Hadrons as Skyrmions in the presence of isospin chemical potential

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The stability of the Skyrmion solution in the presence of finite isospin chemical potential \( \mu \) is considered, showing the existence of a critical value \( \mu_c = 222.8 \) MeV where the Skyrmion mass vanishes. Using the Hamiltonian formulation, in terms of collective variables, we discuss the behavior of different skyrmionic parameters as function of the isospin chemical potential (\( \mu \)), such as the energy density, the isoscalar radius and the isoscalar magnetic radius. We found that the radii start to grow very fast for \( \mu \geq 140 \) MeV, suggesting the occurrence of a phase transition.

The skyrmion picture has attracted the attention of many authors as a possible way for understanding hadronic dynamics and the hadronic phase structure. The behavior of hadrons in a media, i.e. taking into account temperature and/or density effects, can be analyzed according to this perspective.

The Skyrme lagrangian is

\[
\mathcal{L} = \frac{F_\pi^2}{16} Tr \left[ \partial_\mu U \partial^\mu U^\dagger \right] + \frac{1}{32\pi^2} Tr \left[ \left( \partial_\mu U \right) U^\dagger, \left( \partial_\nu U \right) \left( U^\dagger \right) \right]^2,
\]

where \( F_\pi \) is the pion decay constant and and is a numerical parameter. The isospin chemical potential is introduced as a covariant derivative of the form \[\hat{\mu} = \mu \xi \cdot \hat{n} \] where \( \hat{\mu} \) is the pion decay constant and \( \hat{n} = \hat{r} \) is the radial profile.

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where \( F_\pi \) is the pion decay constant and \( e \) is a numerical parameter. The isospin chemical potential is introduced as a covariant derivative of the form \[\hat{\mu} = \mu \xi \cdot \hat{n} \] where \( \hat{\mu} \) is the pion decay constant and \( \hat{n} = \hat{r} \) is the radial profile.

The mass of the Skyrmion, for static solutions, develops a dependence on the Isospin Chemical potential as well as on the temperature. Defining \( \hat{r} = eF_\pi r \), the mass of the Skyrmion will be given by

\[
M_\mu = M_{\mu=0} - \frac{\mu^2}{4\pi^2F_\pi} I_2 - \frac{\mu^2}{32\pi^2F_\pi} I_4,
\]

where \( M_{\mu=0} \) is the zero chemical potential contribution. Notice that the chemical potential terms contribute with opposite sign. This implies that the solution will become unstable above certain value of \( \mu \). In the previous equation

\[
M_{\mu=0} = \frac{F_\pi}{4\pi} \left\{ 4\pi \int_0^\infty \hat{r} \sin^2(\xi_1) \right\}
\]

\[
+ 4\pi \int_0^\infty \hat{r} \sin^2(\xi_1) \times \left[ 4\hat{r}^2 \left( \frac{d\xi_1}{d\hat{r}} \right)^2 + 2 \sin^2(\xi_1) \right] \right\}.
\]

Assuming a radial profile \( \xi = \xi(r) \), the integrals \( I_2 \) and \( I_4 \) are given by

\[
I_2 = \frac{4\pi}{3} \int d\hat{r} \hat{r}^2 \sin^2 \xi,
\]

\[
I_4 = \frac{32\pi}{3} \int d\hat{r} \hat{r}^2 \left[ \sin^2 \xi \left( \frac{d\xi_1}{d\hat{r}} \right)^2 + \frac{4}{\hat{r}^2} \sin^4 \xi \right].
\]

In order to minimize the mass, we use a variational procedure which leads us to the following condition for the radial profile

\[
\frac{d}{d\hat{r}} \left( \frac{2}{\sin \xi} \frac{d}{d\hat{r}} \right) + \frac{2}{\sin \xi} = 0.
\]
As usual, the SU(2) critical value of \( \mu \) is the point where the mass vanishes. Figure 1 shows the chemical potential dependence of the mass.

It turns out that this procedure is limited only for small values of \( \mu < 100 \text{ MeV} \), been in this region quite in agreement with the numerical solution.

In order to obtain the mass of the Skyrmion, the profile has to be inserted, numerically, in equation (5). Figure 2 shows the chemical potential dependence of the mass.

The Skyrmion solution we have presented, can be considered as the basic or skeleton structure for the hadronic states. In order to characterize different hadronic states, and following [2], we will introduce collective variables \( A(t) \), computing then the mass spectrum of the nucleons, in an hadronic approach. As usual, the \( SU(2) \) collective coordinates \( A(t) \) are introduced as

\[
U = A(t)U_0A^\dagger(t).
\]  

FIG. 1: Numerical solution for the profile \( \xi(r) \) for different values of \( \mu \). (\( \mu = 12.9 \) (MeV): solid line; \( \mu = 25.8 \) (MeV): dashed line; \( \mu = 38.7 \) (MeV): dotted line.)

where

\[
A(t) = a_0(t)\sigma_0 + i\vec{a}(t) \cdot \vec{\sigma},
\]

where the \( a \)'s obey the constraint

\[
a_0^2(t) + \vec{a}^2(t) = 1.
\]

Introducing (9) and (10) into (12), a rather length computation leads us to the Lagrangian

\[
L = -M_\mu + 2\lambda \left( \dot{\vec{a}}_i + \mu \frac{\dot{\vec{A}}}{2} \right)^2,
\]

where \( \lambda = (2\pi/3e^3F_e)\Lambda \), with

\[
\Lambda = \int r^2 \sin^2 F \left[ 1 + 4 \left( F'^2 + \frac{\sin^2 F}{r^2} F'' \right) \right].
\]

In equation (12) we have defined \( \vec{A}_i = g_{ij}a_j \), where \( \vec{A}_0 = a_3, \vec{A}_1 = -a_2, \vec{A}_2 = a_1 \) and \( \vec{A}_3 = -a_0 \). \( M_\mu \) is the chemical potential dependent mass given in (6). In this way we get the Hamiltonian

\[
H = M_\mu - 2\lambda \mu^2 + \frac{\pi^2}{8\lambda}
\]

where the canonical momentum is given through a minimal coupling \( \pi_i = p_i - 4\lambda\mu\vec{A}_i \). The Hamiltonian can be expressed as

\[
H = M_\mu + \frac{\vec{p}^2}{8\lambda} - \mu \vec{A}_i p_i
\]
The baryonic charge density for the Skyrmion is given by

\[ \rho_B = 4\pi r^2 B^0(r) = -\frac{2}{\pi} \sin^2 F(r) F'(r). \]  

(19)

Obviously, \( \int_0^\infty dr \rho_B = 1 \), independently of the shape of the skyrmionic profile. The isoscalar mean square radius is defined by

\[ \langle r^2 \rangle_{I=0} = \int_0^\infty dr r^2 \rho_B. \]  

(20)

This radius seems to be quite stable up to the value of \( \mu \approx 120 \text{ MeV} \), starting then to grow dramatically. Although we do not have a formal proof that this radius diverges at a certain critical \( \mu = \mu_c \), the numerical evidence supports such claim, as it is shown in figure 4.

Divergent behavior for several radii, associated to different currents, has also been observed in different hadronic effective couplings as function of temperature in the frame of thermal QCD sum rules [6]. Similar behavior is found for the mean square radius associated to the isoscalar magnetic density

\[ \rho_M^{I=0}(r) = \frac{r^2 F' \sin^2 F}{\int dr r^2 F' \sin^2 F}, \]  

(21)

The divergent behavior of the the effective radii, suggests the occurrence of a phase transition, in reference [7] the behavior of the nucleons magnetic moments is also presented, which also have a divergent behavior.

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