AXIAL ANOMALY AND ATIYAH-SINGER THEOREM

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A B S T R A C T

The fermion integration for gauge theories requires the study of the Euclidean c number Dirac equation in arbitrary external gauge fields. Winding of A_u leads to a trapping for the Euclidean ψ 's. We complete our previous discussion by relating this problem to the Atiyah-Singer index theory. For gauge theories a suitably modified axial anomaly relation provides a new proof of this theorem.

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1. INTRODUCTION

In a recent paper¹⁾ we studied the c-number Dirac equation in an arbitrary classical Abelian or non-Abelian gauge field. The motivation for this investigation was to obtain a structural understanding of functional integration in the presence of fermion fields and non-trivial winding in the gauge field A_μ (see Ref. 2) to go beyond quasiclassical methods³⁾).

In the course of proving the statement of Ref. 1 on the connection between the zero energy space of the Euclidean Dirac equation^{*)}

$$i\gamma_\mu D_\mu \psi = E\psi \quad , \quad D_\mu = \partial_\mu - ie A_\mu \quad (1)$$

and the winding (Chern) number of the gauge field we became acquainted with a very deep theorem from algebraic geometry: the Atiyah-Singer (A-S.) index theorem⁵⁾. The present paper is dedicated to an explanation of the role this theorem plays in our problem, and to proving the theorem by simple means in some special cases.

As a simple illustration let us consider the two models QED₂ and QCD₄. The theorem relates the difference γ of the dimension of positive and negative chirality $E = 0$ eigenspaces ($\gamma_5 H_\pm = \pm H_\pm$) to the Chern number n of the gauge field

$$\gamma \equiv \dim H_- - \dim H_+ = a \cdot n \quad (2)$$

Here the integer constant a depends on the number of indices of ψ , i.e. on the representation of quarks. γ represents the analytic index of the Dirac operator (1).

In order to avoid confusion through incorrect mathematics, we always view the Dirac equation (1) in compactified Euclidean space as being the projection from the corresponding equation on S^2 and S^4 , respectively. This one-point compactification is consistent with the physical situation if we demand the gauge field part of the action to be separately finite:

$$\int d^D x \, F_{\mu\nu} F_{\mu\nu} < \infty \quad (3a)$$

As was pointed out⁶⁾, this requirement is met if the A_μ are asymptotically "flat":

$$A_\mu \xrightarrow{x \rightarrow \infty} \frac{i}{e} g^{-1} \partial_\mu g \quad g = \text{gauge transformation} \quad (3b)$$

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*) In the SU(n) case our fields are matrix-valued.

This compactified Euclidean formalism enforces an inner product for the spinors which is different from the usual one. As a result the $E = 0$ eigenstates in QED_2 will be normalizable, whereas in the usual norm some of them lead to logarithmic divergencies for long distances¹⁾.

In Section 2 we will present a direct derivation of the A.S. index γ for the Dirac equation in gauge theories in the case of QED_2 .

In Section 3 we mention some of the deeper mathematical aspects of the A.S. index theorem. The main purpose of that section is to point out some important cohomology concepts and the relevant mathematical references. The other sections do not depend in any essential way on Section 3.

In Section 4 we give a field theoretical derivation of the spinor index theorem⁴⁾. It is based on the modified axial current anomaly relation

$$\partial_\mu j_{\mu 5} = \frac{e^2}{16\pi^2} \epsilon_{\mu\nu\alpha\beta} \text{tr} F_{\mu\nu} F_{\alpha\beta} + 2 \frac{2}{1+x^2} \sum_i u_{0i}^+ \gamma_5 u_{0i} \quad (4)$$

$$i \gamma_\mu D_\mu u_{0i} = 0 \quad (5a)$$

$$\text{and} \quad \int \frac{2}{1+x^2} u_{0i}^+ u_{0i} d^D x = 1 \quad (5b)$$

For the $D = 2$ model one only has to change the second term in (4) to $(e/2\pi) \epsilon_{\mu\nu} F_{\mu\nu}$; everything else remains unchanged. As already pointed out, the usual u_{0i} norm may diverge logarithmically; however, with (5b) this case is as normalizable as the $D = 4$ model.

The index theorem follows from (4) by integration over all space, provided surface terms arising from the total derivative on the left-hand side are absent. Actually it is easier to prove the corresponding statement on the hypersphere, so the hyperspherical formulation is used in our proofs. Equation (4) is similar to Coleman's result⁷⁾. The difference is our direct formulation in the massless theory and the factor $2/(1+x^2)$ in front of the last term. In order to avoid confusion, we should point out that it is only the external field problem which for winding fields to the modified axial current anomaly relation. The relation is, thanks to the appearance of functional determinants in addition to classical Green functions, the functional formulation of the full quantum field theory, not directly applicable here.

Finally, two appendices provide the technical details necessary for the derivation of the results of Sections 2 and 4.

2. COMPUTATION OF INDEX γ IN QED₂

Consider for the QED₂ model the special gauge field [vortex⁸⁾]

$$A_\mu(x) = \frac{x^2}{1+x^2} \frac{i}{e} \bar{g}^{-1} \partial_\mu g = \frac{1}{e} \frac{1}{1+x^2} \varepsilon_{1\mu} x_1 \quad (6a)$$

with
$$g = e^{-iy} = \frac{x_1 - ix_2}{\sqrt{x_1^2 + x_2^2}} \quad (6b)$$

The Euclidean Dirac equation has now the form

$$i\gamma_\mu D_\mu \psi \equiv i \begin{pmatrix} 0 & , & i\partial_1 + \partial_2 - i\frac{x_1 - ix_2}{1+x^2} \\ -i\partial_1 + \partial_2 - i\frac{x_1 + ix_2}{1+x^2} & , & 0 \end{pmatrix} \psi = 0 \quad (7)$$

and leads to the single-value decreasing solution

$$u_0 = c \begin{pmatrix} (1+x^2)^{-1/2} \\ 0 \end{pmatrix} \quad (8)$$

It was pointed out¹⁾ that this solution, whose ordinary norm has a logarithmic divergence, is projected onto a normalizable S²-hypersphere solution. The method is that of stereographic mapping⁹⁾.

The formal zero energy solution (8) becomes normalizable and its $\hat{\psi}$ representative is

$$\hat{u}_0 = \frac{1}{2} \frac{1}{\sqrt{1+r_3}} \begin{pmatrix} 1+r_3 \\ r_1 + ir_2 \end{pmatrix} \quad (9)$$

One easily checks that u_0 is the only $E = 0$ eigenstate of the hyperspherical Dirac operator obtained from (7) by replacement of l_{ab} with L_{ab}^{*}). As a generalization, let us consider the winding n -class representation

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*) See Appendix A.

$$\begin{aligned} A_{\mu}^{(n)} &= \frac{x^2}{1+x^2} i g^{-1} \partial_{\mu} g \\ &= n \frac{1}{1+x^2} \varepsilon_{\lambda\mu} x_{\lambda} \end{aligned} \quad (10)$$

The formal solutions of the $E = 0$ Dirac equation are

$$\begin{aligned} u_{0i}^{(n)}(x) &= c_i \left(g^{-1} \sqrt{x^2} \right)^{i-1} \binom{(1+x^2)^{-n/4}}{0} \\ i &= 1, \dots, n \end{aligned} \quad (11)$$

The corresponding u_{0i} are easily demonstrated to exhaust all zero energy solutions of the corresponding hyperspherical Dirac operator. We therefore obtain (omitting the mapping symbol for the maps into the hypersphere):

$$\begin{aligned} \dim H_- &= \dim \ker i \mathcal{D} / \mathcal{D} = - = n \\ \dim H_+ &= \dim \ker i \mathcal{D} / \mathcal{D} = + = 0 \end{aligned} \quad (12)$$

and therefore (2).

For an arbitrary field A_{μ} with the winding number n given by (A.11) one gets all formal solutions of the Dirac equation by multiplication of (11) with the factor

$$\begin{aligned} \exp \left[-e \int d^2 z \mathcal{D}(x-z) (i \partial_{\mu} A_{\mu} - F_{12})(z) \right. \\ \left. + \frac{e}{4\pi} \ln \frac{1+x^2}{2} \cdot \int d^2 z F_{12}(z) \right] \end{aligned} \quad (13)$$

which leaves the asymptotic behaviour unchanged. Thus, the index theorem (2) remains true in the form (12).

3. REMARKS ON THE ATIYAH-SINGER THEOREM

The A.S. index theorem⁴⁾ is the outgrowth of a long development in algebraic geometry; the Gauss-Bonnet theorem¹⁰⁾, the Hodge theory¹¹⁾ of harmonic differential forms on manifolds, the Hirzebruch signature theorem⁵⁾ and the general Hirzebruch-Riemann-Roch theory⁵⁾ on the index of algebraic varieties are some milestones in the historical development of concepts which culminated in the general A.S. index theorem.

The general idea is to relate analytic properties of elliptic differential operators to topological invariants of the underlying manifold and the vector-bundle structure affiliated with the differential operator.

Recall the definition of an general linear differential operator of r^{th} degree in Euclidean space R^n . It is defined by an expression

$$Df = \sum_{|\alpha| \leq r} g_\alpha(x) D^\alpha f \quad (14a)$$

Here α is a multi-index $(\alpha_1, \dots, \alpha_n)$:

$$|\alpha| = \sum \alpha_i, \quad D^\alpha = (-i)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \quad (14b)$$

is a column vector of length k and the g_i are $\ell \times k$ matrices of smooth functions. It is customary to assign a "symbol" to such a differential operator:

$$\sigma(D, y) = \sum_{|\alpha|=r} g_\alpha(x) y^\alpha \quad (15)$$

Here $y = (y_1, \dots, y_n)$; the summation only extends over the highest order terms. σ defines a homomorphism of $R^\ell \rightarrow R^k$ and the operator D is called elliptic if for $y \neq 0$ this homomorphism turns into an isomorphism, i.e. if $\ell = k$ and $\sigma^{-1}(D, y \neq 0)$ exists. Since the most important operators of mathematical physics are at most of 2^{nd} degree, this property is easily checked in concrete cases. Interesting elliptic operators are, however, in two aspects more complicated. On the other hand, the manifold may unlike R^n not be parametrizable in terms of one system of coordinates. An example is the sphere of the compactified R^n . On the other hand, the objects on which D acts may have a complicated internal structure. For example, they may be sections in a vector bundle which is associated with a principle $U(q)$, $SU(q)$ gauge bundle. In that case a coordinate description does not only have the manifold patches but also may have gauge patches. Accordingly, the classical framework of elliptic differential equations has to be generalized to:

$$D : \Gamma(\xi) \longrightarrow \Gamma(\eta) \quad (16)$$

where $\Gamma(\xi)$, $\Gamma(\eta)$ are the spaces of sections in vector bundles⁵⁾. In local coordinates this just looks like the classical situation (14); however, the mathematical concepts imply automatically certain consistency requirement in the coordinate patches between two such situations. A Riemann metric on the vector bundles ξ

and η leads to the concept of the adjoint D^* and the analytic index γ of D is defined as follows:

$$\gamma(D) = \dim \ker D - \dim \ker D^* \quad (17)$$

i.e. a difference dimension between (zero energy) null spaces.

The corresponding topological invariant for such a situation is much harder to explain. A famous example may illustrate its meaning. Consider a two-dimensional surface S as our basic manifold. For such a situation we know the classical Gauss-Bonnet theorem¹⁰⁾:

$$\chi(S) = \frac{1}{2\pi} \int_S K \, dS \quad (18)$$

Here $\chi(S)$ is the Euler Poincare index of S

$$\chi(S) = 1 - b_1 + b_2 \quad (19)$$

where the positive integers b_i are the i^{th} Betti numbers of S . This index of S is easily indentified with the index of a certain elliptic differential operator on differential forms⁵⁾:

$$D = d + \delta \quad (20)$$

where

- d = exterior derivation,
- δ = its Hodge adjoint,

Defining as ξ and η the vector bundle of even and odd degree differential forms over S , one can see that in this case

$$\gamma(D) = \chi(S) \quad (21)$$

The Gauss-Bonnet theorem (18) tells us that the index is also equal to the total integral over a two form namely the K curvature form. Looking again at the left-hand side we see that the dependence on metric, curvature, etc., dropped out; the result only depends on the topology of the manifold. In such a situation one should express the right-hand side as the evaluation of a cohomology class (physically: a flux over all space of a certain density (modulo densities which are themselves total derivatives) which does not depend on metric, curvature, etc. This

idea underlies the general form of the right-hand side of the A.S. theorem. It can be expressed in terms of two well known cohomology classes in algebraic geometry: the Todd class $td(x)$ of the tangent bundle and a relative Chern cohomology class having to do with the way the two bundles ξ and η communicate via the symbol σ of the elliptic differential operator. The general A.S. index theorem states

$$\gamma(D) = \{ch(D) td(X)\} [X] \quad (22)$$

where the bracket denotes evaluation of the product of cohomologies over the entire manifold X .

The situation simplifies if one considers the specialization to gauge theories. In that case one deals with two groups, the $SO(n)$ acting on the tangent space and the $U(q)$ or $SU(q)$, respectively, acting in the vector bundle. Since the objects on which the differential operator D acts are spinor wave functions in local coordinates, it is desirable to pass from $SO(n)$ to the two-fold covering groups $spin(n)$ and to consider Dirac wave functions as sections in the $U(q)$ gauge spin bundle which is associated with the spin corresponding principle bundles. Then the product $ch(D) td(X)$ goes over into the generalized Hirzebruch $\hat{A}$ genus⁵⁾

$$\gamma(D) = \hat{A}(X, \eta, \gamma) \quad (23)$$

where η , the gauge group bundle, is related to the gauge aspects of the Dirac-wave function. It is easy to see that $\gamma = 0$ for $\dim X = \text{odd}$; this is also true for the general theorem (22). Let us look at two extreme cases:

a) $X = S^{2n}$

In this case the Hirzebruch $\hat{A}$ genus goes over into the Chern class evaluated over all space X . i.e. the Chern number

$$\gamma(D) = ch_n [S^{2n}] \quad (24)$$

Since the index is an integer, we obtain an "integrality" theorem. In physical language the gauge winding number, which by its mathematical definition⁵⁾ could have been a rational number, is in fact an integer if the Euclidean space which underlies the functional integral for gauge theories can be one-point compactified. There is no change by going from $U(q)$ to $SU(q)$, since the $U(1)$ part does not contribute in (24).

b) Consider a situation of a Dirac spinor in general relativity¹²⁾, where there is no internal gauge bundle η but just the tangent bundle with its affiliated spin bundle. In that case we have:

$$\gamma(D) = \hat{A}(X) \quad (25)$$

i.e. the index of the Dirac operator measures the Hirzebruch genus of the manifold X .

In the next section we will give a proof based on the axial anomaly relation, which will be mainly geared towards the case a. It is conceivable, however, that similar techniques may work for the more general formula (23) which corresponds in physics terminology to a Dirac particle in an external gravitational field as well as an external Yang-Mills field.

4. THE CONNECTION BETWEEN THE AXIAL VECTOR CURRENT ANOMALY AND THE INDEX THEOREM IN GAUGE THEORIES

The proof of the index theorem (2) on the basis of the axial vector current anomaly falls in two steps. First, one has to prove the anomaly in the form of Eq. (4). Secondly, one must verify that the left-hand side of the equation indeed vanishes after the integration over all space.

The proof given below of Eq. (4) and the corresponding relation in QED₂ is rather similar, at least in spirit, to the original treatment of the axial vector current anomaly in QED₄ due to Schwinger¹³⁾ (see also Ref. 14) and does like that not rely on arguments from ordinary perturbation theory¹⁵⁾.

The eigenvalue equation for the Dirac operator on the hypersphere is, according to (A.8) and (A.12), in the case of QED₂:

$$(S_{ab} L_{ab} + 1) \hat{u}_i(r) = \lambda_i \hat{u}_i(r) \quad (26)$$

For QCD₄ the term 1 in the bracket is replaced by 2; all other changes are the obvious ones.

The Dirac operator projected onto a hypersphere will, since the hypersphere is compact, have a discrete spectrum. This is in fact the very reason why the hypersphere is suited for our considerations since it ensures that the zero eigenvalues of the Dirac operator are separated from the other eigenvalues by a gap in the pseudoenergy.

The eigenstates $\hat{u}_i$ in Eq. (26) are complete, so they can be normalized to fulfill:

$$\sum_i \hat{u}_i(r) \hat{u}_i^+(r') = \delta(r-r') \quad (27a)$$

When the states are transformed back to Euclidean space by (A.13) and (A.22) this becomes

$$\sum_i u_i(x) u_i^\dagger(y) = \frac{1+x^2}{2} \delta(x-y) \quad (27b)$$

whereas the eigenvalue equation (26) combines with (A.7), (A.21) and (A.12a) to yield

$$i \not{x} \not{D}_\mu u_i(x) = \frac{2}{1+x^2} \lambda_i u_i(x) \quad (28)$$

In the presence of zero modes, we can only define the Green's function of the Dirac operator in the part of the Hilbert space orthogonal to the null space; it becomes

$$G'(x, y; A) = \sum' \frac{u_i(x) u_i^\dagger(y)}{\lambda_i} \quad (29)$$

where the prime means that the summation runs over all eigenfunctions of the Dirac operator having non-zero eigenvalues. This Green's function fulfills, because of (27) and (28):

$$i \not{x} \not{D}_\mu G'(x, y; A) = \delta(x-y) - \frac{2}{1+x^2} \sum_i u_{0i}(x) u_{0i}^\dagger(y) \quad (30)$$

where the sum in the last term on the right-hand side runs over all zero-modes.

The induced axial current to be considered is in the case of QED₂:

$$j_{\mu 5}(x) = \lim_{\epsilon \rightarrow 0} \text{tr} \not{x} \not{\gamma}_5 G'(x+\frac{\epsilon}{2}, x-\frac{\epsilon}{2}; A) \exp -ie \int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} d\xi_\mu A_\mu \quad (31)$$

tr = trace in γ and (for nonabelian case) colour space.

The only change in the case of QCD₄ is that the exponential has to be ordered in such a way that the total expression is gauge invariant. In both cases one must average over $\pm \epsilon$ in order to remove the singularity in ϵ before $\epsilon \rightarrow 0$.

The divergence of Eq. (31) is evaluated by means of Eq. (30), and one first gets

$$\partial_\mu j_{\mu 5}(x) = \lim_{\epsilon \rightarrow 0} \text{tr} \not{x} \not{\gamma}_5 G'(x+\frac{\epsilon}{2}, x-\frac{\epsilon}{2}; A) e \epsilon_\lambda F_{\lambda 2}(x) + 2 \frac{2}{1+x^2} \sum_i u_{0i}^\dagger(x) \not{\gamma}_5 u_{0i}(x) \quad (32)$$

as a result for both QED₂ and QCD₄. The further developments depend on the singularity structure of the Green's function, which we determine in Appendix B. The outcome is, as expected, independent of the presence of zero modes and is given by Eq. (B.6) for QED₂ and by Eq. (B.9) for QCD₄, respectively. Thus, the first term on the right-hand side of Eq. (34) becomes in the two cases:

$$\frac{e}{2\pi} \varepsilon_{\lambda\mu} F_{\lambda\mu}(x) \quad (33)$$

and [by means of the identity (B.10)]:

$$\frac{e^2}{16\pi^2} \varepsilon_{\mu\nu\alpha\beta} \text{tr} F_{\mu\nu} F_{\alpha\beta} \quad (34)$$

i.e. the familiar results from the $n = 0$ sectors. Note that the non-Abelian character of the field strengths in Eq. (34) causes no trouble for the derivation.

Having thus found the divergence of the induced current, we proceed to the second half of the proof of Eq. (2). It is most convenient to carry that out on the hypersphere.

In the case of QED₂, the axial vector current on the hypersphere J_a^5 is obtained by a relation similar to (A.9), whereas in the case of QCD₄ an extra factor $[(1+x^2)/2]^2$ has to be applied upon the expression projected according to (A.9) in order to get $\hat{J}_a^5$. From (32) to (34) it follows that the projected currents fulfill

$$\mathcal{L}_{ab} \hat{J}_b^5 = -i \hat{J}_a^5 - i r_a \frac{e}{3\pi} \varepsilon_{bcd} \hat{F}_{bcd} + 2i r_a \sum_i u_{ai}^+ \Gamma_b \gamma_b u_{ai} \quad (35)$$

for QED₂, and

$$\mathcal{L}_{ab} \hat{J}_b^5 = -i \hat{J}_a^5 - i r_a \frac{e^2}{16\pi^2} \text{tr} \hat{F}_{bcd} \hat{F}_{bcd} + 2i r_a \sum_i u_{ai}^+ \Gamma_b \gamma_b u_{ai} \quad (36)$$

for QCD₄. The symbols occurring here are all explained in Appendix A. These two equations could, of course, also have been proved directly in the hyperspherical formalism, but the proof would then have been less simple than the one given above, and the connection with the customary results on the axial vector current¹³⁻¹⁵⁾ less obvious.

It follows in both cases from the construction that J_a^5 fulfils:

$$r_a \hat{J}_a^5 = 0 \quad (37)$$

Also u_{0i} is an eigenfunction of $\Gamma_a r_a$ if and only if the corresponding spinor in Euclidean space is an eigenfunction of γ_5 , and the eigenvalues have opposite signs. Thus the index theorem (2) follows from Eqs. (35) and (36) if we can show:

$$\int d\Omega r_a l_{ab} \hat{J}_b^5 = 0 \quad (38)$$

This condition can be transformed to a convenient form, if we apply $\ell_{ab} \ell_{ab}$ to both sides of Eq. (37); the outcome is the relation

$$r_c l_{ab} l_{ab} \hat{J}_c^5 = 4i r_a l_{ab} \hat{J}_b^5 \quad (39)$$

We now use that the currents $\hat{J}_a^5$ are gauge invariant; this means that they are single valued.

The currents $\hat{J}_a^5$ are by construction also regular. It is thus possible to express the quantities $\ell_{ab} \ell_{ab} \hat{J}_c^5$ in terms of $\hat{J}_c^5$ themselves by means of the Green's functions of the operators $\ell_{ab} \ell_{ab}$, which are known as sums over the usual three-dimensional, or five-dimensional, spherical harmonics, respectively. Doing so and exploiting that one of the spherical harmonics can be chosen to be the vector r_a itself, one finds that Eq. (37) implies

$$\int d\Omega r_c l_{ab} l_{ab} \hat{J}_c^5 = 0 \quad (40)$$

whence Eq. (38) follows by Eq. (39).

The argument clearly fails for the gauge-dependent axial vector symmetry currents, obtained by writing the anomalous terms (33) and (34) as total derivations and transferring them to the left-hand sides of the respective divergence equations, since the projected potential is singular for a non-zero winding number, so arguments based on spherical harmonics cannot be applied.

To summarize the contents of this section, the index theorem (2) has been proved in the form

$$\gamma = \frac{e}{4\pi} \int d^2x \varepsilon_{\lambda\mu} \bar{F}_{\lambda\mu} = \frac{e}{6\pi} \int d\Omega \varepsilon_{abc} \hat{F}_{abc} \quad (41)$$

for QED₂, and

$$\gamma = \frac{e^2}{32\pi^2} \int d^4x \varepsilon_{\mu\nu\alpha\beta} \text{tr} F_{\mu\nu} F_{\alpha\beta} = \frac{e^2}{32\pi^2} \int d\Omega \text{tr} \hat{F}_{abc} \hat{F}_{ab} r_c \quad (42)$$

for QCD_4 , where the trace runs over matrices belonging to the quark representation of the gauge group.

The essential idea in our proof was the use of the gauge-invariant short-distance limit and the existence of an operator (γ_5 in the Dirac case) which anti-commutes with the elliptic differential operator and has eigenvalues ± 1 .

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APPENDIX A

STEREOGRAPHIC PROJECTIONS OF THE DIRAC EQUATION⁹⁾

Mapping from the south pole, we obtain for $D = 2$

$$r_\mu = \frac{2x_\mu}{1+x^2} \quad , \quad r_3 = \frac{1-x^2}{1+x^2} \quad (\text{A.1})$$

such that $r^2 = r_1^2 + r_2^2 + r_3^2 = 1$

The relation between the volume elements is

$$\left(\frac{2}{1+x^2}\right)^2 d^2x = d\Omega \quad (\text{A.2})$$

The ordinary Euclidean $D = 2$ Dirac matrices

$$\begin{aligned} \gamma_1 &= -\sigma_2 = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix} \quad , \quad \gamma_2 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \gamma_5 &= i\sigma_1\sigma_2 = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \quad (\text{A.3})$$

are taken into

$$\Gamma_1 = -i\gamma_1\gamma_5 = \sigma_1 \quad , \quad \Gamma_2 = -i\gamma_2\gamma_5 = \sigma_2 \quad , \quad \Gamma_3 = \sigma_3 \quad (\text{A.4})$$

Using

$$S_{ab} = \frac{1}{4i} [\Gamma_a, \Gamma_b] = \frac{1}{2} \varepsilon_{abc} \sigma_c \quad (\text{latin indices } 1,2,3) \quad (\text{A.5})$$

and

$$L_{ab} = r_a p_b - r_b p_a \quad , \quad p_a = \frac{1}{i} \frac{\partial}{\partial r_a} \quad (\text{A.6})$$

or

$$L_{\mu\nu} = -ix_\mu \partial_\nu + ix_\nu \partial_\mu \quad , \quad L_{3\mu} = -ix_\mu x_\nu \partial_\nu - \frac{i}{2}(1-x^2)\partial_\mu$$

one can check the algebraic relation

$$(S_{ab} L_{ab} + 1) (1 + i\gamma_\mu x_\mu) = \frac{1+x^2}{2} (1 + i\gamma_\mu x_\mu) i\gamma_\lambda \partial_\lambda \quad (\text{A.7a})$$

or

$$\left(\frac{1+x^2}{2}\right)^2 i\gamma_\mu \partial_\mu = \frac{1}{2} (1-i\gamma_\mu x_\mu) (S_{ab} L_{ab} + 1) \quad (A.7b)$$

$$= (1+i\gamma_3 x_3)$$

This relation converts the Dirac operator into the hyperspherical operator

$$i\gamma_\mu \partial_\mu \rightarrow S_{ab} L_{ab} + 1 \quad (A.8)$$

The external potential, the field strength and the winding number defined on the hypersphere are related to the corresponding quantities in Euclidean space by the formulae

$$\hat{A}_\mu = \frac{1+x^2}{2} A_\mu - x_\mu x_\nu A_\nu, \quad \hat{A}_3 = -x_\mu A_\mu \quad (A.9)$$

$$\hat{F}_{123} = \left(\frac{1+x^2}{2}\right)^2 F_{12} \quad (A.10)$$

and

$$n = \frac{e}{2\pi} \int d^3x F_{12}(x) = \frac{e}{2\pi} \int d\Omega \hat{F}_{123}(\Omega) \quad (A.11)$$

Equation (A.7) is maintained in the presence of the external field with (A.8) modified by

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - ie A_\mu \quad (A.12a)$$

$$L_{ab} \rightarrow L_{ab} = L_{ab} - e r_a A_b + e r_b A_a \quad (A.12b)$$

Clearly (A.8) with (A.12) is a Hermitian compact operator with the inner product

$$(\hat{\psi}_1, \hat{\psi}_2)_H = \int \hat{\psi}_1^\dagger \hat{\psi}_2 d\Omega$$

The Hilbert space of the Euclidean compactified operator is generated by states

$$\psi(x) = \frac{2}{1+x^2} \frac{1}{r_2} (1-i\gamma_\mu x_\mu) \hat{\psi}(r) \quad (A.13)$$

with the inner product

$$(\psi, \psi_2) \equiv (\hat{\psi}_1, \hat{\psi}_2)_H = \int \psi_1^+(x) \psi_2(x) \frac{2}{1+x^2} d^4x \quad (\text{A.14})$$

The discussion for the QCD₄ model parallels the previous considerations. The changes are (apart from obvious ones due to the increase of the number of coordinates):

$$d\Omega = \left(\frac{2}{1+x^2}\right)^4 d^4x \quad (\text{A.15})$$

$$\Gamma_\mu = -i\gamma_\mu \gamma_5, \quad \Gamma_5 = -\gamma_5 \quad (\text{A.16})$$

$$i\gamma_\mu \partial_\mu \rightarrow S_{ab} L_{ab} + 2 \quad (\text{A.17})$$

[this relation remains true in the presence of an external field, if we make the replacements corresponding to those carried out in Eq. (A.12)]. The projected field strength is given by

$$\hat{F}_{\mu\nu\lambda} = \frac{1+x^2}{2} [x_\mu F_{\nu\lambda} + x_\nu F_{\lambda\mu} + x_\lambda F_{\mu\nu}] \quad (\text{A.18a})$$

$$\hat{F}_{\mu\nu\delta} = \frac{1+x^2}{2} \left[\frac{1-x^2}{2} F_{\mu\nu} - x_\mu x_\nu F_{\delta\delta} + x_\nu x_\delta F_{\mu\delta} \right] \quad (\text{A.18b})$$

with the dual field strength

$$\hat{\tilde{F}}_{ab} = \frac{1}{6} \varepsilon_{abcde} \hat{F}_{cde} \quad (\text{A.19})$$

and the topological quantum number

$$n = \frac{e^2}{32\pi^2} \int d^4x \varepsilon_{\mu\nu\alpha\beta} \text{tr} F_{\mu\nu} F_{\alpha\beta} = \frac{e^2}{32\pi^2} \int d\Omega \text{tr} \hat{F}_{ab} \hat{\tilde{F}}_{ab} r_c \quad (\text{A.20})$$

The algebraic relation replacing (A.7b) becomes

$$\left(\frac{1+x^2}{2}\right)^4 i\gamma_\mu \partial_\mu = \frac{1}{2} \frac{1+x^2}{2} (1-i\gamma_\mu x_\mu) (S_{ab} L_{ab} + 2) (1+i\gamma_\mu x_\mu) \times \frac{1+x^2}{2} \quad (\text{A.21})$$

leading to the replacement of (A.13) by

$$\psi(x) = \left(\frac{2}{1+x^2} \right)^2 \frac{1}{\sqrt{2}} (1 - i \gamma_5 \gamma_\mu) \hat{\psi}(r) \quad (\text{A.22})$$

APPENDIX B

In the following it is shown how the short-distance singularities of the induced Green's function (29) can be determined and how they lead to the anomalous terms (33) and (34) in the divergence equations of the induced axial vector currents of QED₂ and QCD₄, respectively. It is emphasized that the methods used below, though apparently similar to ordinary perturbation theory¹⁵⁾, make sense for arbitrary external fields (see Ref. 16).

The Green's function in question is first written:

$$G'(x, y; A) = i \gamma_\mu D_\mu G'(x, y; A) \quad (B.1)$$

where the new Green's function G' according to Eq. (32) fulfils

$$(-D^2 - \frac{e}{2} \sigma_{\mu\nu} F_{\mu\nu}) G'(x, y; A) = \delta(x-y) - \frac{2}{i\pi^2} \sum_i u_{0i}(x) u_{0i}^+(y) \quad (B.2)$$

with

$$\sigma_{\mu\nu} = \frac{1}{2i} [\gamma_\mu, \gamma_\nu]$$

$G'(x, y; A)$ is compared with the massless scalar Green's function $D(x-y)$ which fulfils

$$-\square D(x-y) = \delta(x-y) \quad (B.3)$$

Writing

$$G'(x, y; A) = D(x-y) + \tilde{G}'(x, y; A) \quad (B.4)$$

we get the following differential equation for $\tilde{G}'(x, y; A)$:

$$-\square \tilde{G}'(x, y; A) = \frac{e}{2} \sigma_{\mu\nu} F_{\mu\nu}(x) \tilde{G}'(x, y; A) - \frac{2}{i\pi^2} \sum_i u_{0i}(x) u_{0i}^+(y) + \dots \quad (B.5)$$

where terms which are not of interest for our purposes have been left out. Equations (B.4) and (B.5) show that $\hat{G}'$ for $x \rightarrow y$ has to be less singular than G' itself, i.e. than D . The presence of the last term on the right-hand side of (B.5) does not change this conclusion, since it is regular, provided there are only a finite number of zero modes, as is always assumed here.

For the two-dimensional case, it is sufficient for the derivation of the axial anomaly to isolate the leading singularity, which according to the argument given above coincides with the zero-field singularity. Since the two-dimensional scalar propagator D is

$$D(x-y) = - \frac{1}{4\pi} \ln(x-y)^2$$

the result is

$$G'(x, y; A) \approx - \frac{1}{4\pi} \frac{2iy(x-y)}{(x-y)^2} \quad (\text{B.6})$$

by means of which the first term on the right-hand side of (32) immediately can be converted to (33).

When we turn to the four-dimensional case, the singularity of the Green's function implying the axial anomaly can be found by integration of the differential equation (B.5):

$$\begin{aligned} \hat{G}'(x, y; A) &\approx \int d^4z \, D(x-z) \frac{e}{2} \sigma_{\mu\nu} F_{\mu\nu}(z) \hat{G}'(z, y; A) \\ &\approx \frac{e}{2} \sigma_{\mu\nu} F_{\mu\nu}\left(\frac{x+y}{2}\right) \int_V d^4z \, D(x-z) D(z-y) \end{aligned} \quad (\text{B.7})$$

Here the terms left out in the transition to the last expression are all regular for $x \rightarrow y$. The integration volume V is finite and contains both x and y . Inserting here

$$D(x-y) = \frac{1}{4\pi^2(x-y)^2}$$

one finds

$$\hat{G}'(x, y; A) \approx - \frac{1}{16\pi^2} \frac{e}{2} \sigma_{\mu\nu} F_{\mu\nu}\left(\frac{x+y}{2}\right) \ln(x-y)^2 \quad (\text{B.8})$$

and hence

$$G'(x, y; A) \approx \dots - \frac{1}{16\pi^2} \frac{2iy(x-y)}{(x-y)^2} \frac{e}{2} \sigma_{\mu\nu} F_{\mu\nu}\left(\frac{x+y}{2}\right) \quad (\text{B.9})$$

In Eq. (B.9) several terms have been omitted which are as singular or more singular than the recorded term, but contain a smaller number of γ -matrices and hence are unable to contribute to the axial anomaly.

Inserting (B.9) into the first term of Eq. (32) and making use of the identity:

$$4 \varepsilon_{\rho\sigma\mu\tau} F_{\mu\lambda} F_{\rho\tau} = \delta_{\lambda\sigma} \varepsilon_{\rho\mu\tau} F_{\rho\mu} F_{\rho\tau} \quad (\text{B.10})$$

one is led to (34).

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