Strength of interaction for information distribution

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Let us consider two quantum systems: system A and system B. Suppose that a classical information is encoded to quantum states of the system A and we distribute this information to both systems by making them interact with each other. We show that it is impossible to achieve this goal perfectly if the strength of interaction between the quantum systems is smaller than a quantity that is determined by noncommutativity between a Hamiltonian of the system A and the states (density operators) used for the information encoding. It is a consequence of a generalized Wigner-Araki-Yanase theorem which enables us to treat conserved quantities other than additive ones.

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Let us consider two quantum systems, system A and system B. Suppose that a (classical) bit is encoded to a pair of quantum states of the system A. To distribute (or broadcast) the information to both systems, one needs interaction between them. How strong the interaction should be?

Let us begin with a detailed explanation of the problem. The system A (resp. B) is described by a Hilbert space, \( \mathcal{H}_A \) (resp. \( \mathcal{H}_B \)). A classical information, 0 or 1, is encoded to a pair of pure distinguishable quantum states of the system A. 0 is encoded to \( |\psi_0\rangle\langle\psi_0| \), and 1 is encoded to \( |\psi_1\rangle\langle\psi_1| \), where \( |\psi_0\rangle \) and \( |\psi_1\rangle \) are the normalized vectors in \( \mathcal{H}_A \). We assume that the vectors \( |\psi_0\rangle \) and \( |\psi_1\rangle \) are orthogonal with each other. The composite system, \( \mathcal{H}_A \otimes \mathcal{H}_B \), is assumed to be a closed system. The time evolution of the closed system is determined by a Hamiltonian, \( H := H_A + H_B + H_{int} \), where \( H_A \) (resp. \( H_B \)) is an operator acting only on \( \mathcal{H}_A \) (resp. \( \mathcal{H}_B \)), and \( H_{int} \) is an interaction term.

Before the interaction, the system B is assumed to be in a state \( \rho_B \) independent of the state of the system A. Our purpose is broadcasting the classical information to both systems. That is, states after the interaction should be perfectly distinguishable on both systems. Let us write \( \rho_0 \) and \( \rho_1 \) the states of the composite system after the interaction. If we put \( T \) a time interval for the interaction, they can be written as \( \rho_j := U(|\psi_j\rangle\langle\psi_j| \otimes \sigma)U^* \) for \( j = 0, 1 \) with \( U := e^{-iHT} \). Their restriction to the system A (resp. B) defines \( \rho_j^A \) (resp. \( \rho_j^B \)) for \( j = 0, 1 \). To discuss distinguishability, we make use of a measure called fidelity [1,2]. The fidelity between two states \( \sigma_0 \) and \( \sigma_1 \) is defined as \( F(\sigma_0, \sigma_1) := \text{tr}(\sqrt{\rho_0^{1/2} \sigma_1 \rho_0^{1/2}}) \) which takes 1 iff the states coincide with each other and takes a smaller nonnegative value as they are more distinguishable. The following lemma proved by [3,4] is not only useful but also justifies that the fidelity indeed represents distinguishability of states.

**Lemma 1** The fidelity equals the minimum overlap of the square root of coefficient between two probability distributions \( p_0 \) and \( p_1 \):

\[
F(\rho_0, \rho_1) = \min_{\{E_\alpha\}: \text{POVM}} \sum_{\alpha} \sqrt{p_0(\alpha)p_1(\alpha)},
\]

where \( p_0 \) and \( p_1 \) are defined by \( p_0(\alpha) = \text{tr}(\rho_0 E_\alpha) \) and \( p_1(\alpha) = \text{tr}(\rho_1 E_\alpha) \). The minimum is taken over all the possible positive operator valued measures (POVMs), where a POVM \( \{E_\alpha\} \) is a family of the positive operators satisfying \( \sum_\alpha E_\alpha = 1 \). Moreover, the minimum is attained by a projection valued measure (PVM), where a PVM \( \{E_\alpha\} \) is a family of the projection operators satisfying \( \sum_\alpha E_\alpha = 1 \).

This lemma plays an essential role in the proof of our theorem. The following theorem can be regarded as a generalized version of the Wigner-Araki-Yanase theorem on the distinguishability [5].

**Theorem 2** Let us consider a dynamics of the composite system, \( \mathcal{H}_A \otimes \mathcal{H}_B \), described by a unitary operator \( U \).
Suppose that there exists a conserved quantity, \( L := L_A + L_B + L_{int} \), where \( L_A \) (resp. \( L_B \)) is an observable acting only on \( \mathcal{H}_A \) (resp. \( \mathcal{H}_B \)), and \( L_{int} \) is an overlapping term. The following inequality holds:

\[
|\langle\psi_0|L_A|\psi_1\rangle| \leq \|L_B\|F(\rho_0^A, \rho_1^A) + \|L_A\|F(\rho_0^B, \rho_1^B) + 2\|L_{int}\|,
\]

(1)
where $\| \cdot \|$ is the operator norm defined as $\| v \| := \sup_{\| \phi \| \neq 0, \| \phi \| \in \mathcal{H}} \frac{\| v(\phi) \|}{\| \phi \|}$ for an operator $v$ on a Hilbert space $\mathcal{H}$.

**Proof:** By the purification of $\sigma$, we obtain a dilated Hilbert space and a vector state of the system $B$. We write the dilated Hilbert space as $\mathcal{H}_B$ for simplicity and the vector state as $|\Omega\rangle$. The dilated unitary operator $U$ is abbreviated as $\bar{U}$. Let us define initial vector states $|\Psi_1\rangle := |\psi_1\rangle \otimes |\Omega\rangle$ for $i = 0, 1$. Since $L$ is conserved with respect to the dynamics, $U L U^* = L$ holds. Thus, as Wigner, Araki, and Yanase’s original discussion [6–8], we have,

$$\langle \psi_0 | L_A | \psi_1 \rangle = \langle \psi_0 | L_A | \psi_1 \rangle + \langle \Psi_0 | L_{int} | \Psi_1 \rangle$$

$$= \langle \Psi_0 | U^*(L_A + L_B + L_{int}) U | \Psi_1 \rangle$$

$$= \langle \Psi_0 | U^* L_A U | \Psi_1 \rangle + \langle \Psi_0 | U^* L_B U | \Psi_1 \rangle + \langle \Psi_0 | U^* L_{int} U | \Psi_1 \rangle.$$  (2)

Now we consider an arbitrary PVM $\{E_\alpha\}$ on the system $A$ and an arbitrary PVM $\{P_j\}$ on the system $B$. Since $\sum E_\alpha = \sum P_j = 1$ holds, the right hand side of (2) can be written as $\sum_j \langle \Psi_0 | U^* \alpha P_j A U | \Psi_1 \rangle + \sum \langle \Psi_0 | U^* \alpha B \alpha E_\alpha U | \Psi_1 \rangle$. By using commutativity $[P_j, A] = [E_\alpha, B] = 0$, we obtain,

$$\langle \psi_0 | L_A | \psi_1 \rangle + \langle \Psi_0 | L_{int} | \Psi_1 \rangle = \sum_j \langle \Psi_0 | U^* \alpha P_j A P_j U | \Psi_1 \rangle$$

$$+ \sum \langle \Psi_0 | U^* \alpha B \alpha E_\alpha U | \Psi_1 \rangle + \langle \Psi_0 | U^* L_{int} U | \Psi_1 \rangle.$$  

Taking absolute value of both sides, we obtain,

$$\| \langle \psi_0 | L_A | \psi_1 \rangle + \langle \Psi_0 | L_{int} | \Psi_1 \rangle \| \leq \sum_j \| \langle \Psi_0 | U^* \alpha P_j A P_j U | \Psi_1 \rangle \| + \sum \| \langle \Psi_0 | U^* \alpha B \alpha E_\alpha U | \Psi_1 \rangle \|$$

$$\leq \| L_A \| \sum_j \sqrt{\langle \Psi_0 | U^* \alpha P_j A | \Psi_1 \rangle \langle \Psi_1 | U^* \alpha P_j A | \Psi_1 \rangle}$$

$$+ \| L_B \| \sum \sqrt{\langle \Psi_0 | U^* \alpha B \alpha U | \Psi_1 \rangle \langle \Psi_1 | U^* \alpha B \alpha U | \Psi_1 \rangle} + \| L_{int} \|.$$  

We here choose the particular PVMs, $\{E_\alpha\}$ and $\{P_j\}$, which attain the fidelity. Thanks to the lemma 1 and the triangular inequality, we obtain,

$$\| \langle \psi_0 | L_A | \psi_1 \rangle - \langle \Psi_0 | L_{int} | \Psi_1 \rangle \| \leq \| L_B \| F(\rho_0^A, \rho_1^A) + \| L_A \| F(\rho_0^B, \rho_1^B) + \| L_{int} \|.$$  

Thus we obtain,

$$\| \langle \psi_0 | L_A | \psi_1 \rangle \| \leq \| L_B \| F(\rho_0^A, \rho_1^A) + \| L_A \| F(\rho_0^B, \rho_1^B) + 2 \| L_{int} \|.$$  

It ends the proof. Q.E.D.

The left hand side of the equation (1) is related with noncommutativity between $L_A$ and the density operators $E_0 := |\psi_0\rangle \langle \psi_0|$ and $E_1 := |\psi_1\rangle \langle \psi_1|$. In fact, if we put $N_j := |E_j, L_A\rangle$ ($j = 0, 1$), $|\psi_0 | L_A | \psi_1 \rangle|^2 = \text{tr}(E_0 L_A E_1 L_A E_0)$ is expressed as:

$$\text{tr}(E_0 L_A E_1 L_A E_0) = \text{tr}((L_A E_0 + N_0) E_1 L_A E_0)$$

$$= \text{tr}(N_0 E_1 L_A E_0)$$

$$= \text{tr}(N_0 (L_A E_1 + N_1) E_0)$$

$$= \text{tr}(N_0 N_1 E_0) = \langle \psi_0 | N_0 N_1 | \psi_0 \rangle.$$  

In contrast with the discussions on the Wigner-Araki-Yanase theorem so far [6–9], the conserved quantity is not restricted to be the additive one. Therefore we can treat the Hamiltonian itself as the conserved quantity. That is, we put $L_A = H_A$, $L_B = H_B$, and $L_{int} = H_{int}$.

**Theorem 3** Let us consider a composite system $\mathcal{H}_A \otimes \mathcal{H}_B$ that evolves by a Hamiltonian $H = H_A + H_B + H_{int}$ for an arbitrary time interval. Suppose that an information bit is encoded to a pair of orthogonal states, $|\psi_0\rangle$ and $|\psi_1\rangle$, of the system $A$. For any choice of the initial state of the system $B$ that is independent of the bit encoded, the following inequality holds,

$$\| \langle \psi_0 | H_A | \psi_1 \rangle \| \leq \| H_B \| F(\rho_0^A, \rho_1^A) + \| H_A \| F(\rho_0^B, \rho_1^B) + 2 \| H_{int} \|,$$  

where $\rho_j^A$ is the final states of system $A$ and $\rho_j^B$ is the final states of the system $B$ for the encoded bit $j$. 

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Proof: It is a direct consequence of the theorem 2. Q.E.D.

Thus we obtain the strength of the interaction that is required for the perfect information distribution.

Corollary 4 Under the condition of the theorem 3, if $2\|H_{int}\| < |\langle \psi_0 | H_A | \psi_1 \rangle|$ holds, the perfect information distribution cannot be attained.

Proof:
The vanishing fidelities in (3) contradict with the nonvanishing left hand side. Q.E.D.

This corollary provides us a kind of no-go theorems. That is, even classical information cannot be copied if the strength of the interaction is smaller than a quantity determined by noncommutativity between the system Hamiltonian and the states (density operators) used for the information encoding. On the other hand, in case an approximate information distribution suffices, large systems can compensate the weakness of the interaction.

Let us consider the simplest example. The system $A$ is a spin $1/2$ system. The Hamiltonian of the system $A$ is the $z$-component of the spin, $H_A = S_z$, which is written with the eigenvectors, $|1\rangle$ and $|-1\rangle$, as $S_z = \frac{1}{2}(|1\rangle\langle 1|-|{-1}\rangle\langle -1|)$. A pair of the orthogonal normalized vectors, $|\psi_1\rangle, |\psi_0\rangle$ can be written as $|\psi_1\rangle := \alpha|1\rangle + \beta|-1\rangle$ and $|\psi_0\rangle := \beta|1\rangle - \alpha|-1\rangle$ in general with neglect of irrelevant phase, where $|\alpha|^2 + |\beta|^2 = 1$ is satisfied. The noncommutativity between these encoded states (density operators) and $H_A$ can be calculated as, $\langle \psi_0 | H_A | \psi_1 \rangle = \alpha\beta$. Thus, to achieve the perfect information distribution, the strength of the interaction must satisfy, $\|H_{int}\| \geq |\alpha||\beta|/2$ which is nonvanishing in case $\alpha \neq 0$ and $\beta \neq 0$ hold. On the other hand, if $\alpha = 1, \beta = 0$ holds, the term $\langle \psi_0 | H_A | \psi_1 \rangle$ vanishes. In such a case, one can construct whatever weak interactions to achieve perfect information distribution. Let us take the system $B$ as a spin $1/2$ system. If we put the initial state of the system $B$ as $|\Omega\rangle := \frac{1}{\sqrt{2}}(|1\rangle + |{-1}\rangle)$ and the Hamiltonian as $H_B = 1$ and $H_{int} := \epsilon(|1\rangle\langle 1| \otimes |1\rangle\langle 1| + |{-1}\rangle\langle -1|\langle -1|\langle {-1}|) = (|1\rangle \otimes \frac{1}{\sqrt{2}}(|1\rangle + e^{i\epsilon T}|{-1}\rangle))$ in time $T$, and $|{-1}\rangle\otimes |\Omega\rangle$ evolves into $|{-1}\rangle \otimes \frac{1}{\sqrt{2}}(|1\rangle + e^{-i\epsilon T}|{-1}\rangle)$, where we neglected the phase. Thus in time $T = \frac{\pi}{2\epsilon}$, we achieve the perfect information distribution. Since $\|H_{int}\| = \epsilon$ holds and $\epsilon > 0$ is arbitrary, the strength of the interaction can be arbitrarily small.

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