + \beta |-\rangle \varphi_- (t)\rangle\). In the end, as a result of the measuring process, the quantum state collapses into one of these eigenstates.

Coherence can be observed by measuring the spin in the \(x\) direction to obtain \(|S_x\rangle\). We know that spin coherence is lost as the state evolves in time and the two spatial parts become orthogonal. Since the measurement process naturally involves the partial trace over the spatial part of the initially pure global state the remaining spin part becomes a mixture.

But how is this spin coherence lost in the SG experiment? Is it lost just when the two beams are far away? These are interesting questions that address the very origin of the entanglement between the states representing different degrees of freedom of the particles.

Here we intend to study the evolution of the spin coherence in the SG experiment and how the entanglement between the spin and spatial coordinates is affected in the course of time to show the double role played by entanglement and coherence.

II. STERN-GERLACH MODEL

We are going to consider the usual SG model, where we just take into account one direction of the magnetic field. For discussions on these approximations see \[1,2\]. Within this model our Hamiltonian is

\[ H = \frac{p^2}{2m} - f \sigma_z, \]

where \(f = \mu (\partial B/\partial z)\) and we are not considering the uniform part of the magnetic field, that is just responsible for the spin precession. This Hamiltonian takes us to the following propagator \[2\]

\[ K_{ss'} (z, t; z', 0) = \langle S | S' \rangle \sqrt{\frac{m}{2\pi i \hbar t}} \exp \left\{ \frac{i}{\hbar} \left( \frac{m}{2t} (z - z')^2 - \frac{m}{t} \Delta z (t) (z - z') - \Delta p (t) z' - \frac{f^2}{24m} \right) \right\} \]

with

\[ \Delta p (t) = ft, \quad \Delta z (t) = \frac{ft^2}{2m}, \]

and

\[ \overline{\Delta z (t)} = \frac{t \Delta p (t)}{m} - \Delta z (t). \]

This propagator can be used to perform the temporal evolution of the physical state in the Schrödinger prescription and we employ the superposition below as the initial state:

\[ |\psi\rangle = (\alpha |+\rangle + \beta |-\rangle) \otimes |\varphi\rangle. \quad (1) \]

Here \(|\varphi\rangle\) is the spatial part of the physical state which is not initially entangled with the spin. Considering this
initial spatial part as a wave packet with minimum uncertainty we have
\[ \langle z | \varphi \rangle = \varphi(z, 0) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{z^2}{2\sigma^2}}, \]  
(2)
which is a Gaussian packet with width \( \sigma \) and centered at the origin. Performing the temporal evolution we get
\[ |\psi(t)\rangle = \alpha \varphi_+(t) |+\rangle + \beta \varphi_- (t) |-\rangle , \]  
(3)
with
\[ \varphi_{\pm} (z, t) = \frac{e^{i\theta(t)}}{\sqrt{\sigma(t) \sqrt{2\pi}}} \exp\left\{ -\frac{1}{4\sigma^2(t)} \left[ z + \Delta z(t) \right]^2 + \right. 
+ i \left. \frac{m}{2\hbar t} z^2 \pm \frac{m}{\hbar t} \Delta z(t) z + \frac{f^2 \lambda^3}{24m\hbar^2} + \right. 
- \frac{m}{2\hbar t} \left( \frac{\sigma}{\sigma(t)} \right)^2 \left[ z + \Delta z(t) \right]^2 \right\} \]  
(4)
and
\[ \sigma(t)^2 = \sigma^2 + \left( \frac{\hbar t}{2m \sigma} \right)^2 . \]

Since the function \( \theta(t) \) appears only as a time dependent exponent of a complex phase it will not contribute to the evaluation of the probability amplitudes.

As expected, after having interacted with the magnetic field the spatial and spin degrees of freedom of the particle are now entangled. Within a time interval \( t \) the spatial part of the wave function is a Gaussian whose center follows the classical trajectory with a time dependent width given by \( \sigma(t) \).

**III. COHERENCE AND ENTANGLEMENT**

As we are interested in the spin coherence we should trace over the spatial coordinates and look at the off-diagonal elements of the density operator in the spin space which reads, for \( \alpha = \beta = 1 / \sqrt{2} \),
\[ \rho_{+-} (t) = \int dz \varphi_+ (z, t) \varphi_- (z, t) . \]
\[ \rho_{-+} (t) = \exp \left\{ - \frac{1}{2} \left[ \frac{\Delta p(t)}{\hbar / 2\sigma} \left( \frac{\sigma}{\sigma(t)} + \frac{\sigma(t)}{\sigma} \right) \right]^2 + \right. 
- \frac{1}{2} \left( \frac{\Delta z(t)}{\sigma(t)} \right)^2 \right\} . \]

In this expression we can see that we have two terms contributing to the loss of coherence. The first one is the distance between the centers of the packets in momentum space, as measured in units of the spread of the packets in this space, namely \( \hbar / 2\sigma \). The second term also measures the distance between the packets but in coordinate space instead. In this case this measure is taken using the spread of the packet in coordinate space, \( \sigma(t) \), as the standard. We are interested in knowing how long it takes for the spin coherence to be lost and what is the contribution of each of those two above-mentioned terms. Defining a “decoherence” time, \( \tau \), as the time scale within which the off-diagonal element decay to \( 1/e \), we can show that
\[ \tau = \sqrt{\frac{2\sqrt{2m \sigma}}{f} \left[ - \frac{2\sqrt{2m \sigma}^3}{\hbar^2} + \sqrt{1 + \frac{8f^2m^2\sigma^6}{\hbar^4}} \right]^{1/2}} . \]

As this expression does not tell us much if it is analyzed in its full extent we are going to do it for two particular limits,
\[ \frac{8f^2m^2\sigma^6}{\hbar^4} \]  
\[ \approx 1 \]  
first case
\[ \frac{8f^2m^2\sigma^6}{\hbar^4} \]  
\[ \ll 1 \]  
second case

In the first case we approximate the decoherence time by \( \tau_1 = \hbar / \sqrt{2f \sigma} \) whereas in the second case we do it by \( \tau_2 = \sqrt{2m \sigma / f} \).

Now we want to investigate how the separation between the packets evolves during the loss of coherence. This distance, in the coordinate representation, is given by the fraction \( \Delta z(t) / \sigma(t) \) which assumes the following values
\[ \frac{\Delta z(t_1)}{\sigma(t_1)} \approx \frac{1}{\sqrt{2}} \frac{\hbar^2}{2\sqrt{2f m \sigma}^3} \ll 1 \]
and
\[ \frac{\Delta z(t_2)}{\sigma(t_2)} \approx \sqrt{\frac{2f m \sigma^3}{\hbar^2}} \ll 1 . \]

We see that in both cases the packets are not well separated in space when the coherence is lost, and, therefore, the separation of the packets in momentum space must be responsible for the loss of spin coherence in the SG experiment. This can be viewed in the figures 1, 2 and 3, where we plot the loss of coherence, as given by the off-diagonal elements of the density operator in the spin space, and the probability amplitudes for the two spatial parts. These plots were made using the typical values of a SG experiment: \( m = 1.8 \times 10^{-25} K g \) (cooper atom mass), \( \partial B / \partial z = 10^4 T / m \) and \( \sigma = 10^{-5} \text{m} \).

One should also notice that in the first case the spread of the packets in coordinate space is not relevant, \( \sigma(t) \approx \sigma \), whereas in the second case it has to be taken in account since \( \sigma(t) \gg \sigma \). Another way to understanding why the momentum separation is responsible for the loss of coherence is to note that in the beginning, when \( \sigma(t) \approx \sigma \),
while at long times, when \( \sigma(t) \gg \sigma \), one has

\[
\frac{\Delta z(t)}{\sigma(t)} \approx \frac{f \sigma}{\hbar} t. \tag{6}
\]

Nevertheless, the momentum separation is given by

\[
\frac{\Delta p(t)}{\hbar/2\sigma} = \frac{2f \sigma}{\hbar} t
\]

being always linear in \( t \). The only possibility the space separation between the packets exceeds their momentum separation is for long times when the space separation is no longer quadratic but linear instead.

![Figure 1](image1.png)  
**Figure 1:** Off-diagonal elements of the density operator in the spin space showing the loss of coherence (solid curve) and the entanglement between the spin and coordinates degrees of freedom (dashed curve). Plotted with the typical values mentioned in the text.

![Figure 2](image2.png)  
**Figure 2:** The probability amplitudes for the spatial part of the wave function at 2 ns, when coherence is very small.

Another interesting thing to look at is the behaviour of the entanglement between the spin and spatial coordinates. As we are dealing with a global pure state we can use either the von Neumann entropy or the linear entropy of one of the subsystems as a measure of entanglement. We shall develop the latter in what follows.

![Figure 3](image3.png)  
**Figure 3:** Probability amplitudes for the spatial part of the wave function at 10 ns, time interval beyond which the packets start to become separated.

Usually we have a bipartite system with the global state given by

\[
|\psi(t)\rangle = \alpha \varphi_+ (z, t) |+\rangle + \beta \varphi_- (z, t) |-\rangle
\]

and as the coefficients \( \alpha \) and \( \beta \) vary the degree of entanglement of the state of the system also changes. Two extremes cases are the state of maximum entanglement, the Bell state, in which the coefficients are both \( 1/\sqrt{2} \) and the separable state (non-entangled) in which one of the coefficients is zero.

In our case we have a different physical situation since what is varying is not the coefficients but the states themselves. In other words, we are varying the states \( \varphi_+ (z, t) \) and \( \varphi_- (z, t) \) that start "parallel" to one another and become orthogonal as time evolves. The entanglement between the spatial and spin degrees of freedom can be given, in terms of the linear entropy of the spin subsystem, by \( E_L = 1 - \rho^2_{zz} (t) \). In this expression we can see the close connection between entanglement and coherence; as one of them increases the other is found to decrease. This behaviour can be observed in Fig. 4 where we have plotted \( E_L (t) \) and \( \rho_{zz} (t) \) in the typical SG experiment. As expected, the entanglement vanishes at the beginning and increases as the states become orthogonal, having as a consequence the loss of spin coherence.

This example clearly shows the double role played by entanglement; it allows for the creation of superpositions of macroscopic objects but at the same time makes subsystems lose their quantum mechanical coherence.

For the sake of completeness, we should mention that one could also think about tracing out the spin degrees of freedom and see the coherence in the coordinate representation as an interference pattern between the two beams in a double slit experiment. The only "problem" here is that the spin degrees of freedom are always orthogonal which makes the reduced density matrix in coordinate space diagonal and destroy the possible coherence at any time.
Finally, since the present analysis makes the connection between entanglement and coherence clear only for pure states it would be desirable to extend it to mixed states as well.

**SUMMARY**

We have given a simple example of the connection between entanglement and coherence showing how the spin coherence is lost in a SG experiment as the spatial and spin degrees of freedom become entangled. We have also observed that, contrary to the expectations, the spin coherence is lost much faster than the two beams become clearly separated. It is worth commenting that we have used the word decoherence with a somewhat different meaning from that used in the current literature. Here we do not have a real environment as the cause of decoherence, and therefore are referring to a possible case of reversible decoherence. Recovery of coherence should be achieved simply by recombining the two beams (See Refs. [2, 3, 4] for more details on this).

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