Dynamical Eigenfunction Decomposition of Turbulent Pipe Flow

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The results of an analysis of turbulent pipe flow based on a Karhunen-Loève decomposition are presented. The turbulent flow is generated by a direct numerical simulation of the Navier-Stokes equations using a spectral element algorithm at a Reynolds number \( Re_\tau = 150 \). This simulation yields a set of basis functions that captures 90\% of the energy after 2,453 modes. The eigenfunctions are categorised into two classes and six subclasses based on their wavenumber and coherent vorticity structure. Of the total energy, 81\% is in the propagating class, characterised by constant phase speeds; the remaining energy is found in the non-propagating shear and roll modes. The four subclasses within the propagating modes form a basis that provides support for the horseshoe (hairpin) structure as the dominant, self-sustaining mechanism of turbulence.

Keywords: Direct numerical simulation, Karhunen-Loève decomposition, turbulence, pipe flow, mechanism

1 Introduction

Turbulence, hailed as one of the last major unsolved problems of classical physics, has been the subject of numerous publications as researchers seek to understand the underlying physics, structures, and mechanisms inherent to the flow [1]. The standard test problem for wall-bounded studies historically has been turbulent channel flow because of its simple geometry and computational efficiency. Even though much insight has been achieved through the study of turbulent channel flow, it remains an academic problem because of its infinite (computationally periodic) spanwise direction.

The next simplest geometry is turbulent pipe flow, which is of interest because of its real-world applications and its slightly different dynamics. The

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Table 1. Summary of existing DNS of turbulent pipe flow. As seen, most approaches use a 2nd-order radial discretization. The reason is that the standard spectral methods, in the presence of the coordinate singularity, achieve only 2nd-order convergence instead of geometric convergence.

<table>
<thead>
<tr>
<th>Authors</th>
<th>Radial Discretization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eggels et al. (1994) [8]</td>
<td>2nd-order FD - first pipe DNS</td>
</tr>
<tr>
<td>Verzicco and Orlandi (1996) [9]</td>
<td>2nd-order FD flux based</td>
</tr>
<tr>
<td>Loulou et al. (1997) [7]</td>
<td>B-spline</td>
</tr>
</tbody>
</table>

Three major differences between turbulent pipe and channel flow are that pipe flow fails to conform to the logarithmic law until much higher Reynolds numbers (Re = 3000 for a channel versus Re = 7442 for a pipe [2,3]), has a higher critical Reynolds number, and is linearly stable to an infinitesimal disturbance [4,5]. Unfortunately, few direct numerical simulations of turbulent pipe flow have been carried out because of the complexity in handling the numerical radial singularity at the origin. Although the singularity itself is avoidable, its presence causes standard high-order spectral methods to fail to converge exponentially. As a result, only a handful of studies have produced turbulent pipe flow, as shown in Table 1, and typically use a low-order expansion in the radial direction. Only Shan et al. [6] using concentric Chebyshev domains (“piecemeal”) and Loulou et al. [7] using basis spline (B-spline) polynomials provide a higher-order examination of turbulent pipe flow. Using a spectral element method, this study provides not only a high-order examination but the first exponentially convergent investigation of turbulent pipe flow through direct numerical simulation (DNS).

With this DNS result, one of the studies that can be performed with the full flow field and time history it provides is an analysis based on an orthogonal decomposition method. In such methods, the flow is expanded in terms of a natural or preferred turbulent basis. One method used frequently in the field of turbulence is Karhunen-Loève (KL) decomposition, which uses a two-point spatial correlation tensor to generate the eigenfunctions of the flow. This is sometimes referred to as proper orthogonal decomposition, empirical orthogonal function, or empirical eigenfunction analysis.

Work in this area was pioneered by Lumley, who was the first to use the KL method in homogeneous turbulence [11,12]. This method was later applied to turbulent channel flow in a series of papers by Ball, Sirovich, and Keefe [13,14] and Sirovich, Ball, and Handler [15], who discovered plane waves and propagating structures that play an essential role in the production of turbulence through bursting or sweeping events. To study the interactions of the propagating structures, researchers have examined minimal expansions of a turbulent flow [16–18]. These efforts have led to recent work by Webber et al. [19,20],
who examined the energy dynamics between KL modes and discovered the energy transfer path from the applied pressure gradient to the flow through triad interactions of KL modes.

This present study uses a spectral element Navier-Stokes solver to generate a globally high-order turbulent pipe flow data set. The Karhunen-Loève method is used to examine the turbulent flow structures of turbulent pipe flow. This work provides a baseline understanding for future studies that can compare the effect of forcing functions on the KL structures, such as drag reduction by spanwise wall oscillation.

2 Methodology

The direct numerical simulation of the three-dimensional time dependent Navier-Stokes equations is a computationally intensive task. By fully resolving the necessary time and spatial scales of turbulent flow, however, no subgrid dissipation model is needed, and thus a turbulent flow is calculated directly from the equations. DNS has one main advantage over experiments, in that the whole flow field and time history are known, enabling analysis such as the Karhunen-Loève decomposition.

Because of the long time integration and the grid resolution necessary for DNS, a high-order (typically spectral) method is often used to keep numerical round-off and dissipation error at bay. Spectral methods and spectral elements use trial functions that are infinitely and analytically differentiable to span the element. This approach decreases the global error exponentially with respect to resolution, in contrast to an algebraic decrease with standard methods such as finite difference or finite element methods [21].

2.1 Direct Numerical Simulation

2.1.1 Numerical Methods. This study uses a spectral element Navier-Stokes solver that has been developed over the past 20 years [22,23]. This solver employs a geometrically flexible yet exponentially convergent spectral element discretization in space, in this case allowing the geometry to be fitted to a cylinder. The domain is subdivided into elements, each containing high-order (usually 11–13) Legendre Lagrangian interpolants. The resulting data localisation allows for minimal communication between elements, and therefore efficient parallelization. Time discretization is done with third-order operator splitting methods, and tensor-product polynomial bases are solved by using conjugate gradient iteration with scalable Jacobi and hybrid Schwarz/multigrid preconditioning [24].
2.1.2 Geometry. Spectral elements are effective in cylindrical geometries [25] and avoid the radial singularity at the origin by solving the equations in Cartesian coordinates. The mesh is structured as a box near the origin and transitions to a circle near the pipe walls (Figure 1), maintaining a globally high-order method at both the wall and the origin. In addition to avoiding the numerical error associated with the singularity, the method also avoids the time-step restriction due to the smaller element width at the origin of a polar-cylindrical coordinate system, which could lead to potential violations of the Courant-Friedrichs-Levy stability criteria.

Each slice, as shown in Figure 1, has 64 elements, and there are 40 slices stacked in the streamwise direction, adding up to a length of 20R. Each element has 12th-order Legendre polynomials in each direction for $Re_\tau = U_\tau R/\nu = 150$, where $U_\tau = \sqrt{\tau_w/\rho}$ is the shear velocity, $\tau_w$ is the wall shear stress, and $\rho$ is the density. This discretization results in 4.4 million degrees of freedom. Near the wall, the grid spacing in wall units ($y^+ = U_\tau y/\nu$) is $\Delta r^+ \approx 0.78$ and $\Delta \theta^+ \approx 4.9$, where $r$ is the radius and $\theta$ is the azimuthal angle. Near the centre of the pipe, the spacing in Cartesian coordinates is $\Delta x^+ = \Delta y^+ \approx 3.1$. The streamwise grid spacing is a constant $\Delta z^+ = 6.25$ throughout the domain.

2.1.3 Benchmarking at $Re_\tau = 180$. Benchmarking was performed at $Re_\tau = 180$ with our DNS and experiments of Eggels et al. [8] and DNS of Fukagata and Kasagi [10]. For this higher Reynolds number flow, 14th-order polynomials were used, giving grid spacings near the wall of $\Delta r^+ \approx 0.80$ and $\Delta \theta^+ \approx 5.0$, $\Delta x^+ = \Delta y^+ \approx 3.2$ in the centre, and $\Delta z^+ = 6.42$. Eggels et al. and Fukagata and Kasagi used a spectral Fourier discretization in the azimuthal and axial directions and then a 2nd-order finite difference discretization in the radial direction. Also, both groups used a domain length of 10R and grid sizes
in their DNS studies of $96 \times 128 \times 256$ for $r$, $\theta$, and $z$ directions, respectively.

The mean flow profiles in Figure 2 correspond well, but as seen in the root-mean-squared (rms) statistics shown in Figure 3, the spectral element calculation shows a lower peak $u_{z,rms}$ and higher peak $u_{\theta,rms}$ and $u_{r,rms}$. These results are in contrast to those with channel flow, as reported by Gullbrand [26], where the 2nd order finite difference methods undershoot the spectral method wall-normal velocity rms.

When compared to the experimental results of Eggels et al. [8], the spectral element code follows much closer than the 2nd-order finite difference results. We note that Eggels et al. report that their particle image velocimetry system had difficulties capturing $u_r$ near the wall and near the centerline of the pipe due to reflection, which could explain the deviation of all of the DNS results in that area.

A second major difference our spectral element method and the previous pipe DNS using 2nd-order finite difference is the domain size. With the spectral element method, which results in a global high-order convergence, a domain size of 10R yielded instabilities in the flow, as seen in the azimuthal $u_{\theta,rms}$ in Figure 4 at $y^+=55$ and $y^+=120$. This instability would arise even with...
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Figure 3. Benchmark of the rms profiles for $Re_\tau = 180$ comparing the spectral element algorithm vs the low-order methods of Eggels et al. [8] and Fukagata and Kasagi [10]. As seen near $y^+ = 20$ in the $u_{z, \text{rms}}^+$ and near $y^+ = 50$ in the $u_{r, \text{rms}}^+$, the higher-order method is closer to the experimental results than the 2nd-order methods.

A more refined grid. When the domain of the spectral element method was extended to $20R$, this problem disappeared. We surmise that the 2nd-order finite difference method dissipated the large-scale structures after $10R$ that the higher-order spectral element case appropriately resolves. This structure, because of the periodic boundary conditions, re-enters the inlet and causes the unphysical bulges in the azimuthal rms profile. This result is also supported by the work of Jiménez [27] in turbulent channel flow ($Re_\tau = 180$) that shows large-scale structures that extend well past the domain size of $z^+ = 1800 = 10R$.

This benchmark confirms that the spectral element algorithm, at the given grid resolution and domain size, will generate the appropriate turbulent flow field and time history to perform the KL decomposition.

2.2 Karhunen-Loève Decomposition

For completeness, the Karhunen-Loève Decomposition method is briefly described here, but for more detail, see [11–16, 30–33].
2.2.1 Numerical Method. The two-point velocity correlation tensor is defined by

\[ K_{ij}(\mathbf{x}, \mathbf{x}') = \langle u_i(\mathbf{x})u_j(\mathbf{x}') \rangle \]  

(1)

where \( u(\mathbf{x}) \) is the fluctuating component of the velocity and where the mean is determined by averaging over both homogeneous planes and time. The angle brackets represent an average over many time steps, on the order of \( t^+ \approx 40,000 \sim 50,000 \), to sample the entire attractor. The \( i \) and \( j \) represent the three orthogonal velocities, establishing the matrix of velocity correlation and cross-correlations between every spatial point \( \mathbf{x} \) and \( \mathbf{x}' \).

For turbulent pipe flow, with two homogeneous directions providing translational invariance in the \( \theta \) (azimuthal) and \( z \) (streamwise) direction, Eq. (1) becomes

\[ K_{ij}(\mathbf{x}, \mathbf{x}') = K_{ij}(r, r', \theta - \theta', z - z'). \]  

(2)
Thus, given the kernel in Eq. (2), the eigenfunctions have the form

\[ \Phi_i = \Psi_i(n, m; r) \exp(n\theta + \frac{2\pi imz}{L_z}), \]

where \( n \) is the azimuthal wavenumber and \( m \) the streamwise wavenumber. The determination of \( \Psi_i \) is then given by

\[ \int_0^R r' dr' \kappa_{ij}(m, n; r, r') \Psi_j^\star(m, n; r') = \lambda(m, n) \Psi_i(m, n; r), \]

where \( \kappa_{ij} \) is the Fourier transforms of \( K_{ij} \) in the two homogeneous directions, the \( \star \) denotes the complex conjugate, and \( \lambda \) is the eigenvalue.

In the present problem, 2,100 snapshots of the flow field were taken, corresponding to one snapshot every eight viscous time steps \( (t^+ = R^+/U, \tau) \). The results of each snapshot were projected to an evenly spaced grid with 101 × 64 × 400 points in \( r, \theta, \) and \( z \), respectively. The Fourier transform of the data was then taken and the kernel built. This kernel was averaged over every snapshot to generate the final kernel to be decomposed. Since the dimension of \( K \) is 303 (given by three velocity components on 101 radial grid points) there are 303 eigenfunctions and eigenvalues. The eigenfunctions are ranked in descending order with the quantum number \( q \) to specify the particular eigenfunction associated with the corresponding eigenvalue. Thus, it requires a triplet \( k = (n, m, q) \) to specify a given eigenfunction. The notation \( \lambda_{(m,n,q)} = \lambda_k \) will be adopted for the remainder of this paper.

The eigenfunctions \( \Psi^k(r) \) are complex vector functions and are normalised so that the inner product is of unit length \( (\Phi^k, \Phi^l) = \delta_{kl} \), where \( \delta_{kl} \) is the Kronecker delta. The eigenvalues physically represent the average energy of the flow in the direction \( \Phi^k \);

\[ \lambda_k = \langle |u, \Phi^k| \rangle. \]

We note that the eigenfunctions, as an orthogonal expansion of the flow field, retain the properties of the flow field, such as incompressibility and boundary conditions of no slip at the wall.

2.2.2 Symmetry Considerations. The pipe flow is invariant under azimuthal reflection,
and taking advantage of it reduces the total number of calculations as well as the memory and storage requirements. We note that, because of its geometry, turbulent channel flow has two more symmetries – a vertical reflection, and a x-axis rotation – that are not present in the pipe, since a negative radius is equivalent to a 180 degree rotation in the azimuthal direction.

A major consequence of this symmetry is that the resulting eigenfunctions are also symmetric, and the modes with azimuthal wave number $n$ will be the azimuthal reflection of the modes with wave number $-n$,

$$R_\theta : (\Phi^k_r, \Phi^k_\theta, \Phi^k_z), \mathbf{k} = (m, n, q) \rightarrow (\Phi^k_r, -\Phi^k_\theta, \Phi^k_z), \mathbf{k} = (m, -n, q),$$

thus reducing the total computational memory needed for this calculation.

### 2.2.3 Time-Dependent Eigenfunction Flow Field Expansion.

The KL method provides an orthogonal set of eigenfunctions that span the flow field. As such, the method allows the flow field to be represented as an expansion in that basis,

$$\mathbf{u}(r, \theta, z, t) = \sum_k a_k(t) \Phi^k(r, \theta, z),$$

with

$$a_k(t) = (\Phi^k(r, \theta, z), \mathbf{u}(r, \theta, z, t)).$$

Since the Fourier modes are orthogonal to each other, equation 9 becomes

$$a_k(t) = 2\pi L_z \int_0^R \tilde{\Phi}^k(r) \tilde{\mathbf{u}}^\star(m, n; r, t) r dr,$$

with $\tilde{\mathbf{u}}$ being the Fourier transform of $\mathbf{u}$ in the azimuthal and streamwise direction with wavenumbers $n$ and $m$, respectively.

The time history of the eigenfunctions can be used to examine their interactions, such as the energy interaction examined by Webber et al. [20] and bursting events by Sirovich et al. [15].
Figure 5. Variations in bulk mean velocity $u_m(t^+)$ show that the flow is indeed turbulent.

3 Results

This section presents the results of our analysis of turbulent pipe flow based on KL decomposition.

3.1 Mean Properties of Flow and Flow statistics

DNS was performed for $Re_\tau = 150$. The Reynolds number based on the centreline velocity is $Re_c \approx 5700$ and based upon the mean velocity is $Re_m \approx 4300$. This range is above the critical Reynolds number for a pipe and exhibits self-sustaining turbulence, as seen from the fluctuating time history of the mean velocity shown in Figure 5. The velocity profile, seen in Figure 6 shows the mean velocity with respect to wall units ($y^+ = (r - R)U_\tau/\nu$). The profile fits the law of the wall but fails to conform to the log law, and as mentioned in Section 1, the log law is not expected for turbulent pipe flow until much higher Reynolds number.

The rms velocity fluctuation profiles and the Reynolds stress profile is shown in Figures 7a and 7b. The streamwise fluctuations peak near $y^+ = 16$. The azimuthal and axial velocities show a weaker peak near $y^+ = 39$ and $y^+ = 55$, respectively, and then remain fairly flat throughout the pipe. The Reynolds stress $\overline{u_z u_r}$ has a maximum of 0.68 at $y^+ = 31$. 
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3.2 Eigenvalue Spectrum

As discussed in Section 2.2, the eigenvalues represent the energy of each eigenfunction. By ordering the eigenvalues from largest to smallest, one can minimize the number of eigenfunctions, $N$, needed to capture a given percentage of the energy of the flow,
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Table 2. First 25 eigenvalues ranked in descending order of energy.

<table>
<thead>
<tr>
<th>Index</th>
<th>m</th>
<th>n</th>
<th>q</th>
<th>Eigenvalue</th>
<th>Energy (% Total)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>6</td>
<td>1</td>
<td>1.61</td>
<td>2.42 %</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>5</td>
<td>1</td>
<td>1.48</td>
<td>2.22 %</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>1.45</td>
<td>2.17 %</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>4</td>
<td>1</td>
<td>1.29</td>
<td>1.93 %</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1.26</td>
<td>1.88 %</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>5</td>
<td>1</td>
<td>0.936</td>
<td>1.40 %</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>6</td>
<td>1</td>
<td>0.917</td>
<td>1.37 %</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0.902</td>
<td>1.35 %</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>0.882</td>
<td>1.23 %</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0.805</td>
<td>1.20 %</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>7</td>
<td>1</td>
<td>0.763</td>
<td>1.14 %</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0.683</td>
<td>1.02 %</td>
</tr>
<tr>
<td>13</td>
<td>0</td>
<td>7</td>
<td>1</td>
<td>0.646</td>
<td>0.97 %</td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>0.618</td>
<td>0.92 %</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>8</td>
<td>1</td>
<td>0.601</td>
<td>0.90 %</td>
</tr>
<tr>
<td>16</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>0.580</td>
<td>0.87 %</td>
</tr>
<tr>
<td>17</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.567</td>
<td>0.85 %</td>
</tr>
<tr>
<td>18</td>
<td>2</td>
<td>7</td>
<td>1</td>
<td>0.524</td>
<td>0.78 %</td>
</tr>
<tr>
<td>19</td>
<td>1</td>
<td>8</td>
<td>1</td>
<td>0.483</td>
<td>0.72 %</td>
</tr>
<tr>
<td>20</td>
<td>2</td>
<td>6</td>
<td>1</td>
<td>0.476</td>
<td>0.71 %</td>
</tr>
<tr>
<td>21</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0.454</td>
<td>0.68 %</td>
</tr>
<tr>
<td>22</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0.421</td>
<td>0.63 %</td>
</tr>
<tr>
<td>23</td>
<td>2</td>
<td>8</td>
<td>1</td>
<td>0.375</td>
<td>0.56 %</td>
</tr>
<tr>
<td>24</td>
<td>1</td>
<td>9</td>
<td>1</td>
<td>0.358</td>
<td>0.54 %</td>
</tr>
<tr>
<td>25</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>0.354</td>
<td>0.53 %</td>
</tr>
</tbody>
</table>

\[
\mathbf{u}(x, t) = \sum_{k}^{N} \mathbf{a}_k(t) \Phi_k(x),
\]

where \( \mathbf{k} = (m, n, q) \) is the wavenumber vector. Table 2 shows the first 25 eigenfunctions, and Figure 8 shows the running total of energy versus modes. The 90% mark is reached at \( D_{KL} = 2,453 \), where \( D_{KL} \) can be considered as a measure of the intrinsic dimension of the chaotic attractor describing the turbulence as discussed by Zhou and Sirovich [17, 33]. These eigenfunctions are the preferred or natural basis function for turbulent pipe flow, and insight is gained by observing their qualitative structure.

### 3.3 Structure of the Eigenfunctions

The eigenfunctions resulting from the KL decomposition can be categorised into two distinct classes and six modes based on their characteristics, as listed in Table 3. It is observed that certain characteristics of the eigenfunction are dependent on the wavenumber. The modes with higher azimuthal than axial...
Figure 8. Percentage of energy retained in the KL expansion with the 90% mark achieved after 2453 modes (Karhunen-Loève dimension).

Table 3. Structures of turbulent pipe flow as classified by wavenumber.

<table>
<thead>
<tr>
<th>Structure</th>
<th>Definition</th>
<th>Energy</th>
<th>Description</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Propagating modes</td>
<td>(m,n,q), m ≠ 0</td>
<td>80.58%</td>
<td>Constant phase speed</td>
<td>Fig. 10a-f</td>
</tr>
<tr>
<td>(a) Wall</td>
<td>n &gt; m</td>
<td>35.22%</td>
<td>Structures that turn azimuthally more than they lift axially found near wall.</td>
<td>Fig. 10a-f</td>
</tr>
<tr>
<td>(b) Lift</td>
<td>m ≥ n, n &gt; 1</td>
<td>29.68%</td>
<td>Structures that lift axially more than they turn azimuthally, extend from near wall to outer region.</td>
<td>Fig. 11f</td>
</tr>
<tr>
<td>(c) Asymmetry</td>
<td>n = 1</td>
<td>9.09 %</td>
<td>Structures that have non-zero radial and azimuthal velocity, typically with coherent structures found in the outer region</td>
<td>Fig. 12f</td>
</tr>
<tr>
<td>(d) Ring</td>
<td>n = 0</td>
<td>6.60%</td>
<td>Ring-like structures in the outer region</td>
<td>Fig. 13f</td>
</tr>
<tr>
<td>Non-propagating modes</td>
<td>(0, n, q)</td>
<td>19.42%</td>
<td>Non-constant phase speed</td>
<td>Fig. 14f</td>
</tr>
<tr>
<td>(a) Roll mode</td>
<td>(0, n, q) n ≠ 0</td>
<td>18.34%</td>
<td>Near wall streamwise vortices</td>
<td>Fig. 14f</td>
</tr>
<tr>
<td>(b) Shear mode</td>
<td>(0, 0, q)</td>
<td>1.08%</td>
<td>Non-zero centerline streamwise velocity</td>
<td>Fig. 15d</td>
</tr>
</tbody>
</table>

Wavenumber turn more than they lift, and as such the near-wall stretching structures are found. Likewise the modes with higher axial than azimuthal wavenumber lift more than they turn, and as such the lifting structures that extend from the near wall to the outer region are found.

In order to show the qualitative and quantitative classifications of these eigenfunctions, the most energetic function of each class is presented in Figures 10a through 15d. For each class, subfigures (a) show iso-contours and slices...
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Figure 9. Subclass spectra according to streamwise ($m$) and spanwise ($n$) wavenumbers.

of coherent vorticity, and a slice displaying the Reynolds stress and coherent vorticity together. The next four subfigures for each class are the plots of the eigenfunctions, both the real (b) and imaginary component (d), and the time history amplitude (c) and phase (e). To complete the class description, the coherent vorticity of two more representative eigenfunctions are shown in subfigures (f) and (g). For the shear modes, since they do not have coherent vorticity, only the eigenfunction and time history are plotted.

The first subclass of interest is the wall eigenmodes. The coherent vorticity plots of three example wall modes are seen in Figures 10a, 10f, and 10g for the (1,5,1), (1,3,1), and (2,4,1) modes, respectively. Each mode consists of a traveling-wave coherent vortex near the wall, and all eigenmodes with greater spanwise wavenumber than streamwise wavenumber demonstrate this same structure. Consistent in this subclass is the Reynolds stress generated near the wall. The individual velocity components of the (1,2,1) mode, the most energetic wall mode, is shown in Figures 10b and 10d for their real and imaginary components, respectively, and in Figures 10c and 10a for their amplitude squared and phase time history. The amplitude history shows the bursting nature of these traveling-waves, and the near constant phase velocity shows why these structures are referred to as propagating or traveling waves. These wall functions constitute 35.22% of the total energy of the flow, and are the most energetic of the propagating modes.

The next structure of interest, still in the propagating mode class, is the lift mode, found whenever the streamwise wavenumber is greater than or equal to the spanwise wavenumber and the spanwise wavenumber is greater than one.
Three examples of this subclass is shown in Figures 11a, 11f, and 11g for modes (2,2,1), (3,2,1), and (3,3,1), respectively. The coherent vorticity and Reynolds stress start near the wall and finish near the centerline in a lifting motion. The real and imaginary velocity components are shown in Figures 11b and 11d, respectively, and the amplitude and phase time history in Figures 11c and 11e, respectively. Again, the constant phase speed classifies these structures as propagating modes, and the amplitude bursts are also characteristic of a turbulent flow field. The lift structures constitutes 29.68% of the total energy of the flow.

The next structures, the asymmetric modes, are the modes responsible for breaking the symmetry of the flow about the axis of the pipe. These structures are found for a spanwise wavenumber of one, for any streamwise wavenumber, and consist of a coherent vortex just outside the log layer. They are asymmetric because of their nonzero radial and azimuthal velocities at the origin, which is physical only for azimuthal wavenumber \( n = 1 \), since a positive radial velocity at the origin, under a rotation of \( \pi \) around the axis, results in a negative radial velocity. The same is true for the azimuthal velocity at the origin. Three examples of these structures are shown in Figures 12a, 12f and 12g for modes (1,1,1), (2,1,1), and (3,1,1), respectively, again having the Reynolds stress between the coherent vorticity. These modes are also propagating and turbulent, as seen in the (1,1,1) mode’s amplitude (Figure 12c) and phase time history (Figure 12e), with its real and imaginary component found in Figure 12a and 12a. This mode constitutes 9.09% of the total energy of the flow.

The last of the propagating modes is the ring mode, named for the ring of coherent vorticity that is found for all modes with zero azimuthal wavenumber. Examples of this structure are shown in Figures 13a, 13f, and 13g for modes (1,0,1), (1,0,2), and (2,0,1), respectively. The real and imaginary velocities are shown in Figures 13b and 13d, and the amplitude and phase in 13a and 13e. This mode constitutes 6.60% of the total energy of the flow.

The last two modes are the non propagating modes. The first and more energetic of the two is the roll mode, found for any mode with zero streamwise wavenumber. This consists of rolls of coherent vorticity, as seen in the three example modes (0,6,1), (0,5,1), and (0,3,1) in Figures 14a, 14f, and 14g respectively. The Reynolds stress is strong between the coherent vortices but is an order of magnitude less than those of the wall or lift structures. The velocity components of the (0,6,1) mode are shown in Figure 14b and 14d. Of note, these structures do not have a constant phase velocity, seen in Figure 14e, and the decay rate of the energy is slower than that of the propagating modes, seen if Figure 14c. This subclass constitutes 18.34% of the total energy of the flow.

The other non propagating mode is the shear mode, found for zero azimuthal and streamwise wavenumber. This mode corresponds to the fluctuation of the
mean flow rate. Since these structures have no coherent vorticity nor imaginary components, only the real velocity components (Figures 15a and 15c) and their amplitude and phase time history (Figures 15b and 15d) are plotted. Like the roll modes, the shear modes do not have a constant phase velocity, and since they do not have an imaginary component, the phase oscillates between zero and $\pi$. The shear modes constitute 1.08% of the total energy of the flow.

The energy spectra of the propagating mode subclasses are shown in Figure 16. This shows that the tail end of the inertial range is populated primarily by the lift modes and the low end of the spectra is populated predominantly by the wall modes.

The effect of higher radial quantum number $q > 1$ is more zero crossings of the velocities in the radial direction. The effect on the coherent vorticity and the number of vortex cores scales with the radial quantum number $q$. The wall and lift modes retain their characteristics, in that even with more vortices, they remain close to the wall for the wall mode and close to the centreline for the lift modes, shown in Figures 17a through 18b.

4 Discussion

Although the Karhunen-Loève method yields a preferred or natural basis for turbulence, one must be careful in the conclusions drawn from the results, as any structure or feature can be reconstructed from any given orthogonal basis. This work is reported as a foundation for further study to examine how these eigenfunctions change with global changes in the flow, such as relaminarization and drag reduction by spanwise wall oscillation. Nevertheless, there are three interesting results that do not require comparisons to other simulations.

The first is the constant phase speed of the propagating modes. This was also found in studies of turbulent channel flow, as the structures represented by the basis advect with a constant group velocity, the same velocity as burst events [13–15]. The normal speed locus of the propagating waves is shown in Figure 19 for the propagating modes found in the top 50 most energetic modes. For this, the phase speed $\omega/||k||$ is plotted in the direction $k/||k||$. The locus is nearly circular, which is evidence that these structures propagate as a wave packet or envelope that travels with speed of 8.41, the point at which the circle intersects the y-axis in Figure 19.

The second interesting result is that none of the modes with azimuthal wave number $n = 1$ exhibit any traveling waves near the wall, and are without exception a streamwise or inclined streamwise vortex in the outer region. The reason is that the $n = 1$ mode does not allow for a near-wall traveling wave [28], and as such, the basis expansions for these modes have only near-centreline structures.
Figure 10a. Most energetic propagating wall mode (1,5,1). Coherent vorticity and a slice of coherent vorticity with Reynolds stress.

Figure 10b. Real component of (1,5,1).

Figure 10c. Time history amplitude of (1,5,1).

Figure 10d. Imaginary component of (1,5,1).

Figure 10e. Time history phase of (1,5,1).

Figure 10f. Propagating wall mode (1,3,1).

Figure 10g. Propagating wall mode (2,4,1).
Figure 11a. Most energetic propagating lift mode (2,2,1). Coherent vorticity and a slice of coherent vorticity with Reynolds stress.

Figure 11b. Real component of (2,2,1)

Figure 11c. Time history amplitude of (2,2,1)

Figure 11d. Imaginary component of (2,2,1)

Figure 11e. Time history phase of (2,2,1)

Figure 11f. Propagating lift mode (3,2,1).

Figure 11g. Propagating lift mode (3,3,1).
Figure 12a. Most energetic propagating asymmetric mode (1,1,1). Coherent vorticity and a slice of coherent vorticity with Reynolds stress.

Figure 12b. Real component of (1,1,1).

Figure 12c. Time history amplitude of (1,1,1).

Figure 12d. Imaginary component of (1,1,1).

Figure 12e. Time history phase of (1,1,1).

Figure 12f. Propagating asymmetric mode (2,1,1).

Figure 12g. Propagating asymmetric mode (3,1,1).
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Figure 13a. Most energetic propagating ring mode (1,0,1). Coherent vorticity and a slice of coherent vorticity with Reynolds stress.

Figure 13b. Real component of (1,0,1).

Figure 13c. Time history amplitude of (1,0,1).

Figure 13d. Imaginary component of (1,0,1).

Figure 13e. Time history phase of (1,0,1).

Figure 13f. Propagating ring mode (1,0,2).

Figure 13g. Propagating ring mode (2,0,1).
Figure 14a. Most energetic non-propagating roll mode (0,6,1). Coherent vorticity and a slice of coherent vorticity with Reynolds stress.

Figure 14b. Real component of (0,6,1).

Figure 14c. Time history amplitude of (0,6,1).

Figure 14d. Imaginary component of (0,6,1).

Figure 14e. Time history phase of (0,6,1).

Figure 14f. Non-propagating roll mode (0,5,1).

Figure 14g. Non-propagating roll mode (0,3,1).
The third result of note is that the different structures, as observed by the KL decomposition, can be seen qualitatively as an expansion that represents the horseshoe (hairpin) vortex representation of turbulent structures. This horseshoe structure is supported by a large number of researchers in the turbulence community as the self-sustaining mechanism for turbulence [29]. In the representative horseshoe structure (see, for example, the figure by Theodorsen [34]), the structures found in the KL can be seen. The wall modes represent the leg structure and its perturbation near the wall. The lift modes represent the structure lifting off the wall. The asymmetric modes represent the secondary and tertiary horseshoes that are formed. The turn modes represent the spanwise head of the horseshoe. As mentioned before, since the KL decomposition forms a basis, any flow can be recreated from these eigenmodes, so this result is reported as an interesting qualitative result, and as an understanding that the propagating modes are the structures of interest in turbulence.
5 Conclusions

We have presented the use of the Karhunen-Loève expansion method with the results of a globally high-order direct numerical simulation of turbulent pipe flow. The results reveal the structure of the turbulent pipe flow as propagating (80.58% total energy) and non propagating modes (19.42% total energy). The propagating modes are characterised by a constant phase speed and have four distinct classes: wall, lift, asymmetric, and ring modes. These propagating modes form a traveling wave envelope, forming a circular, normal-speed locus, with advection speed of 8.41. Together, the propagating structures as a basis represent well the horseshoe vortex structure as the prevalent turbulent structure. The non propagating modes have two subclasses: streamwise roll and shear modes. These represent the energy storage and mean fluctuations of the turbulent flow, respectively. This eigenfunction expansion, using both their structure and their time-dependent coefficients, provides a framework for further analysis and comparison and has led to understanding the mechanism of drag reduction by spanwise wall oscillation [35] and the mechanism of relaminarization [36].
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Figure 17a. (6,2,3) lift mode, contours of coherent vorticity.

Figure 17b. (6,2,5) lift mode, contours of coherent vorticity.

Figure 18a. (2,6,3) wall mode, contours of coherent vorticity.

Figure 18b. (2,6,5) wall mode, contours of coherent vorticity.

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References
Figure 19. Normal speed locus of top 25 most energetic propagating eigenfunctions, normalised with shear velocity $u_\tau$. The circle is a least-squares fit to the data and represents that the wave packets are acting together as a group with speed of 8.41, the point where the circle intersects the y-axis.


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