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The Dominant Balance at Cosmological Singularities

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Abstract. We define the notion of a finite-time singularity of a vector field and then discuss a technique suitable for the asymptotic analysis of vector fields and their integral curves in the neighborhood of such a singularity. Having in mind the application of this method to cosmology, we also provide an analysis of the time singularities of an isotropic universe filled with a perfect fluid in general relativity.

1. Introduction

There are two approaches to characterizing spacetime singularities in a cosmological context. The first approach may be called *geometric* and consists of finding sufficient and/or necessary conditions for singularity formation, or absence, *independently* of any specific solution of the field equations under general conditions on the matter fields. Methods of this sort include those based on an analysis of geodesic congruences in spacetime and lead to the well known singularity theorems, cf. [1], as well as those which are depend on an analysis of the geodesic equations themselves and lead to completeness theorems such as those expounded in [2], and the classification of singularities in [3].

The second approach to the singularity problem can be termed *dynamical* and refers to characterizing cosmological singularities in a geometric theory of gravity by analysing the dynamical field equations of the theory It uses methods from the theory of dynamical systems and can be *global*, referring to the asymptotic behaviour of the system of field equations for large times, or *local*, giving the behaviour of the field components in a small neighborhood of the finite-time singularity.

In this latter spirit, we present here a local method for the characterization of the asymptotic properties of solutions to the field equations of a given theory of gravity in the neighborhood of the spacetime singularity[1]. We are interested in providing an asymptotic form for the solution near singularities of the gravitational field and understanding all possible dominant features of the field as we approach the singularity. We call this approach the *method of asymptotic splittings.*

[1] Similar approaches have been used in the past to test given systems of equations for integrability under the so-called Painleve test. However, our approach is more geometric in nature and is not related to integrability.

In the following sections, we give an outline of the method of asymptotic splittings with a view to its eventual application to cosmological spacetimes in different theories of gravity. For the sake of illustration, in the last Section we analyze the asymptotic behaviour of a Friedmann-Robertson-Walker (FRW) universe filled with perfect fluid in Einstein's general relativity, which provides the simplest, nontrivial cosmological system.

2. Finite-time singularities: Definitions

It is advantageous to work on any differentiable manifold $\mathcal{M}^n$, although for specific applications we restrict attention to open subsets or $\mathbb{R}^n$ We shall use interchangeably the terms vector field $\mathbf{f} : \mathcal{M}^n \to \mathcal{T}\mathcal{M}^n$ and dynamical system defined by $\mathbf{f}$ on $\mathcal{M}^n$, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, with $(\dot{\cdot}) \equiv d/dt$. Also, we will use the terms 'integral curve' $\mathbf{x}(t, \mathbf{x}_0)$ of the vector field $\mathbf{f}$ with initial condition $\mathbf{x}(0) = \mathbf{x}_0$, and 'solution' of the associated dynamical system $\mathbf{x}(t; \mathbf{x}_0)$ passing through the point $\mathbf{x}_0$, with identical meanings.

Given a vector field $\mathbf{f}$ on the n-dimensional manifold $\mathcal{M}^n$, we define the notion of a *general* solution of the associated dynamical system as a solution that depends on n arbitrary constants of integration, $\mathbf{x}(t; \mathbf{C})$, $\mathbf{C} = (c_1, \cdots, c_n)$. These constants are uniquely determined by the initial conditions in the sense that to each $\mathbf{x}_0$ we can always find a $\mathbf{C}_0 = (c_{10}, \cdots, c_{n0})$ such that the solution $\mathbf{x}(t; \mathbf{C}_0)$ is the unique solution passing through the point $\mathbf{x}_0$. Therefore, a property holds *independently* of the initial conditions if and only if it is a property of a general solution of the system.

A *particular* solution of the dynamical system is any solution obtained from the general solution by assigning specific values to at least one of the arbitrary constants. The particular solutions containing k arbitrary constants can be viewed as describing the evolution in time of sets of initial conditions of dimension k strictly smaller than n. A particular solution is called an *exact* solution of the dynamical system when $k = 0$[2]. Thus, in our terminology, a particular solution is a more general object than any exact solution, the latter having the property that all arbitrary constants have been given specific values. The hierarchy: exact (no arbitrary constants) to particular (strictly less than maximum number of arbitrary constants) to general solutions, will play an important role in what follows.

General, particular, or exact solutions of dynamical systems can develop *finite-time singularities*; that is, instances where a solution $\mathbf{x}(t; c_1, \cdots, c_k)$, $k \leq n$, misbehaves at a finite value t_* of the time t. This is made precise as follows. We say that the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ (equivalently, the vector field $\mathbf{f}$) has a *finite-time singularity* if there exists a $t_* \in \mathbb{R}$ and a $\mathbf{x}_0 \in \mathcal{M}^n$ such that for all $M \in \mathbb{R}$ there exists an $\delta > 0$ such that

$$||\mathbf{x}(t; \mathbf{x}_0)||_{L^p} > M, \tag{1}$$

for $|t - t_*| < \delta$. Here $\mathbf{x} : (0, b) \to \mathcal{M}^n$, $\mathbf{x}_0 = \mathbf{x}(t_0)$ for some $t_0 \in (0, b)$, and $|| \cdot ||_{L^p}$ is any L^p norm. We say that the vector field has a *future* (resp. *past*) singularity if $t_* > t_0$ (resp. $t_* < t_0$). Note also, that t_0 is an arbitrary point in the domain $(0, b)$ and may be taken to mean 'now'. Alternatively, we may set $\tau = t - t_*$, $\tau \in (-t_*, b - t_*)$, and consider the solution in terms of the new time variable τ, $\mathbf{x}(\tau; \mathbf{x}_0)$, with a finite-time singularity at $\tau = 0$. We see that for a vector field to have a finite-time singularity there must be at least one integral curve passing through the point $\mathbf{x}_0$ of $\mathcal{M}^n$ such that at least one of its L^p norms diverges at $t = t_*$. We write

$$\lim_{t \to t_*} ||\mathbf{x}(t; \mathbf{x}_0)||_{L^p} = \infty, \tag{2}$$

[2] There are solutions, called *singular* solutions, which have the property that a certain jacobian vanishes on them, and so they are not obtainable from the general solution of the system like the particular solutions. Below we shall use this term but with a totally different meaning as our notion of a solution with a time singularity (which can be also called a singular solution) is completely different.

to denote a finite-time singularity at t_*.

One of the most interesting problems in the theory of singularities of vector fields is to find the structure of the set of points $\mathbf{x}_0$ in $\mathcal{M}^n$ such that, when evolved through the dynamical system defined by the vector field, the integral curve of $\mathbf{f}$ passing through a point in that set satisfies property (2).

Another important question, of special interest in relativistic cosmology, is to discover the precise relation between the finite-time singularities of vector fields that arise as reductions of the field equations and those that emerge in the form of geodesic incompleteness. The difficulty here is that the finite-time singularities of vector fields appear to be unconnected to geodesic incompleteness and conversely, singularities which arise through the formation of conjugate points do not seem to demand, or require, any dynamical description.

3. Fixed and movable singularities

Finite-time singularities of (general or particular) solutions of linear dynamical systems are located at the singularities of their coefficients and are *fixed* because they are known from the singularities of the coefficients of the system. The fixed singularities in a solution are therefore independent of the choice of initial conditions. In contrast, solutions of nonlinear systems can develop finite-time singularities that are either fixed or movable. A singularity is *fixed* if it is a singularity of $\mathbf{x}(t;\mathbf{C})$ for all $\mathbf{C}$; otherwise, we say it is a *movable singularity.*

Note that any fixed finite-time singularity of a particular solution ($k < n$) cannot be a fixed singularity of a general solution since at least one of the constants appearing in the general solution has been set to zero, and so this singularity is not one of a $\mathbf{x}(t;\mathbf{C})$ for *all* $\mathbf{C}$, that is independent of the initial conditions. Hence, fixed finite-time singularities *in a general solution* cannot be understood by studying fixed singularities in particular solutions. However, movable singularities of a particular solution, if they *are* singularities (in the sense of the definition above) of the general solution, will always be movable ones. Therefore, movable finite-time singularities in particular solutions make a nonzero contribution to the singularity pattern of the vector field and must be taken into consideration in the general study of its singularities.

It may also happen that a dynamical system has no movable singularities in the general solution but still has a singular or particular solution with a movable singularity. Hence, a fixed (or movable) singularity in a general solution can be a fixed (or movable) singularity of some particular solution but not vice versa. In general, movable singularities are more interesting, since the issue of choosing initial conditions plays an important role for them; consequently, we shall restrict our attention to them almost exclusively in what follows.

How should we tackle the geometric problem of describing the behaviour of vector fields and their integral curves – solutions of the associated dynamical system – in the neighborhood of a finite-time movable singularity? Assume that we are given a vector field, and we know that at some point, t_*, a system of integral curves, corresponding to a particular or a general solution, has a (future or past) finite-time singularity in the sense of definition (1). The approach we take in this paper is an asymptotic one. The vector field (or its integral curves) can basically do two things sufficiently close to the finite-time singularity, namely, it can either show some dominant feature or not. In the latter case, the integral curves can 'spiral' in some way around the singularity *ad infinitum* so that (1) is satisfied and the dynamics are totally controlled by the subdominant (lower-order in terms of weight - see below) terms, whereas in the former case solutions share a distinctly dominant behaviour on approach to the singularity at t_* determined by the most nonlinear terms.

To describe both cases invariably, we decompose the vector field into simpler, component vector fields and examine whether the most nonlinear one of these shows a dominant behaviour while the rest become subdominant in some exact sense. Using this picture, we then built a system of integral curves corresponding, where feasible, to the general solution, and sharing

exactly its characteristics in a sufficiently small neighborhood of the finite-time singularity. The construction of these solutions is given as a formal asymptotic series expansion around the singularity and is done term-by-term.

4. Weight-homogeneous vector fields

We begin by introducing some useful notation and terminology. We write

$$\mathbf{x}(\tau) = \mathbf{a}\tau^{\mathbf{p}}, \tag{3}$$

to denote the function

$$\mathbf{x}(\tau) = (\alpha\tau^{p}, \beta\tau^{q}, \gamma\tau^{r}, \cdots),$$

where $\mathbf{a} = (\alpha, \beta, \gamma, \cdots) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, and $\mathbf{p} = (p, q, r, \cdots) \in \mathbb{Q}^n$. A function of the form (3) is *scale invariant* in the sense that a change in the time scale, $\tau' = \eta\tau$, $\eta \in \mathbb{R}$, reveals that we must also have $\mathbf{x}' = \eta^{\mathbf{p}}\mathbf{x}$, and conversely.

Now we demand that a scale invariant function $\mathbf{x}(\tau; \mathbf{x}_0)$ of the form (3) is an integral curve of the vector field $\mathbf{f}$ passing through $\mathbf{x}_0$, or equivalently, the associated dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a particular solution that is scale invariant, valid in a neighborhood of the assumed finite-time singularity. This means that

$$\mathbf{a}\mathbf{p}\tau^{\mathbf{p}-\mathbf{1}} = \mathbf{f}(\mathbf{a}\tau^{\mathbf{p}}). \tag{4}$$

The notation $\mathbf{ap}$ stands for the monomial $a_i p_i$ and is valid for each $i = 1, \cdots, n$ (range convention, no summation).

We say that a vector field $\mathbf{f}$ (or the associated dynamical system) is *scale invariant* if it satisfies

$$\mathbf{f}(\mathbf{a}\tau^{\mathbf{p}}) = \tau^{\mathbf{p}-\mathbf{1}}\mathbf{f}(\mathbf{a}). \tag{5}$$

More generally, a vector field is called *weight-homogeneous* with *weighted degree* $\mathbf{d}$ if there is a vector $\mathbf{p} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, called *the weight*, and a vector $\mathbf{d} = (d_1, \cdots, d_n)$ such that

$$\mathbf{f}(\mathbf{a}\tau^{\mathbf{p}}) = \tau^{\mathbf{d}}\mathbf{f}(\mathbf{a}). \tag{6}$$

We denote the degree by $\deg(\mathbf{f}, \mathbf{p}) = \mathbf{d}$. When $\mathbf{p} = \mathbf{1}$ the vector field is called *homogeneous* of degree $\mathbf{d}$. Note that any scale invariant vector field is a weight-homogeneous field $\mathbf{f} = (f_1, \cdots, f_n)$ such that each component f_i has degree $p_i - 1$, from which it follows that $\partial f_i/\partial x_j$ has weight $p_i - 1 - p_j$. If $\mathbf{f}$ has degree $\mathbf{d}$, then $\partial\mathbf{f}/\partial x_j$ has degree $\mathbf{d} - p_j\mathbf{e}_j$, with $\mathbf{e}_j$ the j-th unit vector. There are weight-homogeneous vector fields that are not scale invariant; for instance, all linear dynamical systems with constant coefficients which are homogeneous vector fields.

Using Eqs. (4), (5) we see that given any nonzero vector $\mathbf{p}$, a scale-invariant vector field $\mathbf{f}$ admits a scale-invariant integral curve provided that the inhomogeneous linear system

$$\mathbf{a}\mathbf{p} = \mathbf{f}(\mathbf{a}), \tag{7}$$

has nontrivial solutions for $\mathbf{a}$. Note that there may be nontrivial solutions for $\mathbf{a}$ with some components zero. When at least one of the components of $\mathbf{a}$ is nonzero, the corresponding solution of the form (3) is a particular solution of the dynamical system. Therefore, in this case, we know the exact asymptotic behaviour of the vector field $\mathbf{f}$ in the neighborhood of a finite-time singularity, given by the solution (3) with suitable $\mathbf{a}$'s and $\mathbf{p}$'s.

5. Vector-field decompositions

Unfortunately, most vector fields are neither scale invariant nor weight-homogeneous. However, since any *analytic* vector field $\mathbf{f}(\mathbf{x})$ can be expanded in a power series in some domain $\mathcal{D} \subset \mathcal{M}^n$, by taking any $\mathbf{x}_0 \in \mathcal{D}$ for any $\mathbf{x} \in \mathcal{D}$ distinct from $\mathbf{x}_0$, any such vector field can be decomposed into weight-homogeneous *components* by taking for instance the first $k+1$ terms in its Taylor expansion around $\mathbf{x}_0$. A simpler example of a vector field admitting a weight-homogeneous splitting is to take $\mathbf{f}$ to be any polynomial vector field; then, a possible decomposition is to split it such that each component is a suitable combination of monomials.

We say that the nonlinear vector field $\mathbf{f}$ on $\mathcal{M}^n$ admits a *weight-homogeneous decomposition* with respect to a given vector $\mathbf{p}$ if it splits as a combination of the form

$$\mathbf{f} = \mathbf{f}^{(0)} + \mathbf{f}^{(1)} + \cdots + \mathbf{f}^{(k)}, \tag{8}$$

where the *components* $\mathbf{f}^{(j)}$, $j = 0, \cdots, k$, are weight-homogeneous vector fields, namely,

$$\mathbf{f}^{(j)}(\mathbf{a}\tau^{\mathbf{p}}) = \tau^{\mathbf{p}+\mathbf{1}(q^{(j)}-1)}\mathbf{f}^{(j)}(\mathbf{a}), \quad j = 0, \cdots, k, \tag{9}$$

for some non-negative numbers $q^{(j)}$ and all $\mathbf{a}$ in some domain $\mathcal{E}$ of $\mathbb{R}^n$. In terms of individual components, condition (9) reads

$$f_i^{(j)}(\mathbf{a}\tau^{\mathbf{p}}) = \tau^{p_i+q^{(j)}-1} f_i^{(j)}(\mathbf{a}), \quad i = 1, \cdots, n, \ j = 0, \cdots, k. \tag{10}$$

In a slightly vague but suggestive manner we can say that a weight-homogeneous decomposition splits the original vector field in parts starting by collecting together the most nonlinear part and then proceeding down to the 'weakest' component such that each term in the splitting is 'less nonlinear' than the previous one in a precise sense.

There are two important features of such a vector-field decomposition (8), (9). Firstly, it is not unique. In general, for the given vector field $\mathbf{f}$, many different vectors $\mathbf{p}$ can be found which each lead to different weight-homogeneous decompositions of the vector field in the same domain $\mathcal{D}$. Secondly, since the *subdominant exponents* $q^{(j)}$, $j = 1, \cdots, k$ can be ordered,

$$0 = q^{(0)} < q^{(j_1)} < q^{(j_2)}, \quad \text{when} \quad j_1 < j_2, \tag{11}$$

the degrees of the component vector fields in the decomposition (9) are also ordered and (only) the first vector field $\mathbf{f}^{(0)}$ appearing in the decomposition is scale invariant.

Therefore, a weight-homogeneous decomposition of a vector field with respect to a vector $\mathbf{p}$ is a splitting into $k+1$ weight-homogeneous components each with degree $\mathbf{p}+\mathbf{1}(q^{(j)}-1)$, $j = 0, \cdots, k$, such that the lowest-order vector field in the decomposition is scale invariant. The lowest-order vector field, $\mathbf{f}^{(0)}$, in a splitting is sometimes called *the dominant part* of $\mathbf{f}$ and includes the most nonlinear terms in $\mathbf{f}$, whereas the remaining sum of parts, $\mathbf{f}^{\text{sub}} \equiv \sum_{j=1}^{k} \mathbf{f}^{(j)}$, is called *the subdominant part.*

Given a vector field $\mathbf{f}$, it is very important to have the complete list of all possible weight-homogeneous decompositions it admits; in other words, to know all the possible dominant and subdominant ways it can be split. The asymptotic method we employ to trace the behaviour of vector fields and their integral curves in a neighborhood of a movable finite-time singularity (of a particular or general solution), begins by finding all weight-homogeneous decompositions of the vector field valid in that neighborhood.

6. The dominant balance

Suppose that there exists a decomposition

$$\mathbf{f} = \mathbf{f}^{(0)} + \mathbf{f}^{(1)} + \cdots + \mathbf{f}^{(k)}, \tag{12}$$

into $(k+1)$ weight-homogeneous components, such that the dominant part $\mathbf{f}^{(0)}$ is scale invariant, and each subdominant component $\mathbf{f}^{(j)}$, $j = 1, \cdots, k$ is weight-homogeneous. The scale invariant solution

$$\mathbf{x}^{(0)}(\tau) = \mathbf{a}\tau^{\mathbf{p}}, \ \mathbf{a} \neq \mathbf{0}, \mathbf{p} \in \mathbb{Q}^n, \tag{13}$$

is a solution[3] of the dominant part $\dot{\mathbf{x}} = \mathbf{f}^{(0)}$ of the vector field provided that $\mathbf{ap} = \mathbf{f}^{(0)}(\mathbf{a})$. Thus, some of the components of the vector $\mathbf{a}$ may vanish. We call the components of the vector $\mathbf{p} = (p_1, \cdots, p_n)$ the *dominant exponents.* In this case, we sometimes say that (13) is an *asymptotic solution* of the original system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

On the other hand, the k subdominant components $\mathbf{f}^{(j)}$ satisfy

$$f_i^{(j)}(\mathbf{a}\tau^{\mathbf{p}}) = \tau^{p_i + q^{(j)} - 1} f_i^{(j)}(\mathbf{a}), \quad i = 1, \cdots, n, \ j = 1, \cdots, k, \tag{14}$$

with the subdominant exponents $q^{(j)}$, $j = 1, \cdots, k$, which are ordered and strictly positive. Dividing both sides by $\tau^{\mathbf{p}-\mathbf{1}}$ and taking the limit as $\tau \to 0$, the true meaning of the subdominant exponents is revealed, and we have

$$\lim_{\tau \to 0} \frac{\mathbf{f}^{\text{sub}}(\mathbf{a}\tau^{\mathbf{p}})}{\tau^{\mathbf{p}-\mathbf{1}}} = 0, \tag{15}$$

which proves that the subdominant part of the vector field, $\mathbf{f}^{\text{sub}}$, is less dominant than the dominant part, $\mathbf{f}^{(0)}$, which of course asymptotes as $\tau^{\mathbf{p}-\mathbf{1}}$.

We say that the pair $(\mathbf{a}, \mathbf{p})$ is a *(dominant) balance* for the vector field $\mathbf{f}$ if the latter admits a decomposition satisfying Eqs. (12)-(14). There may be several different balances for any particular decomposition of $\mathbf{f}$ and a way to classify them is by means of their order. The *order* of a balance $(\mathbf{a}, \mathbf{p})$ is the number of the nonzero components of the vector $\mathbf{a}$. For a vector field on $\mathcal{M}^n$, the highest order of any possible balance is n and in this case the scale-invariant solution (13) corresponds to a possible dominant behaviour of a *general* solution of the original system $\dot{\mathbf{x}} = \mathbf{f}$ near the singularity. On the other hand, balances of a lower order than n describe possible asymptotics of *particular* solutions.

There is an elegant convex-geometric explanation of the dominant balances of a vector field $\mathbf{f}$. This requires us first to express the vector field in the so-called quasi-monomial form. Then, a dominant balance of order d corresponds precisely to a d-dimensional face of the Newton-Puiseux-Bruno polyhedron associated to $\mathbf{f}$ (cf. [6], [7]).

7. The K-matrix

We can now study an important square matrix, the so-called *(K-)ovalevskaya matrix*, associated with a given vector field $\mathbf{f}$. Consider the dominant part, $\mathbf{f}^{(0)}$, of $\mathbf{f}$ which admits an exact solution of the form (13), described by the dominant balance $(\mathbf{a}, \mathbf{p})$ (of any nonzero order). The *K-matrix of the vector field* $\mathbf{f}$ *at the balance* $(\mathbf{a}, \mathbf{p})$ is the square matrix

$$\mathcal{K} = D\mathbf{f}^{(0)}(\mathbf{a}) - \operatorname{diag} \mathbf{p}. \tag{16}$$

The *(K-)ovalevskaya exponents* associated with the balance $(\mathbf{a}, \mathbf{p})$ are the n eigenvalues $(\rho_1, \cdots, \rho_n)$ of $\mathcal{K}$. When the order of the balance is n ($a_i \neq 0$ for all $i = 1, \cdots, n$), the K-exponents are called the *resonances* of $\mathcal{K}$.

Setting $\mathbf{w} = \mathbf{f}^{(0)}(\mathbf{a}\tau^{\mathbf{p}})$, and differentiating with respect to τ, we have

$$\dot{\mathbf{w}} = (\mathbf{ap}\tau^{\mathbf{p}-\mathbf{1}})^{\cdot} = \operatorname{diag}(\mathbf{p} - \mathbf{1})\mathbf{ap}\,\tau^{\mathbf{p}-\mathbf{2}}, \tag{17}$$

[3] due to a theorem of Goriely and Hyde [4], [5], we can restrict attention to real vectors $\mathbf{a}$ only, as complex ones do not describe the behaviour near finite time singularities in the sense of Eq. (2).

while by the chain rule

$$\begin{aligned}\dot{\mathbf{w}} &= D\mathbf{f}^{(0)}(\mathbf{a}\tau^{\mathbf{p}})\mathbf{a}\mathbf{p}\,\tau^{\mathbf{p-1}} \\ &= D\mathbf{f}^{(0)}(\mathbf{a})\mathbf{a}\mathbf{p}\,\tau^{\mathbf{p-2}},\end{aligned} \tag{18}$$

where the last equality is most easily understood if we expand the derivative to obtain $\dot{w}_i = \tau^{p_i-2}\,\nabla f_i^{(0)}(\mathbf{a})\cdot\mathbf{ap}$, for the i-th component. Thus, from the last two equations, we find

$$\mathcal{K}\mathbf{ap} = -\mathbf{ap}; \tag{19}$$

that is, the K-matrix always has the vector $\mathbf{f}^{(0)}(\mathbf{a})$ as an eigenvector with eigenvalue equal to $\rho_1 = -1$. We say that a balance is *hyperbolic* if the remaining $(n-1)$ K-exponents have positive real parts.

Suppose now that we know all the eigenvectors $\mathbf{v}^{(i)}$ and eigenvalues $(-1, \rho_2, \cdots, \rho_n)$ of the K-matrix. By simply inspecting the form of the two sides in the variational equation for the dominant part of the vector field, namely, the equation

$$\dot{\mathbf{w}} = D\mathbf{f}^{(0)}(\mathbf{a}\tau^{\mathbf{p}})\mathbf{w}, \tag{20}$$

we can write the set of fundamental solutions $\mathbf{w}^{(i)}$ of this linear equation for $\mathbf{w}$ in the form

$$\mathbf{w}^{(i)} = \mathbf{g}^{(i)}(\log\tau)\,\tau^{\mathbf{p}+\rho_i}, \tag{21}$$

where, depending on whether or not $\mathcal{K}$ is semi-simple, the $\mathbf{g}^{(i)}(\log\tau)$'s are the eigenvectors $\mathbf{w}^{(i)}$ or, in general, polynomials in $\log\tau$. Hence, the solutions of the variational equation will be appropriate sums of terms by the form (21).

Therefore, any solution of the original system will be well approximated (cf. [8], p.299, for the precise conditions) by a solution of the form (this is a Taylor estimate):

$$\mathbf{x} = \mathbf{x}^{(0)} + \mathbf{w}, \tag{22}$$

where

$$\mathbf{x}^{(0)}(\tau) = \mathbf{a}\tau^{\mathbf{p}}, \quad \mathbf{w} = \sum_{i=1}^{k}\mathbf{h}^{(i)}\tau^{\mathbf{p}+\rho_i}, \tag{23}$$

where the $\mathbf{h}^{(i)}$'s are, in general, polynomials in $\log\tau$. Furthermore, we arrive at the interesting conclusion that in any particular or general solution, the arbitrary constants characterizing it will first appear in those terms whose coefficients have indices equal to a K-exponent. In a general solution, the arbitrary constants normally appear at different places in an expansion, and consequently, a solution in which a K-exponent (different from the -1 value which as we have shown always exists) is either negative, or has nontrivial multiplicity, may or may not be a general solution. We shall see in the next section how we can use Eq. (22) to obtain an important series representation of the solutions to the original dynamical system near a finite-time singularity.

Apart from the principal use made here in unravelling the nature of finite-time singularities, there are various important connections between the K-exponents, first integrals, integrability properties of Hamiltonian systems and complex algebraic geometry, cf. [9], [10].

8. Formal expansions

In the previous subsection, we derived the formal expansion (22) for the solution of the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ near a finite-time singularity by including only the first terms in a power-series expansion of $\mathbf{x}$. That solution corresponds to a given dominant balance $\mathcal{F} = (\mathbf{a}, \mathbf{p})$ and depends on the K-exponents $(-1, \rho_2, \cdots, \rho_n)$ associated with the dominant part, $\mathbf{f}^{(0)}$, of the original vector field $\mathbf{f}$. Writing Eq. (22) in full, that is including terms of all orders, can lead to very complicated expansions in general, and the essential feature is the appearance of logarithmic terms. We note that there are a number of general theorems guaranteeing the existence and convergence of such expansions – see, for instance, [11] for a review.

Let $\mathcal{E}^+$ be the set of all K-exponents with positive real parts. For simplicity we assume that all K-exponents in $\mathcal{E}^+$ are rational, and define the number $1/s$ to be the least common multiple of the denominators of the numbers in the set $\mathcal{H} = \{q^{(1)}, \cdots, q^{(m)}\} \cup \mathcal{E}^+$. In the case where any log terms are absent (for instance, when K is semi-simple), we can write, by (22), the full expansion of the general solution around the finite-time singularity in the form of a *Puiseux series*,

$$\mathbf{x} = \tau^{\mathbf{p}} \left(\mathbf{a} + \sum_{i=1}^{\infty} \mathbf{c}_i \tau^{i/s} \right), \tag{24}$$

where, as we know already from the previous section, each of the n arbitrary constants in (24) will first appear in the term with coefficient $\mathbf{c}_k$, $k = \rho s$ and $\rho \in \mathcal{E}^+$.

Hence, finding the final form of the solution (general or particular, depending on the number of arbitrary constants appearing in the series expansion) can be now reduced to knowing the coefficients $\mathbf{c}_i$ in the expansion. These coefficients are computed by inserting the Puiseux series (24) into the original system. This leads to a set of *recursion relations*, a linear system for the coefficients $\mathbf{c}_i$. For the jth-order coefficient we find

$$\mathcal{K}\mathbf{c}_j - \frac{j}{s}\mathbf{c}_j = \mathbf{P}_j(\mathbf{c}_1, \cdots, \mathbf{c}_{j-1}), \tag{25}$$

where the forms $\mathbf{P}_j$ are polynomial in its variable, read off from the original equation.

There is an important consistency condition to be satisfied for the above analysis to be valid. Multiplying both sides of (25) by $\mathbf{v}$, an eigenvector of the K-matrix, we see that when $j/s = \rho$, an eigenvalue of the K-matrix, we must have the following *compatibility condition* ($\mathbf{v}^\top$ denotes the adjoint eigenvector of $\mathbf{v}$):

$$\mathbf{v}^\top \cdot \mathbf{P}_j = 0, \quad \text{for all } \rho \in \mathcal{E}^+. \tag{26}$$

Therefore, if the above compatibility condition is *violated* at some eigenvalue, then we conclude that no solution in the form of a Puiseux series can exist and we have to search for more general solutions which may contain logarithmic terms. Such a more general series will be of the form of a *ψ-series* (cf. [12] for this terminology): a direct generalization of the form (24)

$$\mathbf{x} = \tau^{\mathbf{p}} \left(\mathbf{a} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbf{c}_{ij} \tau^{i/s} (\tau^\rho \log \tau)^{j/s} \right), \tag{27}$$

where ρ is the first K-exponent for which the compatibility condition is not satisfied and s as defined above. The procedure for the calculation of the coefficients in this more general case is the same as before, leading again to the form of the general solution in a suitable neighborhood of the finite-time singularity as a ψ-series.

9. Relation to unstable manifolds

Suppose that (13) is an asymptotic solution of the vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. We set

$$\mathbf{x}(\tau) = \tau^{\mathbf{p}} \mathbf{X}(s), \quad s = \log \tau, \tag{28}$$

and imagine that an equation of the form

$$\mathbf{X}(s) = \text{const.} = \mathbf{a}, \tag{29}$$

regards the coefficient $\mathbf{a}$ of a given balance $(\mathbf{a}, \mathbf{p})$ of the vector field $\mathbf{f}$ as an equilibrium point of the new system given, in terms of the new variables $\mathbf{X}(s) = (X_1(s), \cdots, X_n(s))$, by

$$\begin{aligned} X_i' &= F_i(X_1, \cdots, X_n), \quad i = 1, \cdots, n \\ X_{n+1}' &= q X_{n+1}, \quad X_{n+1} = e^{qs}, \end{aligned} \tag{30}$$

where $(')= d/ds$ and q is the least common multiple of the denominators of the subdominant exponents in the original system. We call the dynamical system (30) the *companion system* of the original vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Thus, the transformation (30) to the companion system associates a different system of this sort to each one of the balances $(\mathbf{a}, \mathbf{p})$ of the original system provided that any given pair of balances has different dominant exponents $\mathbf{p}$).

Consider now the *linearized system*

$$\mathbf{X}' = D\mathbf{F}(\mathbf{a})\mathbf{X}, \tag{31}$$

which is the variational equation of the companion system around the equilibrium point $\mathbf{X} = (\mathbf{a}, 0)$. This is a constant-coefficient, linear system. From the fundamental theorem of such systems, it follows that the general solution passing through the initial condition $\mathbf{X}(0) = \mathbf{X}_0$ is given by

$$\mathbf{X}(s) = e^{sD\mathbf{F}(\mathbf{a})} \mathbf{X}_0. \tag{32}$$

Now we know (cf. [14]) that for any $n \times n$ matrix $\mathbf{A}$:

$$e^{s\mathbf{A}} \mathbf{v} = \sum e^{\lambda_j s} v_j(s), \tag{33}$$

where λ_j are the eigenvalues of $\mathbf{A}$ and v_j are $\mathbb{C}^n$-valued polynomials, the latter being constant if and only if A is semi-simple (diagonalizable). So

$$\mathbf{X} = \sum v_j(\log \tau) \tau^{\lambda_j}, \tag{34}$$

in terms of the τ-time. Hence, using this form of the local solution to the companion system, we can express the local solution to the original system around its finite-time singularity in the form

$$\mathbf{x}(\tau) = \sum v_j(\log \tau) \tau^{\mathbf{p}+\lambda_j}, \tag{35}$$

which is precisely the Taylor estimate leading eventually to the *psi*-series representation we arrived at in previous Sections. We therefore reach the interesting conclusion that around the equilibrium point $(\mathbf{0}, q)$, origin of the companion system (30), the eigenvalues of that system are simply $(\mathbf{p}, q)$; that is, the dominant exponents of the asymptotic solution of the original system together with the number q characterizing the subdominant part of the vector field. Moreover, around the equilibrium point $(\mathbf{a}, q)$ of the companion system (30), the eigenvalues $\lambda_1, \cdots, \lambda_j, q$ of the companion system are $(\rho_1, \cdots, \rho_n, q)$; that is, they are precisely the K-exponents of the original system together with the subdominant number q.

We have an interpretation of the results of the previous Sections concerning the local behaviour of the original system around its finite-time singularities from a dynamical systems perspective. Using the companion transformation, a local analysis of the companion system $\mathbf{X}' = \mathbf{F}(\mathbf{X})$ *around its equilibrium points* (with $s \to -\infty$ necessarily) will provide the local analysis of the solutions of all possible balances of the original system *around its singularities* ($\tau = e^{qs} \to 0$ since $q > 0$ always, for an acceptable decomposition).

Note that, since we need $s \to -\infty$, we are only interested in the unstable manifold (eigenvalues with positive real parts) of the equilibrium points of the companion system. Therefore, the negative K-exponents (corresponding to the stable manifold of the companion system) are not connected to the behaviour of solutions of the original system at the finite-time singularity, but are associated with its behaviour as $\tau \to \infty$.

For a discussion of the companion transformation in connection with integrability and complex dynamics, see [11] and Refs. therein.

10. The method of asymptotic splittings

The results in the previous subsections suggest a general procedure to uncover the nature of singularities by constructing series expansion representations of particular or general solutions of dynamical systems in suitable neighborhoods of their finite-time singularities. This method consists of building splittings of vector fields that are valid asymptotically and trace the dominant behaviour of the vector field near the singularity. A resulting series expansion connected to a particular dominant balance helps to decide whether or not the arrived solution is a general one and to spot the exact positions of the arbitrary constants as well as their role in deciding about the nature of the time singularity.

The method we suggest below is analogous to the so-called ARS procedure connected with the Painlevé and integrability properties of dynamical systems [13], but here the whole approach and viewpoint are completely different. We are not concerned with notions of integrability but solely with the problem of the nature of finite-time singularities. Also, our systems are real-valued with a real time variable.

To apply this *method of asymptotic splittings* to a particular dynamical system in an effort to discover the nature of its time singularities, we must follow this recipe:

(i) Write the system of equations in the form of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} = (x_1, \cdots, x_n)$, and identify the vector field $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_n(\mathbf{x}))$.

(ii) Find all the different weight-homogeneous decompositions of the system; that is, the splittings of the form

$$\mathbf{f} = \mathbf{f}^{(0)} + \mathbf{f}^{(1)} + \cdots + \mathbf{f}^{(k)},$$

and choose one of these splittings to start the procedure.

(iii) Substitute the scale-invariant solution

$$\mathbf{x}^{(0)}(\tau) = \mathbf{a}\tau^{\mathbf{p}},$$

into the equation $\dot{\mathbf{x}} = \mathbf{f}^{(0)}$. Study the resulting algebraic systems, and find all dominant balances $(\mathbf{a}, \mathbf{p})$ together with their orders.

(iv) Identify the non-dominant exponents, that is the positive numbers $q^{(j)}$, $j = 1, \cdots, k$, such that

$$\tau^{q^{(j)}} \sim \frac{\mathbf{f}^{\,\mathrm{sub},(j)}(\tau^{\mathbf{p}})}{\tau^{\mathbf{p-1}}} \to 0.$$

(v) Construct the K-matrix $\mathcal{K}$:

$$\mathbf{f}^{(0)} \to D\mathbf{f}^{(0)} \to D\mathbf{f}^{(0)}(\mathbf{a}) \to D\mathbf{f}^{(0)}(\mathbf{a}) - \operatorname{diag}\mathbf{p}.$$

(vi) Compute the spectrum of $\mathcal{K}$,

$$\mathrm{spec}(\mathcal{K}) = (-1, \rho_2, \cdots, \rho_n).$$

Is $\mathcal{K}$ semi-simple? Are the balances hyperbolic?

(vii) Find the eigenvectors $\mathbf{v}^{(i)}$ of $\mathcal{K}$.

(viii) Identify s as the multiplicative inverse of the least common multiple of all the subdominant exponents and positive K-exponents.

(ix) Substitute the Puiseux series

$$x_i = \sum_{j=0}^{\infty} c_{ji}\, \tau^{p_i + \frac{j}{s}}$$

into the original system.

(x) Identify the polynomials $\mathbf{P}_j$ and solve for the final recursion relations which give the unknown coefficients $\mathbf{c}_j$.

(xi) Check the compatibility conditions at the K-exponents,

$$\mathbf{v}_\rho^\top \cdot \mathbf{P}_\rho = 0, \quad \text{for each eigenvalue } \rho.$$

(xii) If the Puiseux series is valid, then the method is concluded for this particular splitting. Otherwise, if compatibility conditions are violated at the eigenvalue ρ^*, restart from step ix by substituting the logarithmic series (27).

(xiii) Get coefficient at order ρ^*. Write down the final expansion with terms up to order ρ^*.

(xiv) Verify that compatibility at ρ^* is now satisfied.

(xv) Repeat whole procedure for each of the other possible decompositions.

Note that although the whole spectrum of possible behaviours of the system near a time singularity is concluded once we find valid series expansions corresponding to each balance in each particular decomposition, an additional analysis of the phase space of the companion systems corresponding to each one of the balances of the original system may lead to valuable insights as to the geometric structure of the phase space – *how the orbits behave* – in the neighborhood of the finite-time singularity.

Following the above steps even up to that of calculating a dominant balance in one particular decomposition, can be very useful since it gives you one particular possible asymptotic behaviour of the system near the time singularity. In this respect, the whole method expounded here is truly generic since it helps to decide the generality of any behaviour found in an exact solution – that is, how many arbitrary constants there are in the final solution that shares that behaviour (particular or general solution). It is rare that a Puiseux series is inadequate to describe the dynamics (semi-simplicity of K), but in such uncommon cases one must resort to the more complex logarithmic solutions.

11. A worked example

As a concrete example of the above analysis, consider the homogeneous and isotropic FRW cosmological equations in general relativity for a perfect fluid of pressure $p(t)$, density $\rho(t)$, and equation of state $p = w\rho$, where w is a constant. They read

$$\frac{\ddot{a}}{a} = -\frac{\rho}{6}(1+3w), \tag{36}$$

$$\dot{\rho} + 3H\rho(1+w) = 0, \tag{37}$$

$$3H^2 = \rho - \frac{k}{a^2}, \tag{38}$$

where $a(t)$ is the scale factor, $H(t)$ the Hubble expansion rate and k the constant spatial curvature. Setting $a = x, \dot{x} = y, \rho = z$, this system reads

$$\begin{aligned} \dot{x} &= y, & (39) \\ \dot{y} &= axz, & (40) \\ \dot{z} &= byz/x, & (41) \end{aligned}$$

with

$$a = -(1+3w)/6, \quad b = -3(1+w), \tag{42}$$

and is subject to the integral constraint (the Friedmann equation)

$$3\frac{y^2}{x^2} + \frac{k}{x^2} = z. \tag{43}$$

This system is weight-homogeneous. The unique balance is determined by

$$\mathbf{a} = (\alpha, \beta, \gamma) = \left(\alpha, p\alpha, \frac{2(2+b)}{ab^2}\right), \quad \mathbf{p} = (p, q, r) = \left(-\frac{2}{b}, -\frac{2}{b} - 1, -2\right), \tag{44}$$

and the parameters a, b depend on w via Eq. (42). Since the field is weight-homogeneous (i.e., $\mathbf{f} = \mathbf{f}^{(0)}$), this balance corresponds to an exact, scale-invariant solution of the original system.

Note that there is one arbitrary coefficient, and so we expect that one of the K-exponents will be zero. For the vector field $\mathbf{f} = (y, axz, byz/x)$ the associated K-matrix, $K = D\mathbf{f}(\mathbf{a}) - \mathrm{diag}\mathbf{p}$, has characteristic equation

$$r^3 + (2p-1)r^2 + 2(p-1)r = 0, \tag{45}$$

hence the K-exponents are

$$r = -1,\ 0,\ 2(1-p). \tag{46}$$

We can use the integral constraint (43) to get a value for the coefficient α. Balancing the terms in (43) leads to

$$\alpha = \pm\sqrt{\frac{3\beta^2 ab^2}{2(2+b)}}. \tag{47}$$

In the case when all components of the vector $\mathbf{a}$ are real, the solution $X = (x, y, z) \equiv (a, \dot{a}, \rho)$ of the dynamical system experiences a finite-time singularity. When $1 - p > 0$, that is when either $w < -1$ or $w > -1/3$, solutions are general; while, in the range $-1 < w < -1/3$, we have only behaviours corresponding to particular solutions of the system. Note that when $w = 0$, we have $b = -3$ and we find a behaviour similar to that of dust, but when $w = 1/3$, $b = -4$ the behaviour is that of the standard radiation models. Further, calculating the recursion relations to compute the coefficients of the series expansion term by term, we find the following asymptotic solution for radiation,

$$\begin{aligned} x &= \alpha\tau^{1/2} - \frac{\alpha}{3}\tau^{3/2} - \frac{\alpha}{18}c_{31}\tau^{5/2} + \cdots \\ z &= \frac{3}{4}\tau^{-2} + c_{31}\tau^{-1} + c_{31}^2 + \frac{8}{9}c_{31}^3\tau + \cdots, \end{aligned}$$

while the dust-dominated expansion is found to be

$$\begin{aligned} x &= \alpha\tau^{2/3} - \frac{\alpha}{4}\tau^{4/3} + \cdots \\ z &= \frac{4}{3}\tau^{-2} + c_{32}\tau^{-4/3} + \cdots \end{aligned}$$

The method that we have described in detail here provides a toolkit for the investigation of the general form of a range of finite-time singularities in general relativistic cosmologies. In particular, the 'sudden' singularities introduced by one of us [15], in which $a, \dot{a}$, and ρ remain finite but $\ddot{a}$ and $p \to \infty$ at finite time in situations where no functional relationship is assumed between p and ρ, have been widely studied [16]. Such situations, allow sudden singularities to develop at finite time without violating the strong-energy condition of general relativity and are require less severe conditions than future 'big rip' singularities [17] with $w < -1$. Elsewhere, we will report on the results of applying these methods to determine the general behaviour on approach to these singularities in general relativity and in higher-order gravity theories.

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