Asymmetric Swiss-cheese brane-worlds

László Á. Gergely and Ibolya Képérő

1Departments of Theoretical and Experimental Physics, University of Szeged, 6720 Szeged, Dóm tér 9, Hungary
2Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BW, UK

(Dated: August 29, 2006)

We consider Swiss-cheese brane universes embedded asymmetrically into the bulk. Neither the junction conditions between the Schwarzschild spheres and the surrounding Friedmann brane regions with cosmological constant \( \Lambda \), nor the evolution of the scale factor are changed with respect to the symmetric case. The universe expands and decelerates forever. The asymmetry however has a drastic influence on the evolution of the cosmological fluid. Instead of the two branches of the symmetric case, in the asymmetric case four branches emerge. Moreover, the future pressure singularity arising in the symmetric case only for huge values of \( \Lambda \) becomes quite generic in the asymmetric case. Such pressure singularities emerge also when \( \Lambda = 0 \) is set. Then they are due entirely to the asymmetric embedding. For generic values of \( \Lambda \) we introduce a critical value of a suitably defined asymmetry parameter, which separates Swiss-cheese cosmologies with and without pressure singularities.

I. INTRODUCTION

In the last decade brane-world models, originally motivated by string / M-theory, have developed into classical theories of gravitation, alternative to general relativity and reducing to it under certain circumstances. In these models our observable universe is a brane with tension \( \lambda \) embedded in a 5-dimensional bulk and gravitation has more degrees of freedom than in general relativity. The fifth dimension is not compactified, as in the Kaluza-Klein theories, but instead curved, in the simplest case due to a cosmological constant. Cosmological generalizations of the Randall-Sundrum (RS) second model consist of a moving Friedmann-Lemaître-Robertson-Walker (FLRW) brane embedded into a static Schwarzschild-anti de Sitter bulk. The motion of the brane in the bulk is the cosmological dynamics.

Our homogeneous and isotropic universe certainly contains local inhomogeneities. These can be dealt with the perturbation theory on the FLRW brane. However due mainly to technical problems involving the boundary conditions to be imposed, this is not a completed task for the moment, excepting particularly simple cases. There is still much to learn until a complete perturbative description will be achieved. Therefore exact models of the universe containing inhomogeneities are important.

In general relativity the simplest such model is the Einstein-Straus (Swiss-cheese) model, in which Schwarzschild spheres of constant comoving radius are glued into the FLRW space-time. Recently, the corresponding brane-world model was worked out in the simplest case of spatially flat sections \( k = 0 \) of the FLRW brane. It was also assumed that there is no source term from the bulk Weyl curvature: the electric part of the Weyl curvature on the brane was chosen to vanish. Such an assumption is motivated by the existence of stable blackstring solutions with vanishing electric part of the Weyl curvature in the two-brane models of Ref. [2]. Thus our model consist of black strings penetrating the cosmological brane, resulting in a Swiss-cheese type structure on the brane.

By allowing for a difference \( \Lambda \) in the cosmological constants in the FLRW and Schwarzschild regions, interesting behavior of the cosmological fluid emerged. For a vanishing \( \Lambda \) both the energy density \( \rho \) and the pressure \( p \) approach the general relativistic limits at late times, however they differ at early times. By increasing \( \Lambda \), the behavior of the fluid changes dramatically. For highly enough values of \( \Lambda \) a pressure singularity appears, while the scale factor \( a \) and all of its derivatives stay regular.

In this paper we extend our study on the Swiss-cheese brane-world by allowing for asymmetric embedding of the brane in the bulk. While both in the original RS model and in some of its curved generalizations the symmetry of the embedding was assumed such that the brane is the moving boundary of the bulk, there are many attempts to lift this symmetry, as this introduces interesting new possibilities in the model, like a late-time acceleration (see for example [3]). Asymmetric embedding can arise from allowing for different black hole masses on the two sides on the brane [7], [8], [9], different cosmological constants on the two sides [10], [11] or both types of generalizations [12], [13], [14]. The role of asymmetry in brane-worlds was discussed covariantly in [15]. In the simple case we study here, the asymmetry is achieved by
TABLE I: Domains of positivity, negativity and ill-definedness of $\rho_{++}$ for various values of $\Lambda$ in the small asymmetry regime $\alpha < 1$.

<table>
<thead>
<tr>
<th>$\rho_{++}$</th>
<th>$\tau &lt; \tau_{1,\alpha}$</th>
<th>$\tau = \tau_{1,\alpha}$</th>
<th>$\tau_{1,\alpha} &lt; \tau \leq \tau_{2,\alpha}$</th>
<th>$\tau &gt; \tau_{2,\alpha}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda \leq \Lambda_{1,\alpha}$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\Lambda_{1,\alpha} &lt; \Lambda \leq \Lambda_{2,\alpha}$</td>
<td>+</td>
<td>0</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$\Lambda &gt; \Lambda_{2,\alpha}$</td>
<td>+</td>
<td>0</td>
<td>–</td>
<td>no real solution</td>
</tr>
</tbody>
</table>

TABLE II: For high asymmetry $\alpha > 1$ the energy density $\rho_{++}$, whenever well-defined, is positive.

<table>
<thead>
<tr>
<th>$\rho_{++}$</th>
<th>$\tau \leq \tau_{2,\alpha}$</th>
<th>$\tau &gt; \tau_{2,\alpha}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda \leq \Lambda_{2,\alpha}$</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\Lambda &gt; \Lambda_{2,\alpha}$</td>
<td>+</td>
<td>no real solution</td>
</tr>
</tbody>
</table>

TABLE III: For small asymmetry $\alpha < 1$ the energy density $\rho_{+-}$, whenever well-defined, is negative.

<table>
<thead>
<tr>
<th>$\rho_{+-}$</th>
<th>$\tau \leq \tau_{2,\alpha}$</th>
<th>$\tau &gt; \tau_{2,\alpha}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda \leq \Lambda_{2,\alpha}$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$\Lambda &gt; \Lambda_{2,\alpha}$</td>
<td>–</td>
<td>no real solution</td>
</tr>
</tbody>
</table>

choosing different values of the bulk cosmological constant $\tilde{\Lambda}$ on the two sides of the brane.

In Section II we discuss the evolution of the scale factor as derived from the junction conditions on the brane, given explicitly in [16]. We do not discuss the complicated structure of the bulk, but conjecture, as in [4], that by suitably choosing the corresponding bulk regions, their junction can be done. From the junction conditions on the brane we find that the asymmetry does not modify the cosmological evolution as compared to the symmetric case. However, the Friedmann and Raychaudhuri equations are changed by asymmetry, thus the evolution of $\rho$ and $p$ should change accordingly. Indeed the behavior of $\rho$ and $p$ are changed dramatically, compared to the symmetric case. The two branches of the symmetric Swiss-cheese model split into four, when the asymmetry appears. We introduce a properly chosen dimensionless asymmetry parameter $\alpha$. Then we show that for $\Lambda \leq \kappa^2 \lambda / 2$ ($\kappa^2$ being the brane coupling constant) there is a critical value of $\alpha$, which separates cosmologies with pressure singularities from the ones with regular evolution during the whole lifetime of the universe. The occurrence of pressure singularities is generic for $\Lambda > \kappa^2 \lambda / 2$.

While in the symmetric case it was a single branch allowing for positive energy density, in the asymmetric case there are two. In Section III we study the domains of positivity of the energy density on these two branches. Then we illustrate graphically various typical behaviors of $\rho$ and $p$.

Finally in the Concluding Remarks we point out that the totality of the source terms in this brane-world scenario add up to a dust, establishing the analogy with the general relativistic Einstein-Straus model, as in the symmetric case. We also illustrate here graphically the critical behavior of the Swiss-cheese brane-world models with asymmetry.

II. SWISS-CHEESE BRANE-WORLDS WITH ASYMMETRY

In [16] the junction conditions on spheres of constant comoving radius $\chi_0$ at the interface of Schwarzschild and FLRW regions were derived. As proven in [14] for $k = 0$ they reduce to

$$a^3 = \frac{9m}{2\chi_0^3} \tau^2.$$  \hspace{1cm} (1)

Here $\tau$ is cosmological time, with the origin at the Big Bang and $m$ the mass of the Schwarzschild black hole. This result does not depend on the embedding. Rather, it depends on choosing flat spatial sections for the FLRW space-time and on assuming no electric Weyl source on the brane. As the FLRW regions are homogeneous and
TABLE IV: Domains of positivity, negativity and ill-definedness of $\rho_{\pm}$ for various values of $\Lambda$ and high asymmetry $\alpha \geq 1$.

<table>
<thead>
<tr>
<th>$\rho_{\pm}$</th>
<th>$\tau &lt; \tau_{1,\alpha}$</th>
<th>$\tau_{1,\alpha} &lt; \tau \leq \tau_{2,\alpha}$</th>
<th>$\tau &gt; \tau_{2,\alpha}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda \leq \Lambda_{1,\alpha}$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\Lambda_{1,\alpha} &lt; \Lambda \leq \Lambda_{2,\alpha}$</td>
<td>$-$</td>
<td>$0$</td>
<td>$+$</td>
</tr>
<tr>
<td>$\Lambda &gt; \Lambda_{2,\alpha}$</td>
<td>$-$</td>
<td>$0$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

As $\alpha > 0$, for the positivity of the left-hand side $\beta > 0$ should hold. This is exactly the condition which had to be satisfied in the symmetric case in order that the solutions for $\rho$ exist. No such solutions existed when $\Lambda > \kappa^2\lambda/2$ and $\tau > \tau_2 = 2/3(\Lambda - \kappa^2\lambda/2)$. In the asymmetric case however the square root in Eq. (9) imposes a more severe condition: $\beta \geq \sqrt{\alpha}$. This implies

$$\Lambda - \Lambda_{2,\alpha} \leq \frac{4}{3\tau^2},$$

where we have defined

$$\Lambda_{2,\alpha} = \frac{\kappa^2}{2} (1 - 2\sqrt{\alpha}) .$$

This is identically satisfied for $\Lambda \leq \Lambda_{2,\alpha}$. When $\Lambda > \Lambda_{2,\alpha}$ Eq. (10) holds only for $\tau \leq \tau_{2,\alpha}$, where

$$\tau_{2,\alpha} = \frac{2}{\sqrt{3(\Lambda - \Lambda_{2,\alpha})}} < \tau_2 .$$

Thus asymmetry lowers the range of $\Lambda$ for which the density is well-defined during the whole evolution of the universe. For the rest of $\Lambda$ values, asymmetry shortens the interval of well-definedness.

Stated otherwise, a critical value

$$\alpha_{\text{crit}} = \left(\frac{1}{2} - \frac{\Lambda}{\kappa^2\lambda}\right)^2$$

of the asymmetry parameter can be introduced for each $\Lambda \leq \kappa^2\lambda/2$ such that for any $\alpha < \alpha_{\text{crit}}$ the density is well defined during the whole evolution of the Swiss-cheese brane-world. For high asymmetry $\alpha > \alpha_{\text{crit}}$ and for huge cosmological constant $\Lambda > \kappa^2\lambda/2$ (irrespective of the degree of asymmetry) the evolution of $\rho$ stops at $\tau_{2,\alpha}$.

Whenever $\rho$ is well-defined, it is given by

$$\rho_{\pm} = -1 \pm \sqrt{\beta \pm \sqrt{\beta^2 - \alpha}} .$$

The first subscript refers to the sign of the second term, while the second to the sign under the square root. By employing Eqs. (11) and (13) in (10) we also obtain the
evolution of the pressure in cosmological time $\tau$:

$$
\frac{p_{\pm\pm}}{\Lambda} = 1 \pm \frac{4}{3\kappa^2 \lambda^2 \tau^2 \left( \beta \pm \sqrt{\beta^2 - \alpha} \right)^{1/2} \left[ 1 - \alpha \left( \beta \pm \sqrt{\beta^2 - \alpha} \right)^{-2} \right]}.
$$

(15)

Here the first subscript refers to the signs preceding the second and third terms in Eq. 15 while the second to the signs in the respective terms. One can see that whenever $\rho$ is well defined, $p$ also exists. We remark that when $\beta \to \sqrt{\alpha}$ (at $\tau \to \tau_{2,\alpha}$), the factor $\left[ 1 - \alpha \left( \beta \pm \sqrt{\beta^2 - \alpha} \right)^{-2} \right]$ → 0 in Eq. 15 and in consequence $p_{\pm\pm} \to \infty$. The energy density becomes ill-defined for $\tau > \tau_{2,\alpha}$ because a pressure singularity occurs at $\tau = \tau_{2,\alpha}$.

There are four admissible branches of solutions of both Eqs. (14) and (15). In the symmetric limit $\alpha \to 0$ only the $(\pm\pm)$ branches survive, the other two give unphysical solutions $p_{\pm} \to -\lambda$ and $p_{\pm} \to \infty$.

III. EVOLUTION OF THE FLUID IN THE ASYMMETRIC SWISS-CHEESE BRANE-WORLD

Only two branches, $p_{\pm\pm}$ can give positive energy density. We first discuss this possibility for the range $\alpha < 1$ of the asymmetry parameter, which allows for the symmetric limit. The condition $\rho_{++} \geq 0$ reduces to $2\beta \geq 1 + \alpha$, which gives

$$
\Lambda - \Lambda_{1,\alpha} \leq \frac{4}{3\tau^2},
$$

where we have introduced the notation

$$
\Lambda_{1,\alpha} = -\alpha \frac{\kappa^2 \lambda}{2}.
$$

(16)

(17)

The inequality (16) holds for $\Lambda \leq \Lambda_{1,\alpha}$ at any $\tau$ and for $\Lambda > \Lambda_{1,\alpha}$ at $\tau \leq \tau_{1,\alpha}$ where

$$
\tau_{1,\alpha} = \frac{2}{\sqrt{3(\Lambda - \Lambda_{1,\alpha})}} < \tau_1.
$$

(18)

Here $\tau_1$ is the time separating the positive and negative values of $\rho_{++}$ in the symmetric case. We also remark that the relation $\tau_{1,\alpha} < \tau_{2,\alpha}$ holds irrespective of the value of $\alpha$.

For huge asymmetry $\alpha \geq 1$ the condition $\rho_{++} \geq 0$ gives $\beta \geq 1$ or in terms of $\tau$ :

$$
\Lambda - \Lambda_{1,1} \leq \frac{4}{3\tau^2},
$$

(19)

where $\Lambda_{1,1} = \Lambda_{1,\alpha}=1$. For $\Lambda \leq \Lambda_{1,1}$ the inequality (19) identically holds, while for $\Lambda \geq \Lambda_{1,1}$ it gives $\tau \leq \tau_{1,1} \equiv
FIG. 3: (Color online) As in Fig. 1 but for $\Lambda = 0$. The evolution of $\rho$ and $\rho_{\pm}$ follows the same pattern as in Fig. 1 however in all asymmetric branches the density is well-defined only for $\tau \leq \tau_{2,\alpha}$. The asymmetric branches meet two by two at $\tau_{2,\alpha}$.

FIG. 4: (Color online) As in Fig. 2 but for $\Lambda = 0$. The evolution of $p$ and $p_{\pm}$ follows the same pattern as in Fig. 2, however in all asymmetric branches pressure singularities occur at $\tau = \tau_{2,\alpha}$.

$\tau_{1,\alpha}=1$. However $\tau_{1,1} > \tau_{2,\alpha}$ for any $\alpha > 1$, thus $\rho_{++}$ will be ill-defined before becoming negative. An other way to see this is to remark that $\Lambda_{1,1} > \Lambda_{2,\alpha}$ for $\alpha > 1$.

On the second branch, $\rho_{+-} \geq 0$ holds when $1 \leq \beta \leq (1 + \alpha) / 2$. This condition is never satisfied for $\alpha < 1$ while for $\alpha \geq 1$ it gives

$$\Lambda - \Lambda_{1,\alpha} \geq \frac{4}{3\tau^2}. \quad (20)$$

The inequality (20) holds true only for $\Lambda > \Lambda_{1,\alpha}$ and $\tau \geq \tau_{1,\alpha}$.

We summarize the results for $\rho_{++}$ and $\alpha < 1$ in Table I for $\rho_{+-}$ and $\alpha \geq 1$ in Table II for $\rho_{--}$ and $\alpha < 1$ in Table III and for $\rho_{+-}$ and $\alpha \geq 1$ in Table IV. Inserting $\alpha \rightarrow 0$ in Table II we recover the symmetric solution presented in [4]. There is no correspondent of the $\rho_{+-}$ branch in the symmetric case.

In what follows, we present graphically three typical evolutions of the cosmological fluid. On Figs. 1 and 5 we show the evolution of all four branches $\rho_{\pm\pm}$ as compared to the evolution of $\rho_{\pm}$ in the symmetric case (two branches) and the unique evolution of $\rho$ in the Einstein-Straus model. On Figs. 2 and 6 we compare the evolution of the pressure in all four branches with the symmetric evolution in the Swiss-cheese brane-world and in the Einstein-Straus model.

On Figs. 1 and 2 we have selected a negative cosmological constant with $\alpha \leq \alpha_{\text{crit}}$. In consequence $\rho_{\pm\pm}$ and

FIG. 5: (Color online) As in Fig. 1 but for $\Lambda = 2$. This time the domain of well-definedness is bounded by $\tau_{2}$ in the symmetric and by $\tau_{2,\alpha}$ in the asymmetric case, as opposed to the Einstein-Straus model, where $\rho$ is well-defined through the entire cosmological evolution. Asymmetry both reduces the domain of well-definedness and the magnitude of the energy density.

FIG. 6: (Color online) As in Fig. 2 but for $\Lambda = 2$. Pressure singularities are present both in the symmetric and in the asymmetric case. The pressure singularity occurs faster due to asymmetry.
FIG. 7: (Color online) The critical behavior of the fluid, in the asymmetry parameter $\alpha$, illustrated for $\Lambda = 0$. The sections $\alpha=\text{const.}$ give the four branches $\rho_{\pm\pm}$, either extending to $\infty$ as in Fig. 4 or constrained to a finite domain as in Figs. 5 and 6. The front "edges" where the branches meet two by two represent the pressure singularities, which come together with the ill-definedness of the energy density. The rear "edge" at $\rho = -\lambda$ is at the zero value of the asymmetry parameter, however it does not represent a solution of the symmetric problem, in contrast with the corresponding sections of the top and of the bottom leaves. The energy densities and the time $\tau$ are given in units $\lambda$ and $4/3\kappa^2\lambda$, respectively. In this figure the pressure singularities are due entirely to the asymmetry in the embedding.

$p_{\pm\pm}$ stay well-defined during the whole cosmological evolution. We remark that the $\pm\pm$ branches stay close to the energy density and pressure of the symmetric case and that brane-world effects are milder by the presence of asymmetry, as the asymmetric branches $\pm\pm$ stay closer to the energy density $\rho$ and pressure $p$ of the Einstein-Straus model.

Figs. 4 and 5 represent the case of a vanishing brane cosmological constant. The asymmetry parameter this time is above the critical value $\alpha_{\text{crit}}$, therefore the domain of $\rho_{\pm\pm}$ is restricted. At $\tau_{2,\alpha}$ pressure singularities occur. In this case the evolution of the fluid is drastically modified by asymmetry: the symmetric branches are well-defined throughout the cosmological evolution and there is no pressure singularity in the symmetric case.

On Figs. 6 and 7 we have represented a case with positive cosmological constant, which is above the threshold for having restricted domain of $\rho_{\pm}$ and pressure singularities even in the symmetric case. The effect of asymmetry this time is to speed up the occurrence of pressure singularities.

$\rho_{\pm\pm}$ are four possible branches $\rho_{\pm\pm}$, either extending to $\infty$ as in Fig. 4 or constrained to a finite domain as in Figs. 5 and 6. The front "edges" where the branches meet two by two represent the pressure singularities, which come together with the ill-definedness of the energy density. The rear "edge" at $\rho = -\lambda$ is at the zero value of the asymmetry parameter, however it does not represent a solution of the symmetric problem, in contrast with the corresponding sections of the top and of the bottom leaves. The energy densities and the time $\tau$ are given in units $\lambda$ and $4/3\kappa^2\lambda$, respectively. In this figure the pressure singularities are due entirely to the asymmetry in the embedding.

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We have discussed the effects of the asymmetric embedding of the Swiss-cheese brane-world into the SAdS5 bulk. First, we have shown that such brane-worlds can exist in the presence of asymmetry. On such Swiss-cheese universes the evolution of the scale factor is not affected by asymmetry. The totality of source terms combine to an effective total dust source with

$$\rho = \frac{4}{3\kappa^2\tau^2}, \quad p = 0,$$

establishing the same cosmological evolution as in the Einstein-Straus model. Due to the modified Friedmann and Raychaudhuri equations, however the fluid in the asymmetric brane-world scenario evolves differently. Even in the symmetric case the modifications were substantial. Two branches $\rho_{\pm}$ were present (one positive) and the possibility of occurrence of a pressure singularity has emerged. The regions with negative energy density could be reinterpreted in terms of an other fluid with energy density and pressure modified such that it absorbs the cosmological constant.

In the asymmetric case these features are conserved. However the asymmetry further splits up the possible evolutions of the fluid. In the asymmetric case there are four possible branches $\rho_{\pm\pm}$ (two allowing for positive energy density). Whenever the pressure singularity is present in the symmetric case as well, asymmetry has the effect that these singularities appear faster. Even in the cases when there are no pressure singularities in the symmetric case, if the asymmetry parameter exceeds the critical value $\alpha_{\text{crit}}$ given by Eq. (13), these singularities appear.

We illustrate this behavior on Fig. 7 for $\Lambda = 0$. As can be seen on Fig. 4 the occurrence of pressure singularities is not due there to the difference of the cosmological constants in the brane regions, but rather to the difference in the bulk cosmological constants.

As in the symmetric case, the occurrence of the pressure singularities is due to the modified dynamics imposed by the brane-world scenario. It is remarkable however that the evolution of the scale factor remains regular when the pressure singularities occur. This is a generic feature of the pressure singularities introduced in Ref. [4].

We conclude that pressure singularities appear as consequence of the difference of the cosmological constants either between the vacuum and FLRW regions (symmetric case) or between the left and right bulk regions with respect to the brane (asymmetric case). They may emerge as consequence of enforcing the Swiss-cheese configuration to the brane-world. It is likely, than in certain phase of the cosmological evolutions a more dynamic scenario has to take the place of the Swiss-cheese brane-world, in which either the voids are not static or and their boundaries are not comoving.
V. ACKNOWLEDGEMENTS

This work was supported by OTKA grants no. T046939 and TS044665. LÁG was further supported by the János Bolyai Grant of the Hungarian Academy of Sciences.