CM-P00061722

Ref. TH.2349-CERN

CLASSICAL KINKS AND THEIR QUANTIZATION IN SUPERCRITICAL
REGGEON FIELD THEORY

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A B S T R A C T

The existence of kink states in supercritical Reggeon Field Theory ($\alpha(0) > \alpha_c$) is related to the expanding disc behaviour previously found in a lattice approximation. The classical solitary waves of RFT with $\alpha(0)-1 = \mu > 0$ are classified, for the case of a one dimensional impact parameter space ($D_t = 1$). We find kink-like solutions corresponding to a range of velocities up to a critical one for which the solution has a phonon-like diffusion law (zero mass).

The general methods of kink quantization are adapted to RFT. By then quantizing the theory around the solution with critical velocity we are able to give the low lying spectrum and to construct the quantum kink states, which in RFT are connected to the vacuum by field operators. Their contribution to the Green's function is found to restore the cluster property by cancelling, at large impact parameter, the constant term provided by the second ground state present in the ordered phase. For impact parameter $|B| < 2\sqrt{\alpha'\mu} \log s$ the Green's function tends to a constant proving therefore that, for $D_t = 1$, the expanding disc behaviour is consistent with s-channel unitarity.

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I - INTRODUCTION

The peculiar nature of the critical phenomenon characteristic of the Reggeon Field Theory (RFT) when the reggeon intercept goes through one, has been recently unraveled by using field quantization techniques apt to identify the phase transition mechanism⁽¹⁾⁽²⁾. A lattice is introduced in impact parameter space, the RFT Hamiltonian is diagonalized at each lattice point and finally an analog spin model is obtained by keeping the two lowest-lying states at each site plus the intersite interactions arising from the kinetic terms in the Lagrangian⁽¹⁾⁽³⁾. It was recognized⁽¹⁾, by using this approach, that contrarily to what happens in more orthodox field theories, the vacuum state does not change structure as the reggeon intercept goes through the critical value. There is however a drastic change in the spectrum of the theory, in the sense that a new energy eigenstate shows up that is degenerate with the old perturbative vacuum in the thermodynamic limit. Even more important, field operators have finite matrix elements between these two degenerate states in this limit, which means that these two degenerate vacua belong to the same Hilbert space. This peculiar feature of RFT can be traced, as we shall see later on, to the non-hermiticity of the Hamiltonian.

The two degenerate vacua of the spin model can be thought of as spatially homogeneous states where all spins are up or down, respectively. It has also been recognized⁽²⁾, always in the context of the spin model, the important role played by spatially inhomogeneous states of a kink-like type in which a dislocation - frontier between spins up and down - propagates with a definite speed. These states give rise to a low-lying continuum with zero energy gap with respect to the ground states, and they are therefore responsible for the peculiar asymptotic behavior of the Green functions, characteristic of a disk expanding with rapidity in impact-parameter space. Nevertheless the spin model, explicitly tailored to put in evidence the collective modes responsible for the phase transition, mistreats badly other degrees of freedom like, for example, higher single site excitations which cannot be neglected far away from the critical point. This is therefore reflected in an unphysical dependence of the disk expansion speed on the infrared cutoff-intersite distance a - in the ordered phase of the theory, when $\alpha(0) > \alpha_{\text{crit}}(0)$. Strictly speaking the disk expansion speed goes to zero as a goes to zero, so the continuum limit cannot be taken in the

spin model, without introducing further degrees of freedom. A possible way out, proposed by Bronzan and Sugar⁽⁴⁾, is to introduce a quartic interaction with a coupling $\lambda' = \lambda^2/\mu$, where λ is the triple-Pomeron coupling and $\mu = \alpha(0) - 1$. At this "magic" value the lowest states at each site become exactly degenerate and the limit $a \rightarrow 0$ is harmless. The authors then construct the box states in the continuum limit, by paralleling the treatment of Ref.(2), and get a physical expansion velocity.

The purpose of the present paper is to propose a more general quantization procedure of RFT based on the observation that the non perturbative effects generated by degrees of freedom neglected in the spin model are already present at the classical level. In fact, we will be able to identify in the continuum limit (and for one transverse space dimension, $D_t = 1$) the classical analogues of the kink-like states of the spin model. The fact that the non perturbative effects can be treated in the semi classical limit follows from the property that the RFT Lagrangian scales with λ as $\frac{1}{\lambda^2} \mathcal{L}(\lambda \Psi, \lambda \bar{\Psi})$. Therefore, as in $\lambda \phi^4$, λ plays the role of a semi-classical expansion parameter, and we expect that as $\lambda \rightarrow 0$ the nonperturbative effects are exhausted by the classical solutions. In our study we shall find that there are several classical candidates, and by quantizing around some of them, we shall be able to construct the low-lying quantum states relevant to the large rapidity, large impact parameter, behaviour of Pomeron Green's functions.

Our method is then similar to the ones recently applied to hermitian field theories - as $\lambda \phi^4$ in two dimensions - but there are some peculiar aspects stemming from the non-hermiticity of RFT and/or its lack of Galilean invariance. In fact, as learnt from the spin model, there is just one Hilbert space in the ordered phase, and the kink states are connected to the vacuum by field operators. Moreover, we find that classical kink solutions exist for any speed v , and that for a critical value of v they have a phonon-like dispersion law (zero mass). The quantum kink states have therefore small energy gap for small momenta and contribute in an essential way to the disc's expansion. The critical speed is just the expansion velocity of the disc-actually a box for $D_t = 1$ - and turns out to be the same as in the simple eikonalization of a Pomeron above one.

This paper is organised as follows : in Section II, we rephrase some results of the spin model and recall mainly the properties of the two degenerate ground states. We also present an heuristic argument leading in an intuitive way to the expanding disk behavior of hadronic scattering, based on a generalized coherent

state formalism. The details of these calculations are relegated to Appendix A because they are not essential for the rest of the paper. In Section III, we discuss the classical kink solutions of RFT. We prove the existence of a large class of kink-like solutions and explore its main properties, like dispersion relation and dependence on the propagation speed. Section IV is devoted to a general discussion of the kink quantization of RFT. We have adapted to our case the well-known methods of kink quantization⁽⁵⁾⁽⁶⁾⁽⁷⁾, adding some further understanding to the treatment of the zero-mode problem in a case in which one is forced - as we are - to use the Hamiltonian path integral formulation instead of the Lagrangian one.

In Section V we discuss the dependence of the procedure on the classical solution around which we expand and we show how one is driven to a specific choice in order to avoid linear (tadpole) terms in the quantum fluctuations, which fixes the disk expansion speed. The quantization of the theory is then performed up to order zero in the triple pomeron coupling, and the RFT spectrum in the ordered phase obtained and discussed. Technical details related to the completeness of our eigenfunctions expansions and the calculation of zero-mode states are discussed in Appendix B. Finally in Section VI we use this spectrum to compute the two-point Green function to find the disk behavior. We then conclude by a discussion of our results, emphasizing the peculiarities of the field theory we considered.

II - PRELIMINARY RESULTS AND SEMICLASSICAL APPROXIMATION

We shall adopt in this paper the notations and conventions used before in connection with the spin model⁽¹⁾. The Gribov fields $\Psi(b,y)$ and $\bar{\Psi}(b,y)$ are defined in a D-dimensional impact parameter space b , the rapidity y playing the role of imaginary time. By rescaling the Gribov fields according to

$$q = i\bar{\Psi} \quad ; \quad p = i\Psi \tag{II-1}$$

the triple pomeron coupling term becomes real, and the Lagrangian can be writ-

ten as

$$\mathcal{L} = \frac{1}{2} q \overleftrightarrow{\partial}_y p - \mu q p + \alpha' \vec{\partial}_b q \cdot \vec{\partial}_b p + \lambda q (q+p) p \quad (\text{II-2})$$

where the mass parameter μ is related to the pomeron intercept by

$$\mu = \alpha(0) - 1 \quad (\text{II-3})$$

It must be kept in mind that q and p are creation and destruction operators, respectively, satisfying the equal-time commutation relations

$$[p(b, y), q(b', y)] = -\delta^{(D)}(b-b') \quad (\text{II-4})$$

the wrong sign of the commutator being of course a reflection of the imaginary nature of the triple pomeron coupling, that is to say, of the non hermiticity of the RFT Hamiltonian. The intercept parameter μ in Eq. (II.2) is in principle the bare one. It can be transformed into the renormalized intercept by adding a suitable mass renormalization counterterm, which is perturbative in λ . Since we shall not be concerned in this paper with renormalization effects, we shall ignore counterterms, and accept that the critical intercept is given by

$$\mu_{\text{crit}} = 0.$$

The RFT Hamiltonian reads, in our notation :

$$H = \int db [\mu q p - \alpha' \vec{\partial}_b q \cdot \vec{\partial}_b p - \lambda q (q+p) p] \quad (\text{II.5})$$

and the two states which play in the spin model⁽¹⁾⁽²⁾ the role of degenerate

vacua in the ordered phase $\mu > 0$ can easily be identified. These states are in fact in one to one correspondence with the stationary points of the classical action corresponding to constant field configuration, i.e., $(q,p) = (0,0)$; $(\frac{\mu}{\lambda}, 0)$ and $(0, \frac{\mu}{\lambda})$. They are obtained by remarking that constant field configurations correspond to zero-momentum components of the Gribov fields, and that the zero momentum part of H is just the RFT Hamiltonian in zero transverse dimensions, whose spectrum is well known⁽⁸⁾⁽⁹⁾, with λ replaced by $\lambda / \sqrt{V}^{(*)}$. The lowest-lying state is the perturbative vacuum $|\phi_0\rangle$, defined by

$$p|\phi_0\rangle = 0 \quad ; \quad \langle \phi_0 | q = 0 \quad (\text{II-6})$$

which, because of the ordering properties of H , is always an exact zero-energy eigenstate. This state does not change its structure at the critical point but, as μ becomes positive, a new state which we shall call $|\psi_0\rangle$ shows up on top of it with an energy gap

$$\Delta_v \simeq e^{-\frac{1}{2}\left(\frac{\mu}{\lambda}\right)^2 V} \quad (\text{II-7})$$

which vanishes exponentially in the thermodynamic limit $V \rightarrow \infty$. This second state is associated with the "shifted pomeron" stationary points, and it is given by

$$|\psi_0\rangle = |\phi_0\rangle - |\phi_1\rangle \quad (\text{II-8})$$

where $|\phi_1\rangle$ approaches, as $\frac{\mu}{\lambda} \rightarrow \infty$, a coherent state

$$|\phi_1\rangle = e^{-\frac{\mu}{\lambda} \int q(b,0) db} |\phi_0\rangle \quad (\text{II-9})$$

such that

$$p(b,0) |\phi_1\rangle = \frac{\mu}{\lambda} |\phi_1\rangle \quad (\text{II-10})$$

(*) V denotes the transverse space volume.

Even though the Hamiltonian H is not hermitian, there exists a metric operator⁽¹⁾

$$\mathcal{M} = (-1)^{\int q(\underline{z}) p(\underline{z}) d\underline{z}} \quad (\text{II-11})$$

such that

$$\mathcal{M} H^\dagger \mathcal{M} = H \quad (\text{II-12})$$

Therefore, right and left eigenstates of H are connected by the metric operator $\mathcal{M}$. It follows then that the perturbative vacuum is self-conjugate, while the dual state of $|\Phi_1\rangle$ is

$$\langle \Phi_1 | = \langle \Phi_0 | e^{-\frac{\mu}{\lambda} \int p(\underline{z}, 0) d\underline{z}} \quad (\text{II-13})$$

such that

$$\langle \Phi_1 | q = \frac{\mu}{\lambda} \langle \Phi_1 | \quad (\text{II-14})$$

These right and left eigenstates of H corresponding to the same energy level are therefore associated with the $(\frac{\mu}{\lambda}, 0)$ and $(0, \frac{\mu}{\lambda})$ fixed points of the classical action.

The fact that a coherent state shows up as a new ground state after the phase transition is a characteristic feature of non-relativistic microscopic theories of Bose-Einstein condensation, such as the theory of superfluidity⁽¹⁰⁾. But in hermitian theories this state has an exponentially vanishing overlap with the old ground state in the thermodynamic limit, so it really corresponds to a different phase of the system. This is precisely what does not happen in RFT, due to the nonhermiticity of H . If we compute the norm of the coherent state $|\Phi_1\rangle$, taking care of the q, p commutation rule (II-4), we get

$$\langle \Phi_1 | \Phi_1 \rangle \simeq e^{-\left(\frac{\mu}{\lambda}\right)^2 V} \quad (\text{II-15})$$

while in a hermitian theory one would obtain the opposite sign in the exponential, thereby giving for a normalized coherent state an exponentially vanishing overlap with the perturbative vacuum $|\Phi_0\rangle$. In our case, however, $|\Phi_1\rangle$ is a zero norm state in the thermodynamic limit while $|\Psi_0\rangle$ has negative norm, $|\Phi_1\rangle$ having a finite overlap with the perturbative vacuum $|\Phi_0\rangle$.

Moreover, from the explicit representation of the states one can immediately get the transition matrix elements of the Gribov fields between the two vacua, given by

$$\langle \phi_1 | q | \phi_0 \rangle = \langle \phi_0 | p | \phi_1 \rangle = \frac{\mu}{\lambda} \quad (\text{II-16})$$

These peculiar features of RFT were already recognized in the spin model. In $\lambda \phi^4$ theory, different shifts of the quantum fields define different disconnected Hilbert spaces, corresponding to different phases of the system. Because of the existence of topological conservation laws, the kink sector is also different from the vacuum sector. The old perturbative vacuum disappears because it becomes unstable. In RFT there are no different disconnected phases: all would-be phases are connected by transition matrix elements of the quantum fields that remain finite in the thermodynamic limit. Even the kink states that we shall discuss later on, which interpolate in impact parameter space between the two vacua $|\phi_0\rangle$ and $|\phi_1\rangle$, will have finite overlap with them. Therefore shifting the fields will be in RFT a matter of convenience and not of absolute necessity because the kink sector will be the same as the vacuum sector.

Let us now discuss the coupling to the external particles. Under the hypothesis of eikonal couplings, the RFT scattering amplitude can be written in the Schrödinger picture as the following matrix element of the evolution operator between coherent states

$$S[f, g; \gamma, B] = \langle \phi_0 | e^{-\int f(b-B) p(b) db} e^{-H\gamma} e^{-\int g(b) q(b) db} | \phi_0 \rangle \quad (\text{II-17})$$

where B is the distance in impact parameter space between the colliding hadrons, which act in this theory as external sources of pomerons. The source distributions $g(b)$ and $f(b)$ are taken to be strongly peaked functions near $b = 0$. The hypothesis of eikonal couplings is by no means restrictive, because S is in general the generating functional for Green functions, which are obtained by functional differentiation with respect to the sources.

It was shown in Ref. (2) that, in the context of the spin model, the Green functions have an expanding disk behavior, which means $S = 1$ for $|B| - \sqrt{Y} > 0$. We shall restrict ourselves from now on to the case of a one-dimensional impact parameter space. In the spin model the expansion velocity $\mathcal{V}$ depends on $\mu - \mu_{\text{crit}}$ and on the intersite distance $a^{(2)}$. It is reasonable that, away from the critical point, the dispersion law which is related to the expansion speed $\mathcal{V}$ cannot be correctly obtained with a method that has neglected other excitations not responsible for the critical phenomenon. We shall present in the next sections a quantization technique which is better tailored to give the complete low-energy spectrum in the ordered phase.

We would like to sketch, however, in the rest of this section, an heuristic argument which shows intuitively the emergence of the expanding disk. The argument is heuristic in the sense that some steps are justified "a posteriori" by the results of the following sections, and all the details of the calculations are given in Appendix A. From Eq. (II-17), it is clear that the S-matrix can be computed by solving the Schrödinger equation of RFT

$$H |\Psi(y)\rangle = - \frac{d}{dy} |\Psi(y)\rangle \quad (\text{II-18})$$

with the initial condition that $|\Psi(0)\rangle$ is the initial coherent state, the S-matrix being then the overlap of $|\Psi(Y)\rangle$ with the final state. We then

write the most general state vector of RFT as a generalized coherent state, in terms of an infinite set of correlation functions, and transform the Schrödinger equation into a set of differential equations for these correlation functions.

Needless to say, the problem still contains all the intricacies of the theory, and the next step is to calculate a solution in the semiclassical approximation. The fact that this approximation is extremely relevant in RFT in the ordered phase is indicated by the fact that the RFT Lagrangian has the following scaling property with respect to the triple pomeron coupling

$$\mathcal{L}(p, q; \lambda) = \frac{1}{\lambda^2} \mathcal{L}(\lambda p, \lambda q; 1) \quad (\text{II-19})$$

which means that the weak coupling limit will be correctly described by the semiclassical approximation. Of course the triple pomeron coupling is dimensionful, so $\lambda = 1$ is to be understood in terms of the unit of length, which is fixed by α' .

We then find that, in the semiclassical approximation, the state vector is given

$$|\Psi(y)\rangle = e^{-\int db g(b,y) q(b) + \text{Correlations}} |\phi_0\rangle \quad (\text{II-19})$$

where the correlations are higher order polynomials in q , and $g(b,y)$ satisfies a nonlinear diffusion equation with initial condition $g(b,0) = g(b)$, the initial source function. This diffusion equation is in fact the classical equation of motion for the field $p_{cl}(b,y)$ when $q_{cl}(b,y)$ is set equal to zero. We discuss then its solutions in Appendix A, where it is shown that $g(b,y)$ propagates in y by expanding from $g(b,0)$ and reaching, for large y , a box-like form that expands with a well defined velocity $v = 2\sqrt{\alpha'\mu}$, i.e.

$$g(b,y) \simeq \frac{\mu}{\lambda} \theta(vy + \delta - |b|) \quad (\text{II-20})$$

where δ is a "phase shift" that depends explicitly on the initial condition.

To the extent that correlations can be neglected, the state vector

$|\Psi(Y)\rangle$ is asymptotically in Y a coherent state, and then it is trivial to see that the S-matrix is given by

$$S(B,Y) = e^{-\frac{\mu}{\lambda} \int db f(b-B) \theta(vY + \delta - |b|)} \quad (\text{II-21})$$

which, due to the fact that $f(b-B)$ is localized around $b=B$, is 1 if $|B| > vY + \delta$. This is just the expanding disk behavior of the scattering matrix. Of course the evolution of the state vector in terms of a single coherent state is only approximate, because the classical solutions that dominate in principle the semiclassical limit should be those of the coupled field equations. This effect is taken care of by the correlation functions in

the state vector, that we have neglected in Eq. (II-21). However, from the more careful analysis that we shall perform in the next sections it will turn out that the low-energy part of the spectrum of RFT -and therefore the large Y behavior- will be really controlled by uncoupled field configurations. We shall therefore justify a "posteriori" the validity of the simple picture presented here.

III - CLASSICAL KINK SOLUTIONS OF RFT.

We shall investigate in this section the existence and properties of a large family of kink solutions to the classical equations of motion of RFT, for the case of a one-dimensional impact parameter space. The equations of motion that follow from the Lagrangian (II-2) are

$$\frac{\partial p}{\partial y} - \mu p + \lambda p^2 + 2\lambda pq - \alpha' \frac{\partial^2 p}{\partial b^2} = 0 \quad (\text{III-1a})$$

$$- \frac{\partial q}{\partial y} - \mu q + \lambda q^2 + 2\lambda pq - \alpha' \frac{\partial^2 q}{\partial b^2} = 0 \quad (\text{III-1b})$$

and we are particularly interested in kink-like solutions which depend only on a linear combination of y and b which we define as

$$p = \mu v y - \sqrt{\frac{\mu}{\alpha'}} b \quad (\text{III-2})$$

where the parameter v is the kink propagation velocity. With this choice of variables we write then the classical fields as

$$p_a(b, y) = \frac{\mu}{\lambda} \varphi(p) \quad ; \quad q_a(b, y) = \frac{\mu}{\lambda} \bar{\varphi}(p) \quad (\text{III-3})$$

and obtain for the fields $\varphi(p)$, $\bar{\varphi}(p)$ the coupled differential equations

$$\ddot{\varphi} - v \dot{\varphi} + \varphi - \varphi^2 - 2\varphi\bar{\varphi} = 0 \quad (\text{III-4a})$$

$$\ddot{\bar{\varphi}} + v \dot{\bar{\varphi}} + \bar{\varphi} - \bar{\varphi}^2 - 2\varphi\bar{\varphi} = 0 \quad (\text{III-4b})$$

The fixed points of these equations corresponding to constant (ρ -independent solutions) are given by

$$(\varphi, \bar{\varphi}) = \begin{cases} (0, 0) \\ (0, 1) \\ (1, 0) \end{cases} \quad (\text{III-5})$$

The first one obviously corresponds to the perturbative vacuum associated with unshifted pomeron states, while the two others correspond to the new vacuum associated with shifted pomeron states, as discussed in the previous section. As it is well known, these fixed points are unstable. They are actually saddle points in the sense that in the $(\varphi, \bar{\varphi})$ plane corresponding to phase space they are attractive in some directions and repulsive in others.

From the general structure of Eqs. (III-4) we can see that

$$\varphi_{\nu}(\rho) = \varphi_{-\nu}(-\rho) \quad ; \quad \bar{\varphi}_{\nu}(\rho) = \bar{\varphi}_{-\nu}(-\rho) \quad (\text{III-6})$$

which means that we can restrict our attention only to positive propagation velocities ; solutions with negative ν being then trivially obtained by means of these reflection properties. Moreover, it also follows from Eqs. (III-4) that if $\{\varphi_{\nu}(\rho), \bar{\varphi}_{\nu}(\rho)\}$ is a solution, then $\{\bar{\varphi}_{\nu}(-\rho), \varphi_{\nu}(-\rho)\}$ is also a solution. For all the cases we shall be interested in these two solutions coincide, so we shall have

$$\varphi_{\nu}(\rho) = \bar{\varphi}_{\nu}(-\rho) \quad (\text{III-7})$$

once the point $\rho = 0$ has been fixed by using the translational invariance of the equations of motion. Moreover, a class of solutions of the single-field equation ($\bar{\varphi} = 0$, $\varphi = \varphi(\rho)$ or viceversa) can be proved to exist for $0 < \nu < \infty$. They are discussed in Appendix A.

Some particular kink solutions of Eqs.(III-4) are already known. Cardy and Sugar have found exact analytic solutions for two particular values of the propagation speed. They are given by

$$\begin{pmatrix} \varphi \\ \bar{\varphi} \end{pmatrix} = \begin{cases} \frac{1}{2} \left[1 \mp \tanh \frac{\rho}{\sqrt{8}} - \frac{1}{2} \left(\cosh \frac{\rho}{\sqrt{8}} \right)^{-2} \right] & \text{for } v = \frac{1}{\sqrt{2}} \\ \frac{1}{2} \left[1 \mp \tanh \sqrt{\frac{3}{8}} \rho - \frac{3}{2} \left(\cosh \sqrt{\frac{3}{8}} \rho \right)^{-2} \right] & \text{for } v = \frac{1}{\sqrt{6}} \end{cases}$$

(III-8)

Each of these solutions can be represented in the $(\varphi, \bar{\varphi})$ plane by a phase space trajectory starting at the $(0,1)$ fixed point when $\rho = -\infty$, and ending at the $(1,0)$ one when $\rho = \infty$, as shown in Figure 1. The first solution (III-8a) has both fields always positive and smaller than 1 and is represented therefore by a trajectory internal to the triangle defined by the three fixed points given by Eq.(III-5). The second solution is represented by a trajectory that lies outside the triangle. Due to the reflection property (III-6), one finds for $v = -1/\sqrt{2}$ and $v = -1/\sqrt{6}$ the same trajectories but evolving in the opposite sense, from $(1,0)$ to $(0,1)$.

We are interested in finding out about the existence of some further solutions of the same kind, and in particular the way they may depend on the propagation velocity v . This is a non-trivial problem in RFT due to the lack of Galilean invariance, which means that kink solutions with different velocities are not related by a simple change of variables, and consequently that the functional form of the kink solutions change in a non-trivial way as the propagation velocity v is changed. We shall heavily rely on the existence of a constant of motion for the system (III-4), which is easily obtained by multiplying the first equation by $\dot{\bar{\varphi}}$, the second by $\dot{\varphi}$ and adding to obtain a total derivative which can be integrated to give

$$\dot{\varphi} \dot{\bar{\varphi}} + \varphi \bar{\varphi} (1 - \varphi - \bar{\varphi}) = C$$

(III-9)

where C is a constant. This constant of the motion plays the same role of

energy conservation in an ordinary mechanical system. It severely restricts the shape of the phase space trajectories in the $(\varphi, \bar{\varphi})$ plane. The kink solutions will reach (or leave) asymptotically one of the fixed points (III-5). It then follows that $C = 0$, and then

$$\dot{\varphi} \dot{\bar{\varphi}} = -\varphi \bar{\varphi} (1 - \varphi - \bar{\varphi}) \quad (\text{III-10})$$

for all ρ . Therefore, in all of the $(\varphi, \bar{\varphi})$ plane the sign of $\dot{\varphi} \dot{\bar{\varphi}}$ is fixed, and is exhibited in Figure 2. One can immediately see, for example, that inside the triangle determined by the three fixed points $\dot{\varphi} \dot{\bar{\varphi}}$ is negative, so if one of the fields decreases as a function of ρ , the other must increase and viceversa. Moreover, the phase space trajectory can never be parallel to one of the coordinate axes, except when it crosses one of the three lines of fig. 2, which are the coordinate axes themselves $\varphi = 0$ and $\bar{\varphi} = 0$, and the straight line $\varphi + \bar{\varphi} = 1$. It is easy to deduce in this way that the origin $(0,0)$ can only be reached by infinite looping around it or from one of the coordinate axes which implies, however, that one of the fields vanishes identically for all ρ . We are left in this case with the uncoupled field situation discussed at length in the Appendix A.

III.a) Linearized Equations and numerical results.

Let us now study the behavior of the classical solutions near the $(\varphi, \bar{\varphi}) = (0,1)$ fixed point. We then introduce

$$\chi = 1 - \bar{\varphi} \quad (\text{III-11})$$

and linearize Eqs.(III-4) in φ and χ . This gives

$$\ddot{\varphi} - \nu \dot{\varphi} - \varphi = 0 \quad (\text{III-12a})$$

$$\ddot{\chi} - \nu \dot{\chi} - \chi + 2\varphi = 0 \quad (\text{III-12b})$$

We shall restrict ourselves to $\nu \geq 0$; for $\nu < 0$ the solutions are obtained by the reflection properties expressed by Eq.(III-6). The Eq. (III-12a) implies

$$\varphi(\rho) = A e^{-k\rho} \quad (\text{III-13})$$

with A an arbitrary constant, and

$$k = -\frac{\nu}{2} \pm \sqrt{\frac{\nu^2}{4} + 1} \quad (\text{III-14})$$

In order to solve Eq.(III-12b) we have to distinguish two possibilities, according to whether φ is of the same order or much smaller than χ . We therefore consider the two cases

$$\text{a) } \frac{\varphi}{\chi} \sim O(1) \quad ; \quad \text{b) } \frac{\varphi}{\chi} \ll O(1) \quad (\text{III-15})$$

For case (a), χ is driven by the solution of the φ equation, and we find

$$\chi = B e^{-k\rho} \quad , \quad \text{with} \quad \frac{A}{B} = \nu k \quad (\text{III-16})$$

while for case (b) χ is decoupled of φ and its behavior is then of the form

$$\chi = B e^{-\alpha\rho} \quad (\text{III-17})$$

with B an arbitrary constant and

$$\chi = \frac{\nu}{2} \pm \sqrt{\frac{\nu^2}{4} + 1} \quad (\text{III-18})$$

It is clear that the major difference between cases (a) and (b) is that in the first one the phase space trajectory will reach (or leave) the fixed point (0,1) with a finite slope, while in the second case it will do so with a zero slope, along the φ axis. We still have to consider, however, the various sign possibilities in Eqs.(III-14) and (III-18). Let us first consider trajectories of the class (a), in which case we only have to worry about the two sign possibilities in Eq.(III-14). For the positive sign, $k > 0$ and the linearized solution is relevant for $\rho \rightarrow +\infty$. Since we are restricting ourselves to $\nu > 0$, it follows that $A/B > 0$, and the solution therefore tends towards the (0,1) fixed point as $\rho \rightarrow \infty$ from inside the triangle of Fig. (2). Indeed, the slope with which it reaches the fixed point in the $(\varphi, \bar{\varphi})$ plane is given by

$$S_1 = \lim_{\rho \rightarrow \infty} \left(\frac{\varphi}{-\chi} \right) = \lim_{\rho \rightarrow \infty} \frac{\varphi}{\bar{\varphi} - 1} = -\frac{\nu}{2} (\sqrt{\nu^2 + 4} - \nu) \quad (\text{III-19})$$

which shows that S_1 is always negative, varying monotonically between 0 and (-1) as ν varies from 0 to infinity. In this limiting case the phase space trajectory reaches the fixed point along the hypotenuse of the triangle of Fig. 2.

If instead $k < 0$ (always in the case (a)), the linearized solution is relevant for $\rho \rightarrow -\infty$, and since $\frac{A}{B} < 0$ we conclude that the phase space trajectory leaves the fixed point by going outside the triangle. Indeed, the slope in the $(\varphi, \bar{\varphi})$ plane is given by

$$S_2 = \lim_{\rho \rightarrow -\infty} \left(\frac{\varphi}{-\chi} \right) = \frac{1}{2} \nu (\sqrt{\nu^2 + 4} + \nu) \geq 0 \quad (\text{III-20})$$

Let us now turn to case (b). It is a simple matter to realize that the only sign combination of Eqs.(III-14) and (III-18) compatible with φ being exponentially smaller than χ is

$$k = -\frac{\nu}{2} - \sqrt{\frac{\nu^2}{4} + 1} \quad ; \quad \alpha = \frac{\nu}{2} - \sqrt{\frac{\nu^2}{4} + 1} \quad (\text{III-21})$$

which means that the solution of the linearized problem is relevant when $\rho \rightarrow -\infty$. Since A and B are uncorrelated, the possible solutions leave the fixed point (0,1) with vanishing slope but may end up either inside or outside the triangle. We should also mention that for this class (b) of solutions φ/χ vanishes exponentially in ρ except when $\nu = 0$, in which case φ/χ goes to zero as $1/\rho$. In this case the solution of the linearized equations (III-12) is

$$\varphi = A e^{\pm f} \quad ; \quad \chi = A \rho e^{\pm f} \quad (\text{III-22})$$

with both signs in the exponential being the same.

To sum up, the behavior of the kink solutions of RFT near the (0,1) fixed point for $\nu > 0$ can be summarized as follows

- i) case a, $k > 0$, the solutions arrive to the fixed point from inside the triangle with a finite slope ranging from $S_1 = 0$ for $\nu = 0$ to $S_1 = -1$ for $\nu = \infty$.
- ii) case a, $k < 0$, kink solutions leave the fixed point with a finite positive slope going outside of the triangle.
- iii) case b, there is a unique sign choice for k and α , given by Eq. (III-21), and the kink solution leaves the fixed point with zero-slope.

The behavior of the classical solutions near the other non-perturbative fixed point, namely, $(\varphi, \bar{\varphi}) = (1, 0)$ can be obtained at once from the previous

results, due to the symmetry of the equations of motion, by interchanging φ and $\bar{\varphi}$ and changing ρ into $(-\rho)$. We can also locate the Cardy-Sugar solutions within this classification. The first one, given by Eq. (III-8a) is of type b, leaving $(0,1)$ with zero slope, while the second one corresponds to case (ii), leaving the fixed point and going outside of the triangle with slope $S_2 = + 1/2$.

We have investigated numerically the existence of kink solutions that interpolate between the $(0,1)$ and $(1,0)$ fixed points. These solutions will satisfy the conservation law provided by Eq.(III-10), and, as discussed before, they will not reach the origin $(0,0)$ unless one of the fields vanishes identically. Let us define $\rho = 0$ by the condition

$$\varphi(o) = \bar{\varphi}(o) \quad (\text{III-23})$$

It then follows from Eqs.(III-10) and (III-7) that

$$\dot{\varphi}(o) = -\dot{\bar{\varphi}}(o) = \pm \varphi(o) \sqrt{1 - 2\varphi(o)} \quad (\text{III-24})$$

Therefore, for every given $\varphi(0)$, these relations specify a complete set of initial conditions for the integration of the coupled system of equations of motion. We have performed this integrations numerically by searching, for every given $\varphi(0)$, the value of the propagation speed ν such that the solution approaches steadily the fixed points. We obtained in that way a large class of kink solutions, whose phase space trajectories are either inside or outside the triangle determined by the three fixed points, corresponding to the two different signs of Eq.(III-24). We shall discuss their properties separately.

We present first our results for the classical solutions whose phase space trajectory are inside the triangle. We find in this case kink solutions belonging to the class (a) for all propagation velocities in the range $0 < \nu < \infty$, solutions belonging to the class (b) for velocities in the range $0 < \nu < 2$, and

the solution having the behavior of Eq.(III-22) for $\nu = 0$. Needless to say these numerical solutions match the known behavior near the fixed points. In order to make our results more clear and to make more evident the continuity properties of these solutions as the propagation velocity changes, we turn our class (a) solutions for positive ν into solutions of negative ν ($-\infty < \nu < 0$) by means of the reflection property (III-6). We are therefore left with a family of kink solutions with propagation velocity in the range $-\infty < \nu < 2$, whose properties we now discuss.

The phase space trajectories of this family of solutions are sketched in Fig. 3. We discuss the evolution of the phase - space trajectory as a function of y because in this case the sense of the evolution, indicated by an arrow in Fig. 3, does not change when ν is changed to $(-\nu)$. This is not the case for the evolution in ρ , as follows from Eq. (III-6). As ν tends to $(-\infty)$, the trajectory tends to the hypotenusa of the triangle, evolving in y from the $(1,0)$ to the $(0,1)$ fixed point. As ν begins to increase through negative values, the phase space trajectory begins to sweep the interior of the triangle, always propagating in the same direction and leaving and reaching the fixed points with a non-zero slope. When $\nu = 0$, we find a static solution exhibited in Fig. 3, whose behavior near the fixed points is given by Eqs. (III-22) and for which obviously there is no arrow corresponding to time evolution. When the velocity ν becomes positive, the solutions switch to class (b). They keep sweeping the interior of the triangle as ν increases, but they switch their sense of evolution in the sense that they evolve from the $(0,1)$ to the $(1,0)$ fixed points. Since one of the fields is exponentially smaller than the other near the fixed points, their phase space trajectory is in those regions very close to the φ or $\bar{\varphi}$ axis.

The emergence of the critical propagation velocity $\nu = 2$, which will play a central role in the rest of the paper, can be seen as follows. As ν approaches 2 from below, the phase space trajectories become more and more close to the sides of the triangle (the $\varphi, \bar{\varphi}$ axis) thereby spending a large amount of "time" near the origin, that is to say, the perturbative vacuum. In this limit the coupling between the two field equations becomes less relevant, and the overlap between φ and $\bar{\varphi}$ tends to zero. In the limit $\nu = 2$, the classical fields are solutions of uncoupled field equations, and the phase space trajectory spends

an infinite time in the perturbative vacuum.

We stress again the perfect continuity of this family of solutions that we have found. This is also seen in Fig. 4, where we plot the value of $\varphi(0)$ as a function of ν . The Cardy-Sugar solution (III-8a) is a particular member of this family without any further relevance besides the fact that it is given by a simple analytic expression.

We have also found kink solutions outside of the triangle, corresponding to a choice of initial conditions such that $\varphi(0) < 0$ and $\dot{\varphi}(0) > 0$. They look like the corresponding Cardy-Sugar solution represented in Fig.(1). In agreement with our expectations from the analysis of the linealized equations, we find again a family of class(b) solutions with velocities in the range $0 < \nu < 2$, such that when $\nu \rightarrow 2$ they become very close to the sides of the triangle and eventually merge at $\nu = 2$ with the corresponding solution obtained from the interior of the triangle. Notice, however, that while the classical solutions $\{\varphi(\rho), \bar{\varphi}(\rho)\}$ whose phase space trajectory is inside the triangle are monotonic functions of ρ , those who are outside it necessarily have an oscillation, as functions of ρ .

III.b - Energy-Momentum relationship.

We shall anyhow concentrate our attention on the solutions inside the triangle, which are the ones which are relevant for our quantization procedure. We

turn now to the calculation of the energy, the momentum and the action associated with them. From Eq. (II-5) and (III-3) we obtain, after changing variables from b to ρ :

$$E_{cl} = \left(\frac{\mu}{\lambda}\right)^2 \sqrt{\alpha' \mu} \int_{-\infty}^{\infty} d\rho \left[-\dot{\bar{\varphi}} \dot{\varphi} + \varphi \bar{\varphi} (1 - \varphi - \bar{\varphi}) \right] \quad (\text{III-24})$$

where the dot means as usual the derivative with respect to ρ . The classical momentum is given by $(-i \mathcal{P}_\mu)$, where

$$\begin{aligned}
P_{cl} &= \int db \, q_{cl} \frac{\partial P_{cl}}{\partial b} = \\
&= \left(\frac{\mu}{\lambda}\right)^2 \cdot \frac{1}{2} \int_{-\infty}^{\infty} df \, [\bar{\varphi} \dot{\varphi} - \dot{\bar{\varphi}} \varphi] \quad (III-25)
\end{aligned}$$

and the classical action corresponding to a kink of speed v is

$$A_{cl} = \int dy \, (\mathcal{E}_{cl} - v \sqrt{\alpha' \mu} \, P_{cl}) \quad (III-26)$$

It is worth commenting at this point about the imaginary values of the classical momentum, $(-i P_{cl})$. The corresponding quantum operator defined in terms of the p and q fields is the generator of real translations in impact parameter space and contains an i because p and q satisfy commutation relations adequate to creation and destruction operators. Its classical values are not expectation values but rather transition matrix elements of P between a right coherent eigenstate of p and a left coherent eigenstate of q . Since for our classical solutions p and q have real values, the transition matrix element turns out to be imaginary. It follows immediately from Eq.(III-25) that P_{cl} is just $(\mu/\lambda)^2$ times the area under the phase space trajectory of the classical solution. Therefore, for the family of solutions under consideration, P_{cl} starts from $\frac{1}{2} \cdot (\mu/\lambda)^2$ when $v \rightarrow -\infty$ and decreases monotonically to zero as $v \rightarrow 2$.

By using Eq.(III-10) the classical energy can be written as :

$$\begin{aligned}
\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\sqrt{\alpha' \mu}} \mathcal{E}_{cl} &= -2 \int_{-\infty}^{\infty} df \, \dot{\bar{\varphi}} \dot{\varphi} \\
&= 2 \int_{-\infty}^{\infty} df \, \varphi \bar{\varphi} (1 - \varphi - \bar{\varphi}) \geq 0 \quad (III-27)
\end{aligned}$$

Therefore, $\mathcal{E}_{cl}$ is always positive definite, starting from zero when

$\nu = 2$, then increasing and finally decreasing to zero when $\nu \rightarrow -\infty$. Moreover, by manipulating the equations of motion one can prove that

$$Q_{ce} = \xi_{ce} - \nu \sqrt{\alpha' \mu} P_{ce} = 2 \left(\frac{\mu}{\lambda} \right)^2 \sqrt{\alpha' \mu} \int_{-\infty}^{\infty} d\rho \left[\varphi \bar{\varphi}^2 + \varphi^2 \bar{\varphi} \right]$$

Therefore, for the class of solutions we are considering the action density is always positive definite. It goes to zero as $\nu \rightarrow 2^-$, and increases monotonically to $(+\infty)$ as ν tends to $(-\infty)$.

We are interested in studying in more detail the limit $\nu \rightarrow 2^-$, which will be the relevant one in our quantization procedure. We have seen that in this limit $\varphi(0)$ tends to zero. Moreover, the overlap between φ and $\bar{\varphi}$ is also very small; in fact $\varphi(\rho) \rightarrow 0$ for $\rho < 0$ while $\bar{\varphi}(\rho) \rightarrow 0$ for $\rho > 0$, so the field equations are effectively decoupled. We shall therefore study the properties of the kink solutions by using $\varphi(0)$ as a small expansion parameter. In this case the initial conditions (III-24) become

$$\varphi(0) = \bar{\varphi}(0) \simeq \dot{\varphi}(0) = -\dot{\bar{\varphi}}(0) \quad (\text{III-29})$$

Moreover, for $\rho > 0$ $\bar{\varphi}(\rho)$ is very small, and therefore in this region we can neglect quadratic terms in $\bar{\varphi}(\rho)$. We are therefore led to the following field equations for $\rho > 0$

$$\ddot{\varphi} - \nu \dot{\varphi} + \varphi - \varphi^2 = 0 \quad (\text{III-30})$$

$$\ddot{\bar{\varphi}} + \nu \dot{\bar{\varphi}} + \bar{\varphi}(1 - 2\varphi) = 0 \quad (\text{III-31})$$

Equation (III-31) can be solved by using (III-30). The solution is

$$\bar{\varphi}(\rho) = \dot{\varphi}(\rho) e^{-\nu \rho} \quad (\text{III-32})$$

which satisfies the initial conditions (III-29) and gives $\bar{\varphi}(\rho)$ in terms of the solution of the single field equation for φ , Eq.(III-30). Let us now show that this equation admits a unique solution with $\varphi(0) = 0(2 - \nu)$. In fact, as it is done also in the Appendix, the single field equation can be recast in the form

$$\left(\frac{dW}{d\varphi} - \nu \right) W + \varphi - \varphi^2 = 0 \quad (\text{III-33})$$

where the velocity field $W(\varphi) = \dot{\varphi}$ has a unique phase space trajectory with boundary condition $W'(1) = (1 + \sqrt{2})(1 - \varphi)$. As remarked in the Appendix A $\varphi = 0$ is a spiral point for the trajectory if $\nu < 2$ (damped oscillations) and becomes a regular end point when $\nu = 2$ (critical damping). We show in Fig. 5 the form of $W(\varphi)$. It is then clear that the boundary condition $\varphi(0) = \dot{\varphi}(0)$ is satisfied for $\nu \leq 2$ at the intersection I of $W = W(\varphi)$ and $W = \varphi$; in such a way that $\varphi(0) \rightarrow 0$ as $\nu \rightarrow 2^-$ (see Fig. 5).

The same remarks prove that there are solutions with $\nu \leq 2$ outside the triangle, satisfying $\varphi(0) = -\dot{\varphi}(0)$, given by the intersection II of $W = W(\varphi)$ with $W = \varphi$.

We are now prepared to discuss the order of magnitude, in the small parameter $\varphi_0 = \varphi(0)$ of $\mathcal{E}_a$, $\mathcal{P}_a$, and $\mathcal{Q}_{cc}$. First of all we notice that $\mathcal{P}_a$ is the area under the phase space trajectory in the $\varphi, \bar{\varphi}$ plane, and that it is of $O(\varphi_0^2)$.

Indeed, by using the solution (II-32) for $\bar{\varphi}(\rho)$, we can compute $\mathcal{P}_a$ as follows :

$$\begin{aligned} \left(\frac{\lambda}{\mu} \right)^2 \mathcal{P}_{cc} &= \int_0^\infty [\bar{\varphi} \dot{\varphi} - \dot{\bar{\varphi}} \varphi] d\rho = \\ &= 2 \int_0^\infty \bar{\varphi} \dot{\varphi} d\rho + \varphi_0^2 = \\ &= 2 \int_0^\infty \dot{\varphi}^2 e^{-\nu \rho} d\rho + \varphi_0^2 \end{aligned} \quad (\text{III-34})$$

For small values of ρ , the solution $\varphi(\rho)$ of equation (III-30) grows like

$\varphi_0 \exp\left(\frac{\nu}{2} \rho\right)$ for $\nu \rightarrow 2^-$. Therefore, the integrand in Eq.(III-34) is of order φ_0^2 over a length of order $\log\left(\frac{1}{\varphi_0}\right)$, and outside of this region it is exponentially small. It then follows that ${}^0\mathcal{P}_a$ is of order $O(\varphi_0^2 \ln\left(\frac{1}{\varphi_0}\right))$. We shall neglect logarithms with respect to powers of φ_0 in our estimates.

In what concerns the classical energy ε_a , we can write

$$\begin{aligned} \left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\sqrt{\alpha'\mu}} \varepsilon_a &= -4 \int_0^\infty \dot{\varphi} \dot{\varphi} d\rho = \\ &= 4\nu \int_0^\infty \dot{\varphi}^2 e^{-\nu\rho} d\rho - 4 \int_0^\infty \ddot{\varphi} \dot{\varphi} e^{-\nu\rho} d\rho \end{aligned} \quad (\text{III-35})$$

and, by performing a partial integration in the last term, we obtain

$$\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{\sqrt{\alpha'\mu}} \varepsilon_a = 2\nu \int_0^\infty \dot{\varphi}^2 e^{-\nu\rho} d\rho + 2\dot{\varphi}(0)^2 \quad (\text{III-36})$$

Therefore, using the initial condition (III-30) it follows that

$$\left(\frac{\lambda}{\mu}\right)^2 \left[\varepsilon_a - \nu \sqrt{\alpha'\mu} \mathcal{P}_a \right] = (2-\nu) \varphi_0^2 \quad (\text{III-37})$$

so we conclude that, when $\nu \rightarrow 2^-$, ε_a and $\mathcal{P}_a$ both go to zero, and that the dispersion relation has the form

$$\varepsilon_a = 2\sqrt{\alpha'\mu} P_a + O(P_a^{3/2}) \quad (\text{III-39})$$

The meaning of this result is clear ; it is just the Taylor expansion of the dispersion law $\varepsilon_a = \varepsilon_a(P_a)$ of our family of kink solutions. This dispersion law is not simple because of the lack of Galilean invariance of RFT. In $\lambda \phi^4$ the action, being invariant under Lorentz transformations, does not depend on the propagation speed ν of the moving kinks. The dispersion law for a moving kink is then a direct consequence of Lorentz invariance. For our family of kinks solutions with $-\infty < \nu < 2$, the action does depend on ν , the ν dependence of the solution is non trivial and the general dispersion law cannot be obtained in analytic form. However, as $\nu \rightarrow 2^-$, or, what is the same, for small classical momenta, the dispersion law of the classical solution becomes a phonon-type dispersion law, with a zero energy gap.

Besides the family of classical solutions we have just discussed, we must recall the existence of kink solutions outside the triangle, which have negative action. Moreover, as discussed in Appendix A, there are also classical kink solutions of the uncoupled field equations - with one of the fields identically zero - which connect one of the (0,1) or (1,0) fixed points to the perturbative vacuum. The action, energy and momentum of these kink solutions is obviously zero, and their propagation speed is bigger than 2.^(*) The critical propagation speed is then reached from above when one considers the uncoupled field equations, and from below, as we have discussed at length, when one starts from the coupled case. We summarize in Figure 6 the action per unit rapidity interval of the various families of classical solutions discussed in this paper. As we have done in Fig.(3), we exhibit the perfect continuity in ν by choosing the negative ν branch when convenient.

(*) The ones with $0 < \nu < 2$ have an oscillating field.

IV - QUANTIZATION OF KINK STATES

The purpose of this section is to set up a quantization scheme in RFT by expanding around one of the classical solutions discussed in the previous section. We shall therefore adapt to RFT the methods introduced to quantize $\lambda \varphi^4$ theory in 1 + 1 space time dimensions around a moving kink. The different methods used - canonical quantization or path-integral formulation - are of course all equivalent, and their main task is to isolate the collective coordinates associated with translational invariance and to integrate over the translational mode.

We shall adopt in this paper the path-integral formulation of Gervais and Sakita⁽⁶⁾. In this formalism the global translational invariance is transformed into a local gauge invariance by introducing an extra degree of freedom and a constraint. The collective coordinate degree of freedom can then be easily integrated away when one considers matrix elements of the evolution operator between eigenstates of total momentum. In our case, after projecting total momentum in the initial and final states of Eq. (II-17) we get the following representation for the matrix element

$$S_k = \int \mathcal{D}p \mathcal{D}q \Psi_{f,k} [q(p,y)] \Psi_{i,k} [p(p,0)] \delta[k-p] \delta[q] \cdot \int \sqrt{\frac{\alpha'}{\mu}} \int_0^y d\rho \int_0^y dy [q \dot{p} - \mathcal{H}]$$

x e

(IV-1)

In this equation $\Psi_{k,i}$ and $\Psi_{k,f}$ are the reduced wave functionals for the initial and final state - the plane wave factors of the collective coordinate part of the wave functional have already been used in the collective coordinate integration. The Reggeon fields have already been transformed to the body-fixed system, namely, $p = p(\rho, y)$ and $q = q(\rho, y)$. The first delta functional represents the constraint, and fixes the quantum momentum operator

$$P(y) = \frac{1}{i} \int d\mathbf{f} \, q(\mathbf{f}, y) \frac{\partial P(\mathbf{f}, y)}{\partial \mathbf{f}} \quad (\text{IV-2})$$

equal to the total momentum eigenvalue k . Moreover, $Q \{p, q\} = 0$ is the gauge condition, to be chosen so as to eliminate the zero mode, and J is a Jacobian that depends on the choice of the gauge.

Taking advantage of the presence of the constraint in the functional integration, we can rewrite the S-matrix as :

$$S_k = \int d\mathbf{p} \, d\mathbf{q} \, \Psi_{f,R} \, \Psi_{i,k} \, \delta[k - \mathbf{P}] \cdot \delta[Q] \cdot J \times \int_0^y dy \left[-i\omega\sqrt{\alpha'\mu} k + \sqrt{\frac{\alpha'}{\mu}} \int (q\dot{p} - \mathcal{H}_{\text{eff}}) d\mathbf{p} \right] \quad (\text{IV-3})$$

where the effective Hamiltonian density $\mathcal{H}_{\text{eff}}$ is given by :

$$\mathcal{H}_{\text{eff}} = \mathcal{H} - \mu\omega q \frac{\partial P}{\partial \mathbf{p}} \quad (\text{IV-4})$$

Since $\mathbf{p}$ is the transverse coordinate in the body-fixed system, the classical static fields that make stationary the effective action of Eq.(IV -3) satisfy the equations of motion

$$\frac{\delta \mathcal{H}_{\text{eff}}}{\delta q(s)} = 0$$

(IV-5)

$$\frac{\delta \mathcal{H}_{\text{eff}}}{\delta p(s)} = 0$$

which, given the form (IV-4) of $\mathcal{H}_{\text{eff}}$, are precisely the classical equations of motion (III-1a,b) discussed in the previous section. Their solutions are then the moving kinks propagating with velocity ω . Notice, however, that these moving kinks are local minima of the unconstrained action. One could still try to fix ω , which plays the role of a Lagrange multiplier, in such a way that the constraint $k = P$ is satisfied, but this will not be possible in general given the imaginary nature of the classical momentum of our families of kinks. We shall come back to this problem in the next section, where we shall discuss at length the choice of the classical solution used to shift the reggeon fields.

The classical kink solutions contain both fields q_{cl} and p_{cl} , and it will prove convenient to introduce a spinorial formalism in order to simplify the subsequent expressions. Moreover, we want to treat creation and destruction operators in a symmetrical way. We then introduce a spinor field ξ by

$$\xi = \begin{pmatrix} p \\ q \end{pmatrix} \quad (\text{IV-6})$$

and a metric tensor

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (\text{IV-7})$$

in such a way that

$$\bar{\xi} = \xi^\dagger = \overline{\begin{pmatrix} q & -p \end{pmatrix}} \quad (\text{IV-8})$$

This is typical of "first order" formalisms even for Hermitian theories⁽⁵⁾. Notice that $\bar{\xi}\xi = qp - pq$ is zero at the classical level, but is of course the commutator of p and q at the quantum level. If the scalar product is defined as

$$(\bar{\xi}', \xi) = \int dp (q'p - p'q) = -(\bar{\xi}, \xi') \quad (\text{IV-9})$$

then the transformations that leave this scalar product invariant at the Lagrangian level are equivalent to transformations that leave the commutation relations invariant at the Hamiltonian level, which are of course Bogoliubov transformations. Moreover, from Eq. (IV-9) it follows that for any c-number spinor ξ_n

$$(\bar{\xi}_n, \xi_n) = 0 \quad (\text{IV-10})$$

and therefore, in order to define a non vanishing norm we shall be led to introduce a spinor dual to ξ_n which is not $\bar{\xi}_n$.

Let us now perform a shift of the reggeon fields, by defining the shifted fields according to

$$\xi = \frac{\mu}{\lambda} \xi_{ce} + \left(\frac{\mu}{\alpha'}\right)^{1/4} \xi' \quad (\text{IV-11})$$

where

$$\xi_{cl} = \begin{pmatrix} \varphi(\rho) \\ \bar{\varphi}(\rho) \end{pmatrix} \quad (\text{IV-12})$$

$\varphi(\rho)$ and $\bar{\varphi}(\rho)$ being the solutions of Eqs.(III-4a,b). In terms of the shifted fields the action density $\mathcal{Q}(y)$ becomes :

$$\mathcal{Q} = \mathcal{Q}_{cl} + i\nu\sqrt{\alpha'\mu}k - \frac{1}{2} \int d\rho \left[\bar{\xi}' \dot{\xi}' + \bar{\xi}' \mathcal{H}_2 \xi' \right] + O(\lambda) \quad (\text{IV-13})$$

where $\mathcal{H}_2$ is a differential operator of the form

$$\mathcal{H}_2 = \mu \begin{pmatrix} D(\nu) & B \\ -\bar{B} & -D^T(\nu) \end{pmatrix} \quad (\text{IV-14})$$

where

$$\begin{cases} B = 2\varphi(\rho) & ; \quad \bar{B} = 2\bar{\varphi}(\rho) \\ D(\nu) = \nu \frac{d}{d\rho} - \frac{d^2}{d\rho^2} + 2(\varphi + \bar{\varphi}) - 1 \\ D^T(\nu) = D(-\nu) \end{cases} \quad (\text{IV-15})$$

In order to properly incorporate in the shifted action density (IV-13) the gauge condition and the constraint, let us investigate separately the degrees of freedom associated with the translational invariance of the RFT action. It follows, as usual, from translational invariance that

$$\xi_0(\rho) = N_0 \frac{\partial \xi_{cl}}{\partial \rho} \quad (\text{IV-16})$$

is a zero-energy eigenvector of $\mathcal{H}_2$, i.e.

$$\mathcal{H}_2 \xi_0 = 0 \quad ; \quad \bar{\xi}_0 \mathcal{H}_2 = 0 \quad (\text{IV-17})$$

N_0 being a normalization factor to be discussed later on. Moreover, differentiating with respect to ν the classical equations of motion (III-4ab) we get

$$\mathcal{H}_2 \frac{\partial \xi_a}{\partial \nu} + \mu \frac{\partial \xi_a}{\partial \varphi} = 0 \quad (\text{IV-18})$$

which implies that the spinor χ_0 defined as

$$\chi_0 = L_0 \frac{\partial \xi_{ca}}{\partial \nu} \quad (\text{IV-19})$$

L_0 being a new normalization factor, satisfies

$$\mathcal{H}_2 \chi_0(\varphi) = -\mu \frac{L_0}{N_0} \xi_0(\varphi) \quad (\text{IV-20})$$

From Eqs.(III-17) and (III-20) it follows at once that both ξ_0 and χ_0 are orthogonal to any eigenvector of $\mathcal{H}_2$ with non-zero eigenvalue. Notice that the vector χ_0 just defined is not a zero-frequency eigenvector. It will play however an important role because it is the conjugate vector to the zero mode eigenvector ξ_0 , and it will have to be kept in order to have a complete set of eigenfunctions to expand the shifted field. The arbitrary normalization constant L_0 will be fixed by the condition

$$(\bar{\chi}_0, \xi_0) = 1 \quad (\text{IV-21})$$

We want to emphasize that the formulation we are describing here is by no mean specific to RFT. The same pattern emerges when one performs a kink quantization of $\lambda \varphi^4$ theory, for example, using a first order formalism.

We can now discuss the choice of the gauge condition $Q = 0$. As is usually done, we impose a gauge condition that ensures that the spinor field ξ has no component on the zero mode. Our gauge condition is then, as suggested by Eq. (IV-21)

$$Q = (\bar{\chi}_0, \xi) = 0 \quad (\text{IV-22})$$

The classical part of ξ in Eq. (IV-11) will satisfy by itself the gauge condition provided that the over-all translational parameter a left undetermined by the equations of motion in $\xi_{cl}(p+a)$ is adequately chosen. One can easily check that for the class of kink solutions discussed in the previous section our gauge condition fixes $a = 0$. Therefore, Eq. (IV-22) reduces to a condition for the shifted field ξ' , and the most general solution is

$$\xi = \frac{\mu}{\lambda} \xi_{cl} + \left(\frac{\mu}{\alpha'}\right)^{1/4} [A_0 \chi_0 + \eta] \quad (\text{IV-23})$$

where η is the genuine quantum fluctuation, namely, the part of the shifted field having components on the remaining non-zero modes :

$$(\bar{\chi}_0, \eta) = (\bar{\xi}_0, \eta) = 0 \quad (\text{IV-24})$$

Once the gauge condition is chosen, the Jacobian J of Eq. (IV-3) is given by

$$J = \prod_f \left(\bar{\chi}_0, \frac{\partial \xi}{\partial f} \right) \quad (\text{IV-25})$$

and, by using the expansion (IV-23) we obtain

$$J = \prod_f \left[\frac{\mu}{\lambda} \frac{1}{N_0} + \left(\frac{\mu}{\alpha'}\right)^{1/4} (\partial_f \bar{\eta}, \chi_0) + \left(\frac{\mu}{\alpha'}\right)^{1/4} A_0 (\partial_f \bar{\chi}_0, \chi_0) \right] \quad (\text{IV-26})$$

We can next use the constraint in Eq.(IV-3) to fix the χ_0 component of the shifted field. The constraint equation reads, in our spinor formalism.

$$i(P-k) = \frac{1}{2} (\bar{\xi}, \partial_f \xi) - ik = 0 \quad (\text{IV-27})$$

which in terms of the shifted fields become of the form

$$R - F[A_0] = 0 \quad (\text{IV-28})$$

where we have introduced the notation :

$$R(y) = \mathcal{P}_{ce} + \sqrt{\frac{\mu}{\alpha'}} \frac{1}{2} (\bar{\eta}, \partial_f \eta) - ik \quad (\text{IV-29})$$

$$F[A_0] = -A_0 \left(\frac{\mu}{\alpha'}\right)^{1/4} \left[\frac{\mu}{\lambda} \frac{1}{N_0} + \left(\frac{\mu}{\alpha'}\right)^{1/4} (\partial_f \bar{\eta}, \chi_0) \right] - \sqrt{\frac{\mu}{\alpha'}} A_0^2 \frac{1}{2} (\bar{\chi}_0, \partial_f \chi_0) \quad (\text{IV-30})$$

If the constraint delta functional $\delta[R - F(A_0)]$ is used to perform the A_0 integration, one gets a new Jacobian factor of the form $[\eta F'(A_0)]^{-1}$, evaluated at the value of A_0 that satisfies the constraint, namely

$$A_0(y) = -f(y) + \left[f^2(y) + \frac{2\sqrt{\frac{\alpha'}{\mu}} R(y)}{(\bar{\chi}_0, \partial_f \chi_0)} \right]^{1/2} \quad (\text{IV-31})$$

$$f(y) = \frac{\left(\frac{\alpha'}{\mu}\right)^{1/4} \frac{\mu}{\lambda} \frac{1}{N_0} + (\partial_f \bar{\eta}, \chi_0)}{(\bar{\chi}_0, \partial_f \chi_0)} \quad (\text{IV-32})$$

We have chosen the root of Eq.(IV-28) that guarantees that A_0 vanishes when $R = 0$, that is to say, when the constraint is saturated by the classical momentum and the quantum fluctuation η is zero. By computing $F'[A_0]$ from Eq.(IV-30) and using (IV-31,32) we then find that this new Jacobian exactly cancels -up to a constant- the first one given by Eq. (IV-26), i.e.

$$\frac{J}{\prod_{\vec{y}} F'[A_0]} = \prod_{\vec{y}} \left(\frac{\alpha'}{\mu} \right)^{1/4} = \text{constant} \quad (\text{IV-33})$$

This result confirms the internal consistency of our procedure. In the general formulation of the collective coordinate method the over-all Jacobian one gets in the path integral can be written as the commutator of the gauge condition with the constraint. Calling $\bar{A}_0$ the ξ_0 component of ξ , we have written the constraint and the gauge condition as :

$$\delta \left[A_0(\vec{y}) + \int - \left(\vec{f}^2 + \frac{2R\sqrt{\mu}}{(\bar{x}_0, \partial_\rho \chi_0)} \right)^{1/2} \right] \delta [\bar{A}_0(t)] \quad (\text{IV-34})$$

and the fact that the over-all Jacobian is a constant means that A_0 and $\bar{A}_0$, considered as operators, satisfy canonical commutation relations. Therefore the identification of χ_0 as the conjugate vector to ξ_0 is then fully justified. We then conclude that the constraint and the gauge condition can be integrated, without adding any extra terms to the action density in Eq. (IV-13). All what one has to do is to shift the fields according to Eq.(IV-23), with A_0 fixed by Eqs. (IV-31,32). The η field is to be integrated in the subspace orthogonal to the zero mode.

Let us now turn our attention to the normal modes which diagonalize the quadratic part of the action, and which will provide us with a basis to expand the η field. They are solutions of the eigenvalue equation

$$\mathcal{H}_2 \xi_n = \omega_n \xi_n \quad (n \neq 0) \quad (\text{IV-35})$$

We next define the components of ξ_n by

$$\xi_n(\rho) = \begin{pmatrix} F_n(\rho) \\ \bar{F}_n(\rho) \end{pmatrix} \quad (\text{IV-36})$$

and, using the explicit form of $\mathcal{K}_2$ given by Eqs.(IV-14,15) the eigenvalue equation reads

$$\left[\nu \frac{d}{d\rho} - \frac{d^2}{d\rho^2} + 2(\varphi + \bar{\varphi}) - 1 \right] F_n + 2\varphi \bar{F}_n = \frac{\omega_n}{\mu} F_n \quad (\text{IV-37a})$$

$$- 2\varphi F_n + \left[\nu \frac{d}{d\rho} + \frac{d^2}{d\rho^2} + 1 - 2(\varphi + \bar{\varphi}) \right] \bar{F}_n = \frac{\omega_n}{\mu} \bar{F}_n \quad (\text{IV-37b})$$

By inspection of this system of equations, and using also the fact that the classical solutions we are considering have the property that $\varphi(\rho) = \bar{\varphi}(-\rho)$, it follows that if ξ_n is an eigenfunction of $\mathcal{K}_2$ with eigenvalue ω_n , then

$$\xi_{-n}(\rho) = \begin{pmatrix} \bar{F}_n(-\rho) \\ F_n(-\rho) \end{pmatrix} \quad (\text{IV-38})$$

is an eigenfunction of $\mathcal{K}_2$ with eigenvalue $(-\omega_n)$

$$\mathcal{K}_2 \xi_{-n}(\rho) = -\omega_n \xi_{-n}(\rho) \quad (n \neq 0) \quad (\text{IV-39})$$

This doubling of eigenfrequencies is typical of first-order formalisms, which deal with creation and destruction operators (or coordinate and momenta) as independent variables. We still have to find the left eigenfunctions of $\mathcal{K}_2$. It also follows by inspection that

$$\bar{\xi}_{\pm n} \mathcal{K}_2 = \mp \omega_n \bar{\xi}_{\pm n} \quad (\text{IV-40})$$

which means that the spinor dual to ξ_n is $\bar{\xi}_{-n}$. It is obvious then that the correct normalization conditions to be imposed on our eigenfunctions is

$$(\bar{\xi}_m, \xi_n) = -(\bar{\xi}_n, \xi_m) = \delta_{n,-m} \sigma_n \quad (\text{IV-41})$$

where σ_n is the sign of n , and we adopt the convention that positive n corresponds to positive ω_n .

The diagonalization of the quadratic part of the RFT effective Hamiltonian proceeds then as usual with Bogoliubov transformations. From the completeness relation (c.f. Appendix B)

$$\begin{aligned} \sum_{n>0}^{\infty} \left[\xi_n(p) \bar{\xi}_{-n}(p') - \bar{\xi}_n(p) \xi_n(p') \right] + \\ + \xi_0(p) \bar{\chi}_0(p') - \chi_0(p) \bar{\xi}_0(p') = \pm \delta(p-p') \end{aligned} \quad (\text{IV-42})$$

it follows from Eqs. (IV.20, 35 and 38) that the spectral decomposition of the operator $\mathcal{K}_2$ is

$$\mathcal{K}_2 = \sum_{n>0} \omega_n \left[\xi_n(p) \bar{\xi}_{-n}(p') + \bar{\xi}_n(p) \xi_n(p') \right] + \mu L_0 \xi_0(p) \bar{\xi}_0(p') \quad (\text{IV-43})$$

If we now define, for $n > 0$

$$A_n = (\bar{\xi}_{-n}, \eta) \quad ; \quad \bar{A}_n = (\bar{\xi}_n, \eta) \quad (n > 0) \quad (\text{IV-44})$$

then the expansion for the spinor field η becomes

$$\eta(p, y) = \sum_{n=1}^{\infty} [A_n(y) \bar{\xi}_n(p) - \bar{A}_n(y) \xi_{-n}(p)] \quad (\text{IV-45})$$

and therefore the eigenfunction expansion for the full reggeon field ξ reads

$$\begin{aligned} \xi(p, y) = & \frac{\mu}{\lambda} \xi_{ce}(p) + \left(\frac{\mu}{\alpha_1}\right)^{1/4} \left\{ A_0(y) \chi_0(y) + \right. \\ & \left. + \sum_{n=1}^{\infty} [A_n(y) \bar{\xi}_n(p) - \bar{A}_n(y) \xi_{-n}(p)] \right\} \quad (\text{IV-46}) \end{aligned}$$

The coefficients A_n and $\bar{A}_n$ must be considered as the integration variables in the path-integral formulation, while, as we emphasized before, A_0 is given by the constraint equation (IV-32) in terms of the integration variable η , and the gauge condition has fixed $\bar{A}_0 = 0$ in an extra term of the form $\bar{A}_0 \xi_0$ that could have been added in Eq.(III-46).

In a canonical formalism $\bar{A}_n$ and A_n become, as we shall soon confirm, creation and destruction operators of a quantum of excitation energy ω_n corresponding to the small oscillations around the classical solution. Indeed, they satisfy the commutation relations

$$[A_n, \bar{A}_m] = \delta_{n,m} \quad (\text{IV-47})$$

which are a consequence of the original commutation relations of the reggeon fields and of the completeness of our set of eigenfunctions (c.f. Appendix B). One has to be careful, however, with the ordering of operators, which is unambiguously fixed by the ordering of operators in the original Hamiltonian.

In terms of the shifted fields, the action density (IV-13) becomes

$$\mathcal{A} = -\frac{1}{2} \int d\rho \, \bar{\eta} \frac{\partial \eta}{\partial y} + E \quad (\text{IV-48})$$

where the energy E is given, in the quadratic approximation by

$$E = a_{ce} + i v \sqrt{\alpha' \mu} k - \frac{1}{2} \int d\rho \, [A_0 \bar{\chi}_0 + \bar{\eta}] \mathcal{H}_2 [A_0 \chi_0 + \eta] + \delta E \quad (\text{IV-49})$$

where δE are ordering terms that we shall discuss later. By using Eqs. (IV-20, 21 and 24), one obtains

$$E = a_{ce} + i v \sqrt{\alpha' \mu} k + \frac{1}{2} A_0^2 \mu \frac{L_0}{N_0} - \frac{1}{2} \int d\rho \, \bar{\eta} \mathcal{H}_2 \eta + \delta E \quad (\text{IV-50})$$

and from Eqs. (IV-43, 44 and 47)

$$E = a_{ce} + i v \sqrt{\alpha' \mu} k + \frac{1}{2} \mu A_0^2 \frac{L_0}{N_0} + \sum_{n=1}^{\infty} \omega_n \left(\bar{A}_n A_n + \frac{1}{2} \right) + \delta E \quad (\text{IV-51})$$

The ordering term δE is fixed by remembering that the unshifted Hamiltonian is ordered in such a way that the perturbative vacuum is an exact eigenstate. If one shifts the reggeon fields by keeping the ordering of the operators, one gets the quadratic form

$$\bar{\Psi}' D(r) \Psi' + \bar{B} \Psi'^2 + B \bar{\Psi}'^2 \quad (\text{IV-52})$$

where ψ' and $\bar{\psi}'$ are the components of the shifted spinor ξ' , defined in (IV-11). Therefore, by comparing this quadratic form with the one we have actually diagonalized, one can see immediately that the ordering terms are what is needed in order to replace $\bar{\Psi}' D(r) \Psi'$ by a symmetrized form. Then,

$$\delta E = -\frac{\mu}{2} \text{Tr} \begin{pmatrix} D(r) & 0 \\ 0 & -D(r) \end{pmatrix} = -\frac{1}{2} \sum_n \omega_n^o(r) \quad (\text{IV-53})$$

the $\omega_n^o(r)$ being the eigenvalues of the same Schrödinger problem as before, but without off-diagonal coupling. The final result for the energy is, then

$$E = A_{ce} + i v \sqrt{\alpha' \mu} k + \frac{1}{2} \mu A_0^2 \frac{L_0}{N_0} + \sum_{n \neq 0} \omega_n(r) \bar{A}_n A_n + \frac{1}{2} \sum_{n \neq 0} [\omega_n(r) - \omega_n^o(r)] \quad (\text{IV-54})$$

V - DISCUSSION OF THE SPECTRUM.

V. a - Choice of the parameter ν .

In order to calculate the energy spectrum from Eq.(IV-54) we must calculate the eigenfrequencies ω_n by solving the eigenvalue equations (IV-37a,b) which depend explicitly on the classical solution $\{\varphi_\nu(\rho), \bar{\varphi}_\nu(\rho)\}$ used to shift the reggeon fields. Also Q_{cl} occurring in (IV-54) depends on the classical solution in a rather non-trivial way, as we discussed at length in Section III.

We must therefore face the problem of the actual choice of the classical solution. In principle any one is equally good : quantum fluctuations will span the whole path integral. Our purpose is however to construct eigenstates of the Hamiltonian by treating all nonperturbative effects in the semiclassical limit. We have therefore to require after shifting around the solutions chosen, that the Hamiltonian operator contains the quantum fluctuations in a positive definite quadratic form. This restricts our choice, because for some of them the quadratic form is not positive, and for others linear terms in the quantum fluctuations (tadpoles) can be present due to the momentum conservation constraint.

A problem which is similar in some aspects to ours arises in $\lambda \varphi^4$ theory, related to the arbitrary choice of the kink propagation velocity. Due to Lorentz invariance the classical action in $\lambda \varphi^4$ theory is independent of ν (ν can be changed by a Lorentz transformation). However, the classical energy and momentum depend on ν , the different terms in the expansion of the Hamiltonian are then ν -dependent and to check the independence of ν of the sum is highly non trivial. Since ν is introduced essentially as a Lagrange multiplier (see Eq.(IV-31)), the quantization is usually done by choosing ν in such a way that the constraint is saturated at the classical level, i.e. $\mathcal{P}_\alpha(\nu) = k$. Only in this case the moving kink is a minimum of the constrained action and the expansion of the Hamiltonian does not contain linear terms in the shifted fields. If ν is not chosen in this way the constraint equation introduces tadpole terms which are perturbative in the sense that they are proportional to the small coupling constant λ , and which must be explicitly

summed. The result of this summation has a ν dependence that compensates the ν dependence of the classical energy.

The previous remarks may be intuitively understood as follows :
in $\lambda \varphi^4$ theory, the kink energy is given by

$$E_{\text{kink}}(k) = \sqrt{k^2 + (M_0 + \Delta M)^2} + O(\lambda) \quad (\text{V-1})$$

where $M_0 = O(1/\lambda)$ and ΔM is a renormalization mass^(*). If ν is chosen to saturate the constraint, with the result

$$\nu = \frac{k}{\sqrt{k^2 + M_0^2}} \quad (\text{V-2})$$

then the result for the Hamiltonian that is obtained is equivalent to the expansion in ΔM of Eq.(V-1). For other values of ν , the form of the Hamiltonian is equivalent to an expansion of $E_{\text{kink}}(k)$ in powers of a combination of ΔM and the fraction of k not saturated by p_a .

In our case, the situation is complicated by two facts. In the first place the theory is not Galilean invariant, and therefore the classical action depends on ν . As a consequence, we do not expect a simple particle-like dispersion law to hold for an arbitrary value of ν , as it is the case in $\lambda \varphi^4$ theory. On the other hand, the classical momentum is imaginary, and consequently we cannot hope to saturate the constraint $k = i p_a$ with our classical solutions for physical (real) values of k . This implies that in general we are bound to have tadpoles coming from the momentum conservation constraint.

(*) see, for instance, Eqs.(IV-15) and (IV-24) of Baacke and Rothe, ref.⁽⁷⁾.

Fortunately, it turns out that we can dispose of both difficulties at once by choosing to quantize the theory around solutions of class (a) exhibited in Fig. 3 - namely, the ones inside the triangle - having a propagation velocity ν such that $\nu \rightarrow 2^-$. That is to say, we pick the classical solutions of class (a) having a vanishing classical momentum $P_{cl} = O((2-\nu)^2)$, according to our estimates of section III. We shall then show that no tadpole occurs in this case, for any value of the quantization momentum k , and that the phonon-like dispersion formula

$$E_{cl} - \nu \sqrt{\kappa' \mu} P_{cl} = A_{cl} = O[P_{cl} \cdot (2-\nu)] \rightarrow 0 \quad (V-3)$$

is not destroyed by the quantum fluctuation, in the sense that they do not introduce, for example, an energy gap.

A first intuitive justification for the absence of tadpoles can be given by the following argument. Going back again to the general form of the energy given by Eq.(IV-54), we can see that tadpoles will in general show up through the A_0^2 term, which is fixed by the momentum conservation constraint and given by Eq.(IV-31). Indeed, it is easy to see that A_0^2 contains linear terms in the quantum fluctuation η . Let us first discuss the role played by the A_0^2 term at the purely classical level, by neglecting the quantum fluctuation η . Let us then fix for simplicity $k = 0$ and compare the energy gap $E_{min}(k=0)$ predicted by Eq.(IV-53) for two different values of the velocity ν , i.e. $\nu = 2$ and $\nu < 2$. We expect them to be the same, at the expense of the fluctuation in the direction of χ_0 , that is to say, A_0 , which corresponds to changing in the classical action.

This is indeed what happens, because for $\nu = 2$ both A_{cl} and A_0^2 are identically zero (if $\eta = 0$) and $E_{min}(k=0) = 0$. If, instead, $\nu < 2$, then A_0 as given by (IV-31) turns out to be of order $\sqrt{P_{cl}}$, and the contribution of the A_0^2 is of the form

$$\frac{\partial A_{cl}}{\partial P_{cl}} \cdot P_{cl} \quad (V-4)$$

which corresponds to rewriting the previous result for $\nu = 2$, $E_{min}(k=0) \simeq A_{ce}(P=0)$, by expanding $A_{ce}(P=0)$ around $P = P_{ce} \neq 0$. This shows then that the A_0 contribution is minimized at the classical level if A_{ce} is stationary with respect to P_{ce} , which happens only for $\nu \rightarrow 2^-$. We shall also show later in this section that also the quantum corrections to A_0 vanish in this limit.

The aim of this rather lengthy discussion was to motivate and partially justify our choice of the $\nu \rightarrow 2^-$ kink solutions inside the triangle to shift the reggeon fields. A full justification of our choice will emerge as we proceed along this section. We shall not consider the class outside the triangle. They correspond to classical fields which are not monotonous functions of p , implying that the zero mode wave function has nodes. At the quantum level this means that the zero mode is not the ground state, and therefore they are not stable. Moreover, these solutions do not yield for A_0^2 the result (V-4), not even for $\nu \rightarrow 2^-$, implying that one has also tadpoles in this case.

As for the family of solutions of the single-field equations, the action, as well as the energy and the momentum, vanish for any ν . For $\nu < 2$, the zero mode wave function again has nodes, and we expect them to be unstable. The case $\nu > 2$ and especially $\nu \rightarrow 2^+$ is the one closest to our choice ($\nu \rightarrow 2^-$). The main drawback of this possibility is that it does not satisfy explicitly the $p \leftrightarrow q^+$ symmetry of RFT. Therefore, the quantum states dual to the ones found by shifting around, say, $(p, q) = (\frac{\mu}{\lambda} \varphi, 0)$ are to be obtained by performing a different shift around $(0, \frac{\mu}{\lambda} \bar{\varphi})$. This poses technical problems and we prefer to use an explicitly symmetric procedure.

We would like to mention at this point the entirely different choice of classical solution made by White⁽¹¹⁾, to discuss the same problem we are concerned with. In our language, he chooses to quantize around the limiting solution $\nu \rightarrow -\infty$ of the class (a), which is of the form

$$\begin{aligned} \varphi(p) &\xrightarrow{\nu \rightarrow \infty} \frac{1}{2} (1 - \tanh y) \\ \bar{\varphi}(p) &\xrightarrow{\nu \rightarrow \infty} \frac{1}{2} (1 + \tanh y) \end{aligned} \quad (V-5)$$

i.e. reduces to the "instanton" along the hypotenuse of the triangle in Fig. 3. As we have already discussed in section III, such a solution has the opposite time evolution to the ones we consider, that is to say, it satisfies

$$\bar{\varphi}(y = -\infty) = 0 \quad ; \quad \varphi(y = +\infty) = 0 \quad (V-6)$$

which are to be compared with the perturbative boundary conditions satisfied by our kink solutions

$$\varphi(y = -\infty) = 0 \quad ; \quad \bar{\varphi}(y = +\infty) = 0 \quad (V-7)$$

White claims that, in order to have analyticity in the parameter μ for an infinite volume V in impact parameter space the boundary conditions (V-7) have to be changed, and that (V-6) ends up to be the only possible choice. Our approach in this and in the preceding papers ⁽¹⁾⁽²⁾ has been, on the contrary, to keep the boundary conditions fixed and demand analyticity in μ only at a finite volume V of impact parameter space, and fixed rapidity Y . Analyticity in μ can be proved in this case ⁽⁸⁾⁽⁹⁾, and the matching with ordinary perturbation theory can be argued ⁽⁹⁾. The S-matrix (II-17) is in this case stable against small, μ independent variations of the boundary condition (V-7) because the vacuum $|\Phi_0\rangle$ is the only ground state at finite V . When $V \rightarrow \infty$ a phase transition has been found ⁽¹⁾ due to the ground state degeneracy, and a non-analytic behavior in μ shows up at some critical value μ_{crit} .

We therefore consider White's approach as basically different from ours, and from our point of view is open to criticism. Notice for example that the $\nu \rightarrow -\infty$ kink has an infinite action per unit time that White subtracts away. This infinite action per unit time corresponds, in the language of section II, to the exponentially vanishing normalization factor of the new ground state $|\Phi_1\rangle$, given by Eq.(II-15). In subtracting it away, White keeps a finite norm for states which for us are of zero norm. It is then easily understandable that, if his shifting procedure is convergent, he will get a spectrum completely different from ours.

V. b - Eigenfrequencies and Eigenfunctions for $v \rightarrow 2^-$.

We now discuss the calculation of the eigenfrequencies and eigenfunctions corresponding to quantum fluctuations around a classical kink solution with velocity $v \rightarrow 2^-$. In order to deal with the eigenvalue problem defined by Eqs. (IV-37 a,b) we shall rewrite them in a more suitable form by eliminating the first derivative from the differential operators involved. Let us then define the functions $f_n(\rho)$ and $\bar{f}_n(\rho)$ by

$$\xi_n(\rho) = \begin{pmatrix} F_n(\rho) \\ \bar{F}_n(\rho) \end{pmatrix} = c_n \begin{pmatrix} f_n(\rho) e^{\frac{v}{2}\rho} \\ \bar{f}_n(\rho) e^{-\frac{v}{2}\rho} \end{pmatrix} \quad (V-8)$$

where c_n is a normalization factor to be defined later on. Then, the relation (IV-38) that gives the components of $\xi_{-n}(\rho)$ in terms of those of $\xi_n(\rho)$ implies the obvious relation

$$f_{-n}(\rho) = \bar{f}_n(-\rho) \quad (V-9)$$

between the $(+\omega_n)$ and the $(-\omega_n)$ solutions. Introducing (V-8) in (IV-37 a,b), the system reduces to a coupled Schrödinger problem of the form

$$\left[-\frac{d^2}{d\rho^2} + W(\rho) - \frac{\omega_n}{\mu} \right] f_n(\rho) + 2\varphi(\rho) e^{-\frac{v}{2}\rho} \bar{f}_n(\rho) = 0 \quad (V-10a)$$

$$2\bar{\varphi}(\rho) e^{\frac{v}{2}\rho} f_n(\rho) + \left[-\frac{d^2}{d\rho^2} + W(\rho) + \frac{\omega_n}{\mu} \right] \bar{f}_n(\rho) = 0 \quad (V-10b)$$

where the potential $W(\rho)$, defined by

$$-\frac{d^2}{d\rho^2} + W(\rho) = e^{-\frac{\nu}{2}\rho} D(\nu) e^{\frac{\nu}{2}\rho} \quad (V-11)$$

is given explicitly in terms of the classical solution $\{\psi, \bar{\psi}\}$ by

$$W(\rho) = \frac{\nu^2}{4} - 1 + 2 [\psi(\rho) + \bar{\psi}(\rho)] \quad (V-12)$$

For small values of $(2 - \nu)$, the shape of the potential is sketched in Fig. 7. It looks roughly speaking like a well centered at the origin $\rho = 0$ of height 2 and range $2L$. Due to the fact that $\psi(0) \simeq \bar{\psi}(0) \simeq O(2-\nu)$ and that $\psi(\rho)$ behaves as $\psi(0) \exp(\frac{\nu}{2}\rho)$ in the perturbative region $\psi(\rho) \ll 1$, we can estimate the range L to be

$$L \simeq \log(2-\nu) \quad (V-13)$$

Moreover, at the bottom of the well, near $\rho = 0$, $W(\rho)$ takes a very small value of the order of $(2 - \nu)$.

We notice next that the coupled system of equations (V-10a,b) can be decoupled as $\nu \simeq 2^-$ because in this case one of the non-diagonal potentials is always small. Let us consider for example the region $\rho > 0$. From our previous discussions in Section III, we know that in this region $\psi(\rho)$ grows first like $\psi(0) \exp(\frac{\nu}{2}\rho)$ and then saturates at the value $\psi = 1$. Since the exponential growth of $\psi(\rho)$ is overcompensated by the factor $\exp(-\nu\rho)$, it follows that when $\rho > 0$ the nondiagonal potential $\psi(\rho) \exp(-\nu\rho)$ is of order $\psi(0)$, that is to say, $(2 - \nu)$. Always in the region $\rho > 0$, we can easily see that the other nondiagonal potential, $\bar{\psi}(\rho) \exp(\nu\rho)$, is of order 1. Indeed, from the known approximate solution for $\bar{\psi}(\rho)$ given by Eq.(III-32), it fol-

lows that

$$\bar{\psi}(\rho) e^{\sqrt{\rho}} \approx \dot{\psi}(\rho) \quad (\text{V-14})$$

and in the region $\rho \sim L$ $\dot{\psi}(\rho)$ is of order 1. The opposite results about the magnitude of the nondiagonal potentials hold in the region $\rho < 0$.

Let us start by considering positive energy solutions, $\omega_n > 0$. Because of the previous discussion, in the region $\rho > 0$ Eq.(V-10a) reduces to

$$D_S f_n(\rho) = \omega_n f_n(\rho) \quad (\text{V-15a})$$

where D_S is the Schrödinger operator

$$D_S = - \frac{d^2}{d\rho^2} + W(\rho) \quad (\text{V-15b})$$

while Eq.(V-10b) becomes an inhomogeneous equation that determines $\bar{f}_n$ in terms of f_n . Indeed, D_S being a positive semidefinite operator, there is no non-trivial solution of the homogeneous equation $(D_S + \omega_n) \bar{f} = 0$ when $\omega_n > 0$. Also because of this fact, when we consider the region $\rho < 0$ where the inhomogeneous term - proportional to $\bar{\psi} \exp(\sqrt{\rho})$ - can be neglected, we conclude that the only solution of (V-10b) is

$$\bar{f}_n(\rho) = 0 \quad ; \quad \rho < 0 \quad ; \quad n \neq 0 \quad (\text{V-16})$$

Plugging this result back into Eq.(V-10a), we obtain again Eq.(V-15) for $\rho < 0$. Therefore, Eq.(V-15) is valid everywhere, i.e. in the whole region $-\infty < \rho < \infty$.

We are therefore left with an ordinary Schrödinger equation involving a potential $W(\rho)$ which is approximately a well of height 2 and range L . When $(2 - \sqrt{5})$ is small, L is very large (c.f. Eq. (V-13)) and the low-lying eigen-

values will be approximately given, for $n > 0$, by

$$\omega_n = \mu k_n^2 \quad ; \quad k_n \simeq \begin{cases} \pi m/L & , \quad n = 2m-1 \\ \frac{2m+1}{2L}\pi & , \quad n = 2m \end{cases} \quad (\text{V-17})$$

On the other hand the ground state ω_0 should be of order $O(2 - \nu)$, because we know that the exact coupled equations have an exact zero mode arising from translational invariance, and they differ from the present approximate ones by $O(2 - \nu)$. In deriving the zero-mode eigenfunctions from translational invariance one realizes that the shape of the potential is extremely important in order to have the translational mode correctly placed at $\omega = 0$, so the estimate (V-17) cannot be valid in this case. The fact that the ground state ω_0 of our Schrödinger problem is related to the zero mode arises of course from the observation that the zero-mode eigenfunctions, given by (IV-16), have no nodes. The stability of our classical solutions - and the fact that D_S is positive semidefinite - is then guaranteed.

Since the Schrödinger potential $W(\rho)$ is an even function of ρ , the Schrödinger eigenfunctions $f_n(\rho)$ will satisfy the parity condition

$$f_n(-\rho) = (-1)^n f_n(\rho) \quad (n=0,1,\dots) \quad (\text{V-18})$$

They represent a complete orthonormal set of functions satisfying

$$\sum_{n=0}^{\infty} f_n(\rho) f_n(\rho') = \delta(\rho - \rho') \quad (\text{V-19})$$

$$(f_n, f_m) = \int_{-\infty}^{\infty} d\rho f_n(\rho) f_m(\rho') = \delta_{n,m} \quad (\text{V-20})$$

In what concerns $\bar{f}_n(\rho)$, we have already established that it is determined by the unique solution of Eq.(V-10b)

$$\begin{aligned}
\bar{f}_n(\rho) &= [D_s + \omega_n]^{-1} \cdot 2\bar{\varphi}(\rho) e^{v\rho} f_n(\rho) \\
&= \sum_{m=0}^{\infty} \frac{f_m(\rho) \cdot (f_m, 2\bar{\varphi} e^{v\rho'} f_n)}{\omega_n + \omega_m} \quad (V-21)
\end{aligned}$$

where we have used the completeness relation (V-20) and the obvious notation

$$(f_m, 2\bar{\varphi} e^{v\rho'} f_n) = \int_{-\infty}^{\infty} d\rho' f_m(\rho') 2\bar{\varphi}(\rho') e^{v\rho'} f_n(\rho') \quad (V-22)$$

We notice that, as expected, Eq.(V-22) continues smoothly into Eq.(V-16) for $\rho < 0$. Indeed, the support of $\bar{\varphi}(\rho') \exp(v\rho')$ is ρ' large and positive.

Having discussed the solutions of Eqs.(V-10a,b) for $\omega_n > 0$, we can use Eqs.(IV-38, 39) to derive at once the eigenfunctions for the case $\omega_n < 0$.

$$\xi_{-n}(\rho) = C_{-n} \begin{pmatrix} F_{-n}(\rho) \\ \bar{F}_{-n}(\rho) \end{pmatrix} = C_{-n} \begin{pmatrix} F_n(-\rho) \\ \bar{F}_n(-\rho) \end{pmatrix} = C_{-n} \begin{pmatrix} \bar{f}_n(-\rho) e^{\frac{v}{2}\rho} \\ f_n(-\rho) e^{-\frac{v}{2}\rho} \end{pmatrix} \quad (V-23)$$

Using the reflection properties of $f_n(\rho)$, given by Eq.(V-18), and fixing $C_{-n} = (-1)^n C_n$, we can rewrite $\xi_{-n}(\rho)$ as

$$\xi_{-n}(\rho) = C_n \begin{pmatrix} e^{\frac{v}{2}\rho} \bar{g}_n(\rho) \\ e^{-\frac{v}{2}\rho} f_n(\rho) \end{pmatrix} \quad (n > 0) \quad (V-24)$$

where $\bar{g}_n(p)$ is now given by

$$\begin{aligned}\bar{g}_n(p) &= \sum_{m=0}^{\infty} \frac{f_m(p) (f_m, 2\psi e^{-\psi p'} f_n)}{\omega_n + \omega_m} \\ &= (-1)^n \bar{f}_n(-p)\end{aligned}\tag{V-25}$$

and, of course, $\bar{g}_n(p) \simeq 0$ for $p > 0$. We can now use the normalization condition (IV-41) to determine c_n :

$$\begin{aligned}\delta_{n,m} &= (\bar{\xi}_m, \xi_n) = c_n^2 \int_{-\infty}^{\infty} [f_n(p) f_m(p) - \bar{f}_n(p) \bar{g}_m(p)] \\ &= c_n^2 \delta_{n,m} \Rightarrow c_n = 1\end{aligned}\tag{V-26}$$

where use has been made of the orthogonality relation (V-20) and of the fact that the functions $\bar{f}_n(p)$ and $\bar{g}_n(p)$ have everywhere a negligible overlap, namely, $\bar{f}_n(p) \cdot \bar{g}_n(p) \simeq 0$.

We now turn to a discussion of the zero-mode. We have to identify the zero-mode wave functions ξ_0 and its conjugate vector χ_0 , defined in terms of the classical solutions by Eqs. (IV-16) and (IV-19), with the eigenfunctions we obtained for $n = 0$. Notice that we have two eigenfunctions available, by setting $n = 0$ in ξ_n or ξ_{-n} . We discuss this problem at length in Appendix B, where the approximate forms of ξ_0 and χ_0 are given for $(2 - \nu)$ small. We are more concerned now with an estimate of the normalization factors N_0 and L_0 as $\nu \rightarrow 2^-$, given the role they play as coefficients of the A_0^2 term in the general form of the energy given by Eq. (IV-54).

In order to compute N_0 we recall the fact (see Appendix B) that, for $(2 - \nu)$ small, the correct identification of ξ_0 is

$$\xi_0(p) = \begin{pmatrix} f_0(p) e^{\frac{\sqrt{r}}{2} p} \\ \bar{f}_0(p) e^{-\frac{\sqrt{r}}{2} p} \end{pmatrix} \theta(p) - \begin{pmatrix} \bar{f}_0(-p) e^{\frac{\sqrt{r}}{2} p} \\ f_0(-p) e^{-\frac{\sqrt{r}}{2} p} \end{pmatrix} \theta(-p) \quad (\text{V-27})$$

Therefore, by identifying for $p > 0$ the upper component of ξ_0 with the exact result (IV-16) we obtain

$$f_0(p) = N_0 \dot{\varphi}(p) e^{-\frac{\sqrt{r}}{2} p} ; (p > 0) \quad (\text{V-28})$$

and, since $f_0(p)$ is an even function

$$1 = \int_{-\infty}^{\infty} dp f_0^2(p) = 2 N_0^2 \int_0^{\infty} dp \dot{\varphi}^2 e^{-\sqrt{r} p} \quad (\text{V-29})$$

This integral has already been discussed in section III (c.f. Eq. (III-34)) and proved to be of order $\varphi(0)^2$ up to irrelevant logarithms. Therefore,

$$N_0 \approx O(\varphi_0^{-1}) \quad (\text{V-30})$$

The remaining normalization constant, L_0 , is determined from the normalization condition (IV-21)

$$\begin{aligned} 1 = (\bar{\chi}_0, \xi_0) &= L_0 N_0 \int_{-\infty}^{\infty} dp \left[\frac{\partial \bar{\varphi}}{\partial r} \frac{\partial \varphi}{\partial p} - \frac{\partial \varphi}{\partial r} \frac{\partial \bar{\varphi}}{\partial p} \right] = \\ &= 2 L_0 N_0 \int_{-\infty}^{\infty} \frac{\partial \bar{\varphi}}{\partial r} \cdot \frac{\partial \varphi}{\partial p} dp \end{aligned} \quad (\text{V-31})$$

We have made use of the fact that $\dot{\varphi}(\rho) = -\dot{\varphi}(-\rho)$, and that

$$\frac{\partial \varphi(\rho)}{\partial \nu} = \frac{\partial \bar{\varphi}(-\rho)}{\partial \nu} \quad (\text{V-32})$$

This last statement is just a consequence of the reflection property (III-7).
Going back to Eq.(V-31), we observe next that, for $\rho > 0$

$$\frac{\partial \bar{\varphi}(\rho)}{\partial \nu} = \frac{\partial \bar{\varphi}(0)}{\partial \nu} e^{-\frac{\nu}{2}\rho} - \frac{\nu}{2} \bar{\varphi}(0) e^{-\frac{\nu}{2}\rho} \quad (\text{V-33})$$

The second term of the right-hand side is of order φ_0 , but the first is of order $\varphi_0/(2-\nu)$. Since we know also that $\dot{\varphi} e^{-\frac{\nu}{2}\rho}$ is of $O(2-\nu)$ for $\rho > 0$, we conclude that

$$\frac{1}{L_0 N_0} = O\left(\frac{\varphi_0^2}{2-\nu}\right) \quad (\text{V-34})$$

and therefore, from Eq. (V-29) we obtain

$$\frac{L_0}{N_0} = O(2-\nu) \quad (\text{V-35})$$

We have now all the information we need in order to evaluate the energy spectrum (IV-53) in the limit $\nu \rightarrow 2^-$. We note that :

1) The classical action density $\mathcal{Q}_c = O[\mathcal{P}_c(2-\nu)]$ vanishes as a consequence of the phonon-like dispersion law.

2) The constraint term $(\mu A_0^2 L_0 / N_0)$ also vanishes, because of (V-35). Notice that, away from $\nu = 2$, A_0^2 contains a tadpole which by Eqs.(IV-31,32) is proportional to $N_0^{-1}(\bar{\eta}, \partial_\rho \chi_0)$. It is only for $\nu \rightarrow 2^-$ that - since $N_0^{-1} = O(\varphi_0)$ - A_0^2 is quadratic in η , and moreover gives a vanishing contribution to the energy because of the behavior of (L_0/N_0) given by (V-35).

3) The zero-point energy

$$\Delta E_0 = \frac{1}{2} \sum_{n>0} [\omega_n(\nu) - \omega_n^0(\nu)] \quad (\text{V-36})$$

vanishes as well, because $\omega_n^0(\nu) \rightarrow \omega_n(\nu)$ for $\nu \rightarrow 2^-$. We have in fact previous-

ly shown that $\nu \rightarrow 2^-$ the eigen-value problem for ω_n reduces to (V-15), whose eigenfrequencies, by (V-11), are ω_n^0 . The correction $(\omega_n - \omega_n^0)$ can be computed to first order of perturbation theory, and the result for ΔE_0 is

$$\Delta E_0 = \sum'_{n,k} \frac{(f_n, 2\psi e^{-\nu p} f_k)(f_k, 2\psi f_n)}{\omega_n + \omega_k} = O(2-\nu) \quad (V-37)$$

where Σ' excludes the term $n = k = 0$ in the sum.

By the results above, the final formula for the energy becomes

$$E = 2i\sqrt{\alpha'\mu} k + \mu \sum_{n>0} \omega_n \bar{A}_n A_n + O(\lambda) \quad (V-38)$$

Note that the quantum fluctuations have a small gap $\omega_1 \simeq \pi/l^2$ and for $\nu \rightarrow 2^-$ ($L \simeq |\ln(2-\nu)| \rightarrow \infty$) give rise to a continuum headed by the kink state.

The above formula for the spectrum of RFT represents the main result of this section. Let us analyze the procedure that allowed us to establish it. We have seen the importance of keeping the triple pomeron coupling λ and $(2-\nu)$ as independent parameters. As we discussed at the beginning, the quantization of RFT should be independent of ν and therefore independent of the classical solution chosen to shift the reggeon fields. However, it is only for $\nu \rightarrow 2^-$ that we succeeded in having, after shifting, a bona fide quadratic form, and the reason for this to happen is easily understood. For every $\nu < 2$ there is classical kink solution that represents an unconstrained minimum of the action corresponding to some classical momentum P_{cl} . If the theory could be quantized in such a way that $k = P = i P_{cl}$, then of course the expansion around the classical solution would involve only quadratic terms. This is not possible in our case, but the results of this section show that at the optimal value $\nu = 2$ the term in the Hamiltonian arising from the constraint not only is quadratic in the quantum fluctuation but also gives a vanishing contribution. Truly enough, this is achieved at the expense of having a continuum

spectrum for the quantum fluctuations.

The physical meaning of $\nu = 2$ is that the classical action density has an extremum at $P_c = 0$, where the phonon-like dispersion formula

$$E_c = 2\sqrt{a'\mu} P_c + O(P_c^{3/2}) \text{ holds. This dispersion law is preserved in the quantum case because there is no mass renormalization of the kink arising from zero-point fluctuations. We therefore find that kink states with } \nu = 2$$

give rise to a zero energy gap for $k \rightarrow 0$. It must be remarked that, if our shifting method is properly convergent, the kink states and its quantum fluctuations are bona fide eigenstates of the non-hermitian RFT Hamiltonian. The same conclusion was reached in the spin model⁽²⁾ where, only in the infinite volume limit, the kink states were found to have the dispersion law

$$E(k) \propto [1 - \exp(-ika)] \simeq ika. \text{ However the continuous spectrum arise in that case in a discontinuous way, the finite volume spectrum being highly degenerate, but with a finite } k = 0 \text{ gap. The same situation seems to occur in Ref.(4). Therefore, the occurrence of the phonon-like dispersion law appears more neatly in the present method than in the spin model.}$$

VI - QUANTUM STATES AND THE EXPANDING DISK.

Let us summarize the structure of the lower lying spectrum. For $\mu > \mu_c$ we have the two degenerate ground states discussed in Section I (the vacuum $|\Phi_0\rangle$ and its companion $|\Psi_0\rangle$ of Eq. (II-8)) and, furthermore, two bands of states whose energy gap $E = \pm 2ik \sqrt{\alpha' \mu}$ vanishes for $k \rightarrow 0$. Both $|\Psi_0\rangle$ and the kink states can be thought of as the lowest members of families of states obtained by a quadratic expansion of the Hamiltonian around the corresponding classical solutions.

Let us stress again the difference between our situation and the similar one occurring, e.g., in $\lambda \varphi^4$ theory. In an hermitian field theory the expansions around the space independent fixed points and the kinks define superselected Hilbert spaces. The existence of the topological charge $\varphi(\infty) - \varphi(-\infty)$ is just a sign of that. In RFT we shall find instead that also the kink states, as the ground state $|\Psi_0\rangle$, are connected to the perturbative vacuum $|\Phi_0\rangle$ by field operators. This is due to the peculiar role of the metric operator m explained in Section II, and implies that the vacuum $|\Phi_0\rangle$, $|\Psi_0\rangle$ and excitations and kink and excitations belong to the same (metric) space.

Two logical consequences follow from the previous remark. On one hand all states found so far are to be considered together in the time evolution operator and in the completeness relation. This is what we shall do in the following, and we shall make sure that we do not double-count by checking that energy eigenstates are properly orthogonalized. On the other hand it is not possible, in our case, that the semiclassical expansion we have used be equivalent to a "bona-fide shift". By this we mean that some quantum fluctuations must be large when we compute matrix elements of a general operator. This also means that, in principle, one should be able to find, say, the kink state, starting from vacuum excitations if large fluctuations are allowed (in a sense, this has been done in the spin model).

Nevertheless, the basis of our approach is that some classical solutions exhaust all the non-perturbative content of the quantum states, and that therefore a quadratic expansion is acceptable - at least locally - for the

Hamiltonian operator. This is enough in order to identify the lowest lying states mentioned before, which in this respect are analogous to the breather modes found in the vacuum sector of $\lambda \psi^4$ theory. A different question is how to take into account excited states around the classical solutions. We shall discuss this point briefly later on, and we shall first concentrate on finding the contribution of kink states in the completeness relation.

VI.a - The quantum kink state

The quantum kink states correspond to the ground state of momentum k of the shifted Hamiltonian $\mathcal{H}_2$ in Eq. (V-42). They are defined by

$$\mathcal{H}_2 |k\rangle = 0 \quad , \quad P |k\rangle = k |k\rangle \quad (\text{VI-1})$$

The second condition is automatically implemented by writing the kink states as

$$|k\rangle = \int_{-\infty}^{\infty} da \, e^{-ika} e^{iPa} |\xi_{cl}\rangle \quad (\text{VI-2})$$

and the kink state $|\xi_{cl}\rangle$ is now determined from the first condition by demanding

$$A_n |\xi_{cl}\rangle = 0 \quad (\text{VI-3})$$

We remind that the gauge condition $\bar{A}_0 = (\bar{\chi}_0, \xi) = 0$ has fixed the arbitrary traslation parameter a that occurs in the classical solution to, say, $a = 0$. Notice that Eq. (VI-3) assumes that $\mathcal{H}_2 \approx H - vP - \Delta E_0$ commutes with P , which is true approximately to the order $1/\mu$ to which we are working.

Let us recall that, from Eqs. (IV-44) and (IV-36)

$$A_n = (\bar{\xi}_{-n}, \xi - \xi_{cl}) = (\bar{F}_{-n}, p - \frac{\mu}{\lambda} \varphi) = (F_{-n}, q - \frac{\mu}{\lambda} \bar{\varphi}) \quad (VI-4)$$

where F_{-n} and $\bar{F}_{-n}$ are given explicitly in Eqs. (V-23) and V-24,25) for the $\nu \rightarrow 2^-$ solutions we are using. By representing the operator $p(q)$ by $-\frac{\partial}{\partial q(q)}$, we can look for solutions of Eq. (VI-3) of the form

$$|\xi_{cl}\rangle = \exp\left[-\frac{\mu}{\lambda}(\alpha, q) + \frac{1}{2}(q, \beta q)\right] |\Phi_0\rangle \quad (VI-5)$$

where $\alpha(q)$ is an unknown function and β an operator with symmetric kernel $\beta(q, q')$. Then, by using Eq. (VI-4) we find that Eq. (VI-3) reduces to

$$(\bar{F}_{-n}, \alpha - \varphi) + (F_{-n}, \bar{\varphi}) = 0 \quad (VI-6)$$

$$(\bar{F}_{-n}, \beta q) - (F_{-n}, q) = 0 \quad (VI-7)$$

By using the explicit expressions (V-24,25) for F_{-n} and $\bar{F}_{-n}$, as well as the fact that $\bar{\varphi}(q) \exp(\frac{\mu}{2} q)$ is of order $O(2-\nu)$, it follows that the term $(F_{-n}, \bar{\varphi})$ is of order $O(2-\nu)$. We then find

$$\alpha(q) = \varphi(q) + \text{constant} \cdot \dot{\varphi}(q) + O(2-\nu) \quad (VI-8)$$

where the $\dot{\psi}(g)$ term comes from the fact that Eq. (VI-6) does not determine the zero-mode components of both $\alpha(g)$ and $\beta(g, g')$. These components correspond to an infinitesimal translation in (VI-5). Since we have to perform this translation anyway in order to obtain momentum eigenstates, we can assume that α and β have no zero mode components. With this specification

$$\alpha(g) = \psi(g) + O(2-v) \quad (\text{VI-9})$$

In the same way we obtain from Eq. (VI-7)

$$\beta(g, g') = \sum_{n, m \neq 0} \beta_{nm} e^{\frac{v}{2}g} f_n(g) e^{\frac{v}{2}g'} f_m(g') \quad (\text{VI-10})$$

$$\beta_{nm} = \beta_{mn} = - \frac{(f_m, 2\psi e^{-vg} f_n)}{\omega_n + \omega_m} \quad (\text{VI-11})$$

By putting Eqs. (VI-2), (VI-5) and (VI-9) together, we finally get

$$\begin{cases} |k\rangle = \int da e^{-ika} \exp\left[-\frac{\mu}{\lambda}(\psi_a, g) + \frac{1}{2}(g, \beta_a g)\right] |\Phi_0\rangle \\ \psi_a = \psi(g-a) \quad ; \quad \beta_a = \beta(g-a, g'-a) \end{cases} \quad (\text{VI-12})$$

The field ψ is approximately, for $v \rightarrow 2^-$, a solution of

the single-field equation for $g > 0$:

$$v \dot{\varphi} = \varphi - \varphi^2 + \ddot{\varphi} \quad ; \quad \dot{\varphi}(0) = \varphi(0) \quad (\text{VI-13})$$

and the kernel $\beta(g, g')$ is a solution of the differential equation

$$\begin{aligned} & [v \partial_g - \partial_g^2 + 2\varphi(g) - 1] \beta(g, g') + [v \partial_{g'} - \partial_{g'}^2 + 2\varphi(g') - 1] \beta(g, g') = \\ & = - 2 \varphi(g) \delta(g - g') + \langle 2\varphi \rangle_{\infty} f_0(g) f_0(g') \end{aligned} \quad (\text{VI-14})$$

It is interesting to notice that - aside from the elimination of the zero mode in Eq. (VI-14) - both equations (VI-13) and (VI-14) can be derived from the generalized coherent state approach developed in Appendix A. Indeed, they are essentially the first two correlation functions, in the particular case in which these correlation functions are kink-like, i.e., they depend on y only through the body-fixed variable g . As discussed in Appendix A, these equations are the semiclassical approximations to the exact equations for the correlation functions, and therefore they do not include renormalization effects^(*). This means that, provided we limit ourselves to the null

(*) This indicates a method for including renormalization effects in the construction of the states. For instance, the addition of the term $\beta(g, g')$ in (A-4) to (VI-12) would give a mass renormalization. The ordered phase would occur for $\mu > \mu_c = O\left(\frac{1}{\sqrt{\alpha}\mu}\right)$, as already found in the spin model.

gap states with $v \approx 2$, the approximation of expanding the equations of motion in powers of q is essentially correct for the calculation of right-eigenstates (c.f. Section II).

VI.b - Approximate evolution operator and S-matrix

We want now to investigate the normalization properties of kink states. We first notice that for each kink state with speed $v=2$ and energy $2ik$, which we shall denote by $|k\rangle_+$, there is an antikink state $|k\rangle_-$ with speed -2 and energy $-2ik$, due to the reflection symmetry (III-6) and (III-7). This antikink state $|k\rangle_-$ is simply obtained by the replacement $g \rightarrow -g$ in $\varphi(g)$ and β . The states $|k\rangle_+$ and $|k\rangle_-$ propagate in space in opposite directions: left for $|k\rangle_+$ and right for $|k\rangle_-$, corresponding to a single direction in time, namely, the one for which $p \rightarrow \frac{\mu}{\lambda}$ as $y \rightarrow \infty$. This is exhibited in Fig. 8.

For the purpose of this Section we shall neglect the correlation β with respect to $\frac{\mu}{\lambda} \varphi$, and we shall write

$$|k\rangle_+ = \int da e^{-ik a} \exp\left[-\frac{\mu}{\lambda} \int dg \varphi(g-a) q(g)\right] |\Phi_0\rangle \quad (\text{VI-15})$$

$$|k\rangle_- = \int da e^{-ik a} \exp\left[-\frac{\mu}{\lambda} \int dg \varphi(-g+a) q(g)\right] |\Phi_0\rangle \quad (\text{VI-16})$$

They satisfy:

$$H |k\rangle_{\pm} = \pm i v_0 k |k\rangle_{\pm} \quad (\text{VI-17a})$$

$$P |k\rangle_{\pm} = k |k\rangle_{\pm} \quad (\text{VI-17b})$$

where we have introduced the critical speed $v_c \equiv 2 \sqrt{\hbar^2/\mu}$. We want now to find the left eigenstates corresponding to (VI-17a,b). By introducing the parity operator I , which obviously commutes with H and anticommutes with P , we note that from (VI-17a) and the definition of the metric operator m (see Eq. (II-11)) it follows that

$$\langle -k | I H = \langle -k | H I = i v_c k \langle -k | I \quad (\text{VI-18})$$

$$\langle -k | I P = k \langle -k | I \quad (\text{VI-19})$$

Moreover, one can check that:

$$\langle k | I = \langle k |_+ , \quad \langle -k | I = \langle k |_- \quad (\text{VI-20})$$

which means that in the left eigenstates $+$ and $-$ exchange their roles. This fact is confirmed by the evaluation of the scalar products according to the metric operator (II-11). While $|k\rangle_+$ and $|k\rangle_-$ are themselves null states in the thermodynamic limit

$$\langle k | k' \rangle_+ = \langle k | k \rangle_- = \exp -\frac{1}{2} \left(\frac{y}{\lambda} \right)^2 V \xrightarrow{V \rightarrow \infty} 0 \quad (\text{VI-21})$$

the scalar product $\langle k | k \rangle_+$ is not trivial, because it contains only a partial overlap of $\psi(y)$ and $\psi(-y)$. Indeed, one easily obtains

$$\langle k | k \rangle_+ = 2\pi \delta(k-k') N(k) \quad (\text{VI-22})$$

where

$$N(k) = \int da e^{-ik a} \exp \left[-\left(\frac{\mu}{\lambda}\right)^2 \int_0^L \psi(\xi) \psi(\xi-a) d\xi \right] \approx$$

$$\underset{\frac{L}{\mu} \rightarrow 0}{\approx} \int da e^{-ik a} \Theta(a) = \frac{1}{\epsilon + ik} \quad (\text{VI-23})$$

and we have roughly approximated the integral in the exponent by $|a| \Theta(L-a)$ for $L \ll \mu$ (*).

A final point concerns the orthogonality of $|k\rangle_{\pm}$ with the ground states $|\Phi_0\rangle$ and $|\Psi_0\rangle = |\Phi_0\rangle - |\Phi_1\rangle$ discussed in Section II. Of course the kink states are orthogonal to the ground states for $k \neq 0$, but in order to keep this property for $k=0$ we have to project away the vacuum component by subtracting a δ function part. Then, we define

$$|k\rangle'_+ = |k\rangle_+ - 2\pi \delta(k) |\Phi_1\rangle$$

$$|k\rangle'_- = |k\rangle_- - 2\pi \delta(k) |\Phi_1\rangle \quad (\text{VI-24})$$

(*) This is true if $\psi(\xi)$ is approximated by a step function centered at $\xi=0$. Strictly speaking $\psi(\xi)$ should be centered at the transition region $\xi=L$. This would give rise to an extra factor $\exp 2iLk$ in $N(k)$, which, however, cancels against similar factors in the matrix elements.

in such a way that the new states are automatically orthogonal to both $|\Phi_0\rangle$ and $|\Psi_0\rangle$.

By taking into account the normalization factors (VI-21) and (VI-22) we are now able to write the contribution of ground states and kinks to the completeness relation. We get:

$$e^{-HY} = |\Phi_0\rangle\langle\Phi_0| - |\Psi_0\rangle\langle\Psi_0| + \int_0^\infty \frac{dk}{2\pi} \left\{ -ik e^{i\nu_0 k Y} |k\rangle_+ \langle k|'_+ + ik e^{-i\nu_0 k Y} |k\rangle_- \langle k|'_- \right\} \quad (\text{VI-25})$$

From this completeness relation we can calculate the Pomeron Green's functions. We shall define the propagator as

$$D(B, Y) = - \langle \Phi_0 | p(0,0) e^{iPB} e^{-HY} q(0,0) | \Phi_0 \rangle \quad (\text{VI-26})$$

This definition is stable against addition of time dependent external sources $f(-y_0)$, $g(y_0)$ spread over a volume V , provided we put

$$D^{(\mu)}(B, Y) = \lim_{\mu \rightarrow 0} \lim_{V \rightarrow \infty, y_0 \rightarrow \infty} [\text{analytically continued } D^{(\mu)}(B, Y)_{f(-y_0), g(y_0)}] \quad (\text{VI-27})$$

where the analytic continuation from $\mu < 0$ to $\mu > 0$ is unambiguously made at finite space volume. In fact, $|\Phi_0\rangle$ is the true ground state at finite V and is automatically picked up by the $y_0 \rightarrow \infty$ limit^(*).

(*) Ambiguities both in the analytic continuation and in the ground state may arise only if the $V \rightarrow \infty$ limit is taken first.

With this proviso, the ground state contribution to (VI-26) is, by (VI-25) and (II-16)

$$D(B, Y) \xrightarrow[\substack{\mu Y \rightarrow \infty \\ B \text{ fixed}}]{} - \langle \phi_0 | p | \phi_1 \rangle (-1) \langle \phi_1 | q | \phi_0 \rangle = \quad (VI-28)$$

$$= \left(\frac{\mu}{\lambda} \right)^2$$

The limit (VI-28) is asymptotically reached at finite B . However, one should also have that $D(B, Y) \rightarrow 0$ for $|B| \rightarrow \infty$ at finite Y , by the cluster decomposition of the correlation function. Therefore, we need additional states of small momentum which cancel the contribution (VI-28) at large impact parameters. We shall now show that in fact the kink states provide their cancellation^(*).

By inserting the full expansion (VI-25) into (VI-26) we get in fact:

$$D(B, Y) = \left(\frac{\mu}{\lambda} \right)^2 - \int \frac{d^2 k}{2\pi} e^{-ikB} \left[\varphi_+(k) \bar{\varphi}_-(k) e^{i v_0 k Y} + \text{c.c.} \right] \quad (VI-28)$$

where $\varphi_{\pm}(k) = \langle \phi_0 | p(0) | k \rangle_{\pm}$. It is now easily realized that

$$\varphi_+(k) = \bar{\varphi}_-(k) = \frac{\mu}{\lambda} \left[\int d\alpha e^{-ik\alpha} \Theta(\alpha) - 2\pi \delta(k) \right] = -\frac{\mu}{\lambda} \frac{1}{\epsilon - ik} \quad (VI-29)$$

^(*) The same cancellation mechanism was independently exhibited in the particular case of RFT with "magic" quartic couplings studied in Ref. (4).

$$\psi_-(k) = \bar{\psi}_+(k) = \frac{\mu}{\lambda} \left[\int da e^{-ik a} \Theta(a) - 2\pi \delta(k) \right] = -\frac{\mu}{\lambda} \frac{1}{\epsilon + ik}$$

(VI-30)

Therefore, by using the inverse transforms:

$$\Theta(z) = \frac{1}{2\pi} \int dk e^{ikz} \frac{1}{\epsilon + ik} = \frac{1}{2\pi} \int dk e^{-ikz} \frac{1}{\epsilon - ik} \quad (\text{VI-31})$$

we finally get:

$$D(B, Y) = \left(\frac{\mu}{\lambda} \right)^2 \left[1 - \Theta(B - v_0 Y) - \Theta(-B - v_0 Y) \right] = \left(\frac{\mu}{\lambda} \right)^2 \Theta(v_0 Y - |B|)$$

($\mu Y \gg 1$, $\lambda \ll \mu$)

(VI-32)

This is the final result of our calculation. It clearly shows the role of the kink state in cancelling the ground-state contribution at large $|B|$. The Pomeron propagator of Eq. (VI-32) is a grey disc of opacity $\left(\frac{\mu}{\lambda} \right)^2$ and radius

$$R(Y) = v_0 Y = 2 \sqrt{\alpha' \mu} Y$$

(VI-33)

The J -plane structure of this result, in which kink excitations are neglected, can be analyzed as follows. The contribution of $|\psi_0\rangle$ is a fixed pole at $J=1$, $(J=1-E)$, while the kink and antikink are complex conjugate poles with residues which are singular for $k \rightarrow 0$. They combine together to give

$$\left(\frac{\lambda}{\mu}\right)^2 D(J, k) = \frac{1}{J-1} 2\pi \delta(k) + \left(\frac{1}{ik(J-1+ikv_0)} + c.c. \right) \quad (\text{VI-34})$$

$$\simeq \frac{1}{(J-1)(J-1+iv_0k)} + c.c. = \frac{2}{(J-1)^2 + v_0^2 k^2}$$

which is the $\mathbf{J}$ -plane structure of the expanding disk for $D_c = 1$. The introduction of kink excitations, as discussed later on, gives rise to $\mathbf{J}$ -plane cuts and the kink and antikink poles are really speaking the edge of those cuts.

The S-matrix corresponding to eikonal-type couplings, given by Eq. (II-17), can be evaluated by the same method and, for point-like sources is asymptotically given by^(*)

$$S(f, g; B, \gamma) \sim 1 - [1 - e^{-\frac{\mu}{\lambda} g}] [1 - e^{-\frac{\mu}{\lambda} f}] \Theta(v_0 \gamma - |B|) \quad (\text{VI-35})$$

as in the quantum spin model. This provides inside the disk a factorized opacity bounded by one. Notice also that this result for the S-matrix is really valid in the weak coupling limit, namely $\lambda \ll \mu$. For larger values of λ/μ , renormalization effects neglected here would eventually replace μ/λ by $\sigma \mu/\lambda$, where σ is an order parameter which vanishes at $\mu = \mu_c$. Moreover, the precise form of the asymptotic S-matrix changes due to correlation effects in $|\Psi_0\rangle$ (see Cardy, Ref. (3)).

(*) This simply follows from the approximate representation

$$\exp\left(-\frac{\mu}{\lambda} g \mathcal{H}(\beta)\right) \underset{\lambda \ll \mu}{\simeq} 1 - [1 - \exp(-g \frac{\mu}{\lambda})] \Theta(\beta)$$

VI.c - Excited states

Taking into account excited states is not easy to for several reasons. The first question of principle that arises is whether or not to count both the excitations around $|\Psi_0\rangle$ (shifted Pomerons) and those around the kink states. This difficulty is shared with other theories which have in the same superselection sector several minima of the action which are not perturbatively connected. We can advance some arguments which indicate that the expansion around the kinks could provide, at least approximately, all the elementary excitations of the theory.

Let us in fact try to identify the excited modes around the kink. We have in the first place the lower-lying states inside the well (see Fig. 7) with dispersion law $\omega_n = \mu k_n^2$, ($k_n^2 \ll 1$). These are a band of states with a small gap ω_1 , which collapse into a continuum in the limit $v \rightarrow 2^-$ ($L \rightarrow \infty$). The intuitive picture that we have of these states is that they correspond to unshifted Pomerons (excitations around the perturbative vacuum $|\Phi_0\rangle$) in the presence of the kink, and viewed from a reference frame moving at the critical speed v_0 . First of all, notice that the bottom of the well corresponds to the perturbative vacuum and that, as v approaches 2, our kink solutions spend a very long "time" $2L$ in this state. Secondly, let us see how an unshifted Pomeron looks like in the kink rest-frame. The transformation $b \rightarrow b + v_0 y$ is a Galilean transformation with imaginary velocity $iv_0 = 2i\sqrt{\alpha'\mu}$, because rapidity y corresponds to imaginary time. Correspondingly, the Pomeron dispersion law can be rewritten in terms of momentum k' in the moving frame as follows

$$\omega(k) = -\mu + \alpha' k^2 = -\mu + \alpha' \left(k' + \frac{iv_0}{2\alpha'}\right)^2 = iv_0 k' + \alpha' k'^2$$

(VI-36)

where $iv_0/2\alpha'$ is the center of mass momentum corresponding to a Pomeron "mass" $1/2\alpha'$.

Formally, this is exactly the spectrum that we get by quantizing around the kink if k' is identified with $\sqrt{\frac{\alpha}{2}} k_n$. However, since $k' = k - i v_0 / 2\alpha'$ - as it must be for free Pomerons - Eq. (VI-36) is just a trivial restatement of the original dispersion law. What happens in our case is that unshifted Pomerons are in interaction with the kink, which eats up their momentum in such a way that k' is not fixed by the momentum conservation constraint, and can be interpreted in fact as $\sqrt{\frac{\alpha}{2}} k_n$. Momentum is taken away by the zero mode A_0 which, according to the results of Section V, does not contribute to $\omega(k)$ for $\nu \rightarrow 2^-$. This is the mechanism by means of which the kink stabilizes the unshifted Pomerons.

It is also easy to understand the higher modes $\omega_n > 2\mu$, which correspond to a true continuum above the kink well. They are essentially fluctuations around the value $\varphi = \mu/\lambda$ of the fields, and can therefore be interpreted as shifted Pomerons in interaction with the kink, viewed also from the kink rest frame. Their characteristic frequency is shifted to 2μ by the same mechanism that pushes the unshifted Pomerons up to zero frequencies, namely, the transformation of energies under a Galilean transformation. To sum up, since we have around the kink both shifted and unshifted Pomerons (with dispersion law VI-36) we feel that these excitations can provide all relevant corrections.

Another question of principle that we have to face is the evaluation of corrections introduced by the excited states to the expanding disk just found. The most important are clearly those arising from small ω values (distorted Pomerons). From the point of view of the J plane they give rise, in the limit $\nu \rightarrow 2^-$, to cut singularities at

$$J(\omega) = 1 \pm 2ikv_0 - \omega \quad (\text{VI-37})$$

whose branch points $(\omega = 0)$ coincide with the kink singularities. These contributions will not spoil the disk result provided the cut discontinuity (due to excited states) is less singular than $\omega^{-1/2}$ for any k , including $k = 0$.

We have performed a rough estimate of the contribution of the n^{th} mode to the evolution operator by approximating $f_n(s)$ by eigenfunctions of a square well of range $2L$. The result indicates that the cut discontinuities vanish at the branch points for any real k . Therefore we expect our results to be valid both for $Y \gg \text{Max}(v_0|B|, 1/\mu)$ and for $|B| \gg Y/v_0$. The excitations contribute mostly in the transition region $|B| \sim v_0 Y$. More precisely, in our rough calculations, they are most important outside the disk where they quantitatively reproduce - for $|B| > v_0 Y \gg v_0/\mu$ - the unshifted Pomeron tail, thus correcting the disk result $\sigma_{Tot} \propto Y$ (for $D_t = 1$) by an inverse power of $Y = \log s$. A proof of this indication would show the matching with ordinary perturbation theory, which is expected to be correct outside the disk.

We are still unable to understand whether these distorted Pomerons which in our approach correspond to bona fide eigenstates of the Hamiltonian, are relevant to the fulfillment of t-channel unitarity for $t > 0$ ($k^2 < 0$). A detailed analysis of the role of the excited states is left to future investigation.

VII - CONCLUSIONS

In order to find the lower lying spectrum of RFT, which determines the asymptotic properties of the amplitudes, we have introduced a quantization procedure around classical solutions, suitable for the supercritical phase ($\alpha_c > \alpha_c$). We have been able to identify the relevant solutions in the case of a one-dimensional impact parameter space.

It was shown that kink-like classical solutions exist for a range of values of the speed v up to a critical value λ , for which the dispersion law is phonon like. This particular value is the best suited for our quantization procedure, even if track must be kept of $(2-v)$, which acts as an infrared regulator parameter and can be sent to zero only at the end of the calculation.

We have then obtained the kink-like quantum states and the quantum excitations around them. The spectrum of the kink states is $E = \pm 2i\sqrt{\alpha'} k$, while the excitations give rise to a (real) continuum starting from there.

We have finally computed the contribution to the Green functions of the lower lying states, i.e., the vacuum, its degenerate companion and the kink states. The contribution of the latter was found to cancel the one of the degenerate ground states for $|B| > 2\sqrt{\alpha'} Y$, while for $2\sqrt{\alpha'} Y > |B|$, the Green function was still asymptotically constant. This is just the disk behavior typical of RFT, with the expected expansion velocity.

We argued that the lower lying excited states can be interpreted as perturbative Pomeron excitations viewed in the moving kink frame, and that their contributions do not substantially modify the expanding disk behavior.

On the light of our results we think that a basic characteristic of this theory is the simultaneous interplay of different states which cannot be obtained perturbatively from each other. The quantization around classical solutions is well suited to isolate those nonperturbative effects. In this way we were able to spot the contribution to the evolution operator of various quantum states corresponding to different classical solutions. This

implies that the theory cannot be interpreted as a perturbative expansion around a single, suitably chosen ground state ("shifted" vacuum).

ACKNOWLEDGEMENTS

We wish to thank E. Edberg and K. Kølbig for their help in numerical calculations.

APPENDIX A

We shall first discuss in this Appendix a generalized coherent state approach to the solution of the Schrödinger Equation (II-18) of RFT. After having shown the relevance of the single field equation of motion in the semiclassical limit, we shall give a detailed discussion of its solutions.

a) Generalized Coherent States.

The most general state vector $|\Psi(y)\rangle$ solution of the RFT Schrödinger equation

$$H|\Psi(y)\rangle = -\frac{d}{dy}|\Psi(y)\rangle \quad (\text{A-1})$$

can, without any loss of generality, be written in the form

$$|\Psi(y)\rangle = e^{-A(y)} |0\rangle \quad (\text{A-2})$$

where $|0\rangle$ is the perturbative vacuum, and $A(y)$ an operator given by

$$A(y) = \sum_{N=1}^{\infty} \frac{1}{N!} \int db_1 \dots db_N q(b_1) \dots q(b_N) G_N(b_1, \dots, b_N; y) \quad (\text{A-3})$$

in terms of an infinite number of correlation functions G_N . The Schrödinger equation A(1) can now be transformed into an infinite set of coupled differential equations of first order in "time" y for these correlation functions. These equations are obtained by computing both sides of Eq(A-1), using the RFT Hamiltonian given by Eq(II-5) and the commutation relations (II-4) and by identifying powers of creation operators. Although the general equation for $\partial_y G_N$ is not difficult to obtain, we write below only the first two equations for simplicity

$$-\frac{\partial G_1(b, y)}{\partial y} = -(\mu + \alpha' \nabla_b^2) G_1(b, y) + \lambda G_1^2(b, y) + \lambda G_2(b, b, y) \quad (\text{A-4})$$

$$\begin{aligned} -\frac{\partial G_2(b, b'; y)}{\partial y} = & -(\mu + \alpha' \nabla_b^2) G_2 - (\mu + \alpha' \nabla_{b'}^2) G_2 + 2\lambda \delta(b - b') G_1(b, y) + \\ & + 2\lambda [G_1(b, y) + G_1(b', y)] G_2 + 2\lambda G_3(b, b, b'; y) \end{aligned} \quad (\text{A-5})$$

and so on. In general, $\partial_y G_N$ depends on G_{N+1} and G_{N-1} . For an initial coherent state of the form considered in section II, these equations are to be solved with the initial condition

$$\begin{cases} G_1(b, 0) = g(b) \\ G_N(b_1, \dots, b_N; 0) = 0 \quad \text{for } N \geq 2 \end{cases} \quad (\text{A-6})$$

$g(b)$ being the initial source distribution.

We are still at the starting point in the sense that we have not yet made any approximation that allows us to handle the problem under consideration. We now proceed to compute a solution in the semiclassical approximation which, as discussed in the text, should be good in the weak coupling limit for $\alpha(0) > 1$.

The semiclassical limit for the coupled equations for the correlation functions is easily obtained, because in this case $A(y)$ is just the classical action and, considered as a functional of the dynamical variable $q(b)$ it satisfies the Hamilton-Jacobi equation

$$H \left\{ q(b), - \frac{\delta A}{\delta q(b)} \right\} + \frac{\partial A}{\partial y} = 0 \quad (\text{A-7})$$

Replacing the expansion (A-3) for $A(y)$ and identifying again powers of $q(b)$ one can easily obtain the semiclassical limit for $\lambda_y G_N$. The result is neatly phrased by saying that the semi-classical limit corresponds to dropping the "feedback" terms in the exact equations, that is to say, terms proportional to G_{N+1} in the equation for $\lambda_y G_N$. What is left is then a set of uncoupled equations for the correlation functions of the form

$$\frac{\partial G_1}{\partial y} = (\mu + \alpha' \nabla_b^2) G_1 - \lambda G_1^2 \quad (\text{A-8})$$

$$\begin{aligned} \frac{\partial G_2}{\partial y} = & \left[(\mu + \alpha' \nabla_b^2) + (\mu + \alpha' \nabla_{b'}^2) + 2\lambda (G_1(b, y) + G_1(b', y)) \right] G_2 + \\ & + 2\lambda \delta(b - b') G_1(b, y) \end{aligned} \quad (\text{A-9})$$

The only non-linear equation is that of G_1 ; G_N satisfying a linear differential equation that has as an input the lower correlation functions. By means of an adequate rescaling of the correlation functions one can check that the semiclassical equations are independent of λ , and that the feedback term (radiative corrections) are perturbative in λ . We are therefore naturally led to

study the equation for G_1 , which is exactly the single-field equation for $p_{cl}(b,y)$. Higher correlations take into account the fact that the classical field $q_{cl}(b,y)$ is non zero, and they can in principle be obtained by solving linear equations once G_1 is known.

b) The single-field equation.

We shall henceforth restrict to a one-dimensional impact parameter space, and study the single-field equation (A-8). We want to illustrate how a wave generated by a source approaches asymptotically solitons with a well-defined (critical) speed. Let us consider the well defined boundary value problem for the single-field equation

$$\begin{cases} \frac{\partial p_a}{\partial y}(y,b) = \mu p_a + x' \frac{\partial^2 p_a}{\partial b^2} - \lambda p_a^2 \\ p_a(0,b) = f(b) \end{cases} \quad (A-10)$$

By introducing $t = \mu y$, $x = \sqrt{\frac{\mu}{\lambda}} b$ and $\varphi = \frac{\mu}{\lambda} p_a$ we eliminate all dimensional constants, and the problem reduces to

$$\begin{cases} \dot{\varphi} = \varphi - \varphi^2 + \varphi'' \\ \varphi(0,x) = \frac{\lambda}{\mu} f\left(\sqrt{\frac{\lambda}{\mu}} x\right) = u_0(x) \end{cases} \quad (A-11)$$

Nonlinear diffusion equations of this kind have been widely studied in the mathematical literature⁽¹²⁾, and the qualitative features of (A-11) have already been given in the context of RFT.⁽¹³⁾ They are :

- i) The only fixed points -i.e., static solutions of (A-11)- are $\varphi=0$ and $\varphi=1$
- ii) If the source $f(b)$ is everywhere smaller than $\frac{\mu}{\lambda}$, namely, $0 < u_0(x) < 1$ (as we shall assume), $\varphi(t,x)$ is bounded between 0 and 1 at all times.
- iii) The perturbative solution

$$\varphi_0(t,x) = \frac{1}{\sqrt{2\pi t}} \int e^{t - \frac{(x-x')^2}{4t}} u_0(x') dx' \quad (A-12)$$

is valid for small $u(x)$ and t . Therefore $\varphi(t,x)$ increases with time away from $\varphi=0$, and at any given x approaches $\varphi = 1$ for $t \rightarrow \infty$. The physics described by the nonlinear diffusion equation is, then, an exponential growth with time of $\varphi(t,x)$ arising from the positive φ term in (A-11), followed by a saturation at $\varphi=1$ arising from the negative φ^2 term in the differential equation, provided that the source $u_0(x)$ is everywhere smaller than the saturation value of φ .

We want to clarify how $\varphi=1$ is approached for large times. We shall see that this approach occurs with a well-defined speed, which picks up one among many soliton solutions of (A-11). Let us therefore look for homogeneous waves that interpolate between the two fixed points, of the form^(*)

$$\varphi(\tau) = \varphi(\nu t \pm x) \quad (\text{A-13})$$

such that

$$\varphi(-\infty) = 0 \quad ; \quad \varphi(+\infty) = 1 \quad (\text{A-14})$$

They satisfy the differential equation

$$\ddot{\varphi} = -\varphi + \varphi^2 + \nu \dot{\varphi} \quad (\text{A-15})$$

and solutions to this equation with boundary conditions (A-14) exist for any positive speed $0 < \nu < \infty$ ^(**). In fact, $\varphi(-\tau)$ can be interpreted as the coordinate of a particle of unit mass moving in a potential

$$V(\varphi) = \frac{1}{2} \varphi^2 - \frac{1}{3} \varphi^3 \quad (\text{A-16})$$

and subject also to a friction force with viscosity coefficient ν . Indeed, by reversing the sign of "time" τ the term $\nu \dot{\varphi}$ in (A-15) changes sign and becomes then a dissipative force. The shape of the potential $V(\varphi)$ is exhibited

(*) The propagation velocity ν so introduced is of course dimensionless. When all dimensionful parameters are reintroduced, the propagation velocity becomes $\nu \sqrt{\alpha' \mu}$.

(**) We are indebted to G. Parisi for the argument that follows. More rigorous derivations can be found in the mathematical literature⁽¹²⁾.

in Fig.9. Since for the time reversed motion φ has to start at $\varphi=1$ and end up at $\varphi=0$, it is obvious from the mechanical analogy that this will happen for any value of v . The time reversed motion will correspond to the particle moving downhill and performing damped oscillations around the bottom of the well for small values of the viscosity coefficient v , or reaching very slowly $\varphi=0$ without oscillating for large values of v . The borderline between these two regimes corresponds to a critical velocity v_c that we discuss next. But it is quite clear that solutions of (A-15) exist for any positive v .

When φ becomes small, the potential $V(\varphi)$ approaches a harmonic one, and the problem becomes that of a damped oscillator. Linearizing the equation and looking for solutions of the form $\varphi(\tau) \simeq \exp(k_0 \tau)$ we get

$$k_0^2 - v k_0 + 1 = 0 \quad \therefore \quad k_0^\pm = \frac{v}{2} \pm \sqrt{\frac{v^2}{4} - 1} \quad (\text{A-17})$$

We have therefore damped oscillations for $\varphi(-\tau)$ around $\varphi=0$ for $v < 2$, and at most one oscillation for $v > 2$.

The phase space trajectory corresponding to the solutions of Eq(A-15) can be found by introducing the velocity field $W(\varphi) = \dot{\varphi}$, which satisfies the equation

$$\left(\frac{dW}{d\varphi} - v \right) W + \varphi - \varphi^2 = 0 \quad (\text{A-18})$$

and in order to define $W(\varphi)$ uniquely we shall give a boundary condition at $\varphi=1$. From the linearized form of Eq(A-15) near $\varphi=1$, it follows that φ is of the form

$$\varphi \simeq 1 - c e^{-k_\pm \tau} \quad ; \quad k_\pm = \frac{v}{2} \pm \sqrt{\frac{v^2}{4} + 1} \quad (\text{A-19})$$

but k_- has to be discarded because from (A-14) $\varphi(\tau)$ must reach 1 as $\tau \rightarrow \infty$. Therefore, it follows that

$$\frac{W(\varphi)}{1-\varphi} = k_+ = \frac{v}{2} + \sqrt{\frac{v^2}{4} + 1} \quad \text{as } \varphi \rightarrow 1 \quad (\text{A-20})$$

Equations (A-18,20) define a unique trajectory in the (W, φ) plane that is sketched in Figures (10a) and (10b) for $v < 2$ and $v > 2$ respectively. For $v < 2$, Fig.10a shows the existence of an infinite number of damped oscilla-

tions for the time reversed motion. The origin is then a spiral point for $W(\varphi)$. On the contrary, for $v > 2$ we observe that no damped oscillation can exist. In fact, in such a case the phase space trajectory would cross the $\varphi=0$ axis in Fig(10b) at some point W_0 . But then the corresponding unique trajectory would lie necessarily above the line $W = k_0^+ \varphi$ and could not possibly end up in $\varphi = 1$.

This last statement can also be easily understood by means of the mechanical analogy. Let us think of $\varphi(\tau)$ as the coordinate of a particle. After the particle crosses $\varphi=0$ with a finite speed to the right, it will be subject to a restoring force $(-\varphi)$, an anti-dissipative force $v\dot{\varphi}$ that feeds energy and repels it to the right, and the nonlinear force φ^2 that further repels it to the right. Therefore, the velocity corresponding to the linear problem where φ^2 is neglected gives a lower bound to the actual velocity of the particle. Going back to Fig.(10b), we remark that for $v > 2$ the phase space trajectory is asymptotically tangent to $W = k_0^- \varphi$, k_0^- being the smaller exponent of Eq(A-17), which is asymptotically dominant for $\tau \rightarrow -\infty$.

Therefore, we conclude that the soliton for $v > 2$ lies always in $0 < \varphi < 1$. It is also clear that its steepness increases when v decreases from large values to the critical value $v=2$. Indeed, by identifying the work performed by the non-conservative force $v\dot{\varphi}$ between $\varphi=0$ and $\varphi=1$ with the variation of the mechanical energy of the particle, one gets

$$v \int_{-\infty}^{\infty} \dot{\varphi}^2 d\tau = V(1) - V(0) = \frac{1}{6} \quad (\text{A-21})$$

which shows that, as v decreases the steepness of $\varphi(\tau)$ increases if φ is a monotonous function of τ . The different shapes of solitons of $v \gtrless 2$ are exhibited in Figure 11.

We are now in a position to discuss the asymptotic propagation speed in the original initial value problem (A-11). The basic idea is to rewrite the differential equation (A-11) in a frame moving with speed v in the positive x direction, for example. Then we have

$$\varphi(x, t) = u(vt - x, t) = u(\tau, t) \quad (\text{A-22})$$

$$\begin{cases} \frac{\partial u}{\partial t} = \ddot{u} - v \dot{u} + u - u^2 \\ u(-x, 0) = u_0(x) \end{cases} \quad (A-23)$$

The solitary waves we discussed before are now static solutions of (A-23), and it is clear that they are possible asymptotic limits for $u(\tau, t)$ as $t \rightarrow \infty$ at fixed τ . The mechanism that chooses precisely the $v=2$ soliton for the asymptotic limit of $u(\tau, t)$ is given by the following results which are proved in Ref(12).

- a) For any $v > 2$, if the initial condition $u_0(x)$ vanishes more rapidly than any exponential for $x \rightarrow \infty$, it is possible to find a soliton solution $\phi_2(\tau)$ such that $u(\tau, t) < \phi_2(\tau)$. Moreover, $\partial_t u(\tau, t) < 0$.
- b) For any $v < 2$, there is an (oscillating) soliton solution with $\phi_2(\tau_0) = 0$ such that $u(\tau, t) > \phi_2(\tau) - (\tau > \tau_0)$ - and sufficiently large time. Moreover, $\partial_t u(\tau, t) > 0$.

The situation for $\varphi(x, t)$ is therefore the one depicted in Fig.12. What points (a) and (b) are telling us is that, in a frame moving with velocity $v > 2$ one would see $u(\tau, t)$ -bounded from above by $\phi_2(\tau)$ -decreasing with time t , while in a frame moving with velocity $v < 2$ one would see $u(\tau, t)$ -bounded from below by $\phi_2(\tau)$ for $\tau > \tau_0$ -increasing with time. From (a) and (b) and known theorems, it follows that

$$\begin{aligned} \varphi(vt - \tau, t) &= u(\tau, t) \rightarrow 0 \quad \text{if } v > 2 \\ \varphi(vt - \tau, t) &= u(\tau, t) \rightarrow 1 \quad \text{if } v < 2 \end{aligned} \quad (A-24)$$

In other words, $\varphi(x, t)$ gets trapped between solitons with $v > 2$ and $v < 2$, and has to propagate with the asymptotic speed $v=2$. Without attempting a proof of these statements here, let us notice that the properties of

$u(\tau, t)$ discussed in the preceeding paragraph are indeed shared by the perturbative solution (A.12). For the special case $u_0(x) \propto \delta(x)$ we have in fact

$$\begin{aligned} \varphi(vt - \tau, t) &= \frac{1}{\sqrt{2\pi t}} \exp \left[t - \frac{(vt - \tau)^2}{4t} \right] \\ &= \frac{1}{\sqrt{2\pi t}} \exp \left[t \left(1 - \frac{v^2}{4} \right) + \frac{v^2}{4} \tau - \frac{\tau^2}{4t} \right] \end{aligned} \quad (A-25)$$

Completely similar arguments hold for $x < 0$. For simplicity we shall then summarize the results described above by the statement (not strictly proved in the mathematical literature) that $\varphi(x,t)$ approaches asymptotically the soliton with $v=2$. In other words

$$\varphi(x,t) \xrightarrow[t \rightarrow \infty]{} \left[\phi_2(2t-x-a) + \phi_2(2t+x+b) - 1 \right] \quad (\text{A-26})$$

for some values of the translational parameters a and b which depend on the specific form of the initial condition $u_0(x)$. This is essentially equivalent to the equation (II-20) of the text.

A P P E N D I X B

ZERO MODES AND COMPLETENESS.

The purpose of this Appendix is to relate the zero mode wave function $\xi_0(\rho)$ and its conjugate vector $\chi_0(\rho)$ to the lowest eigenfunction $f_0(\rho)$ of the Schrödinger operator D_s involving the potential $W(\rho)$ in the text. We are particularly interested in understanding their role in the completeness relation, in order to prove the spectral decomposition of the identity given by Eq(IV-42) as well as the spectral decomposition of the Hamiltonian $\mathcal{H}_2$ given by Eq(IV-43). Our strategy will be to prove these results from the known completeness of the eigenfunctions of D_s .

We shall restrict ourselves to kink solutions of the class (a) inside the triangle. Let us recall that $\xi_0(\rho)$ and $\chi_0(\rho)$ are, by Eqs(IV-16) and (IV-19)

$$\xi_0(\rho) = \begin{pmatrix} F_0(\rho) \\ \bar{F}_0(\rho) \end{pmatrix} = N_0 \begin{pmatrix} \dot{\psi}(\rho) \\ \dot{\bar{\psi}}(\rho) \end{pmatrix} \quad (\text{B-1})$$

and

$$\chi_0(\rho) = \begin{pmatrix} G_0(\rho) \\ \bar{G}_0(\rho) \end{pmatrix} = L_0 \begin{pmatrix} \frac{\partial \psi}{\partial \sigma} \\ \frac{\partial \bar{\psi}}{\partial \sigma} \end{pmatrix} \quad (\text{B-2})$$

From the known properties of our classical solutions as functions of ρ and σ (see Eq(V-32) as well as the comment preceeding it) we deduce that the components of ξ_0 and χ_0 have the symmetry property

$$F_0(\rho) = -\bar{F}_0(-\rho) \quad (\text{B-3})$$

and

$$G_0(\rho) = \bar{G}_0(-\rho) \quad (\text{B-4})$$

Therefore, in order to satisfy the symmetry property (B-3) between the upper and lower components of ξ_0 , we identify it with the eigenfunctions intro-

duced in Section V as follows :

$$\xi_o(\rho) = \begin{pmatrix} f_o(\rho) e^{\frac{\nu}{2}\rho} \\ \bar{f}_o(\rho) e^{-\frac{\nu}{2}\rho} \end{pmatrix} \Theta(\rho) - \begin{pmatrix} \bar{f}_o(-\rho) e^{\frac{\nu}{2}\rho} \\ f_o(-\rho) e^{-\frac{\nu}{2}\rho} \end{pmatrix} \Theta(-\rho) \quad (\text{B-5})$$

Likewise, in order to satisfy (B-4) we write $\chi_o(\rho)$ as

$$\chi_o(\rho) = \begin{pmatrix} g_o(\rho) e^{\frac{\nu}{2}\rho} \\ \bar{g}_o(\rho) e^{-\frac{\nu}{2}\rho} \end{pmatrix} \Theta(\rho) + \begin{pmatrix} \bar{g}_o(-\rho) e^{\frac{\nu}{2}\rho} \\ g_o(-\rho) e^{-\frac{\nu}{2}\rho} \end{pmatrix} \Theta(-\rho) \quad (\text{B-6})$$

where $g_o(\rho)$, $\bar{g}_o(\rho)$ are functions yet to be determined. Equations (IV.17) and (IV-20) now read

$$\begin{cases} D_s f_o + 2\varphi e^{-\nu\rho} \bar{f}_o = 0 \\ 2\bar{\varphi} e^{\nu\rho} f_o + D_s \bar{f}_o = 0 \end{cases} \quad (\text{B-7})$$

and

$$\begin{cases} D_s g_o + 2\varphi e^{-\nu\rho} \bar{g}_o = \Omega f_o \\ 2\bar{\varphi} e^{\nu\rho} g_o + D_s \bar{g}_o = -\Omega \bar{f}_o \end{cases} \quad \left(\Omega = -\frac{L_o}{N_o} \right) \quad (\text{B-8})$$

Let us now try to express, for $(2-\nu) \ll 1$, all the components of ξ_o and χ_o in terms of

$$f_o(\rho) = N_o \dot{\varphi}(\rho) e^{-\frac{\nu}{2}\rho} \quad (\rho > 0) \quad (\text{B-9})$$

The function so defined, which is the exact expression for the upper component

of the zero mode for $\rho > 0$, is also approximately equal to the ground state wave functions of D_s introduced in Section V, corresponding to the eigenvalue $\omega_0 = 0(2-\nu)$, and for which we have used the same notation. In fact, both can be identified as $\nu \rightarrow 2^-$. Since the ground state wave function of D_s is an even function, its derivative at the origin vanishes. We check that this is approximately true for $f_0(\rho)$. By using (III-29) and the equation of motion at $\rho=0$, we obtain

$$\dot{f}_0(0) = N_0 \left[\ddot{\varphi}(0) - \frac{\nu}{2} \varphi(0) \right] = N_0 \left[\left(\frac{\nu}{2} - 1 \right) \varphi_0 + \varphi_0^2 \right] = O(2-\nu) \quad (\text{B-10})$$

We therefore extend the definition of $f_0(\rho)$ to $\rho < 0$ by demanding it to be an even function

$$f_0(-\rho) = f_0(\rho) \quad (\rho > 0) \quad (\text{B-11})$$

having then the property that

$$D_s f_0 = O(2-\nu) \quad ; \quad \dot{f}_0(0) \approx O(2-\nu) \quad (\text{B-12})$$

We next go back to Eqs(B-7) which reduce, to leading order in $(2 - \nu)$, and for $\rho > 0$, to

$$\begin{cases} D_s f_0 = 0 \\ 2\dot{\varphi} f_0 + D_s \bar{f}_0 = 0 \end{cases} \quad (\rho > 0) \quad (\text{B-13})$$

where use has been made of the fact that, for $\rho > 0$, $\varphi \exp(-\nu\rho)$ is of $O(2-\nu)$, and that $\bar{\varphi} = \dot{\varphi} \exp(-\nu\rho)$. Due to the fact that, for $\rho > 0$

$$0 = \frac{d}{d\rho} [D_s f_0] = D_s \dot{f}_0 + 2\dot{\varphi} f_0 \quad (\text{B-14})$$

the two independent solutions of (B-13) are

$$(a) \quad \begin{pmatrix} 0 \\ f_0 \end{pmatrix} \quad ; \quad (b) \quad \begin{pmatrix} f_0 \\ \dot{f}_0 \end{pmatrix} \quad (\text{B-15})$$

and the zero mode ξ_0 must be a linear combination of the two, which is determined by using Eq(B-3). This equation implies that, at the origin, $\bar{f}_0(0) = -f_0(0)$. Therefore in order to satisfy this relation at $\rho=0$ between the upper and lower components of ξ_0 , we must have the antisymmetric combination, i.e. :

$$\xi_0(\rho) = N_0 \frac{\partial \xi_4}{\partial \rho} \simeq \begin{pmatrix} f_0 e^{\frac{\nu}{2}\rho} \\ (\dot{f}_0 - f_0) e^{-\frac{\nu}{2}\rho} \end{pmatrix} ; (\rho > 0; \nu \rightarrow 2^-) \quad (B-16)$$

For $\rho < 0$, ξ_0 is automatically given by (B-5), i.e.

$$\xi_0(\rho) \simeq \begin{pmatrix} (\dot{f}_0 + f_0) e^{\frac{\nu}{2}\rho} \\ -f_0 e^{-\frac{\nu}{2}\rho} \end{pmatrix} ; (\rho < 0; \nu \rightarrow 2^-) \quad (B-17)$$

Notice that the upper component of $\xi_0(\rho)$ in $\rho < 0$ contains also f_0 , and therefore it is neither even nor odd under $\rho \leftrightarrow -\rho$ as expected from the known shape of $\varphi(\rho)$.

A similar situation occurs for $\chi_0(\rho)$, in the limit $\nu \rightarrow 2^-$.

In fact, by Eq(V-35)

$$\Omega = -\frac{L_0}{N_0} \simeq \sigma(2-\nu) \quad (B-18)$$

and therefore the inhomogeneous term in the right-hand side of (B-8) vanishes to leading order in $(2-\nu)$. The equations determining the components of $\chi_0(\rho)$ reduce then in this limit to Eq(B-13). In other words, $\chi_0(\rho)$ becomes, for $\nu \rightarrow 2^-$, a second zero mode solution. This is not so surprising if one remembers that the two components of the classical solution, $\varphi(\rho)$ and $\bar{\varphi}(\rho)$, are less and less correlated as $\nu \rightarrow 2^-$, and that in the limit $\nu=2$ they are independent solutions of single field equations. However, this time the boundary condition at $\rho=0$ is given by Eq(B-4), which implies $\bar{g}(0)=g(0)$, and therefore χ_0 is proportional to the symmetric combination of the two independent solutions (B-15) :

$$\chi_0(\rho) = L_0 \frac{\partial \xi_0}{\partial \nu} \approx \frac{1}{2} \begin{pmatrix} f_0 e^{\frac{\nu}{2}\rho} \\ (\dot{f}_0 + f_0) e^{-\frac{\nu}{2}\rho} \end{pmatrix} ; (\rho > 0; \nu \rightarrow 2^-) \quad (\text{B-19})$$

where the factor $\frac{1}{2}$ is determined by the normalization condition

$$\begin{aligned} (\bar{\chi}_0, \xi_0) &= 2 \int_0^\infty [\bar{F}_0(\rho) G_0(\rho) - F_0(\rho) \bar{G}_0(\rho)] d\rho = \\ &= 2 \int_0^\infty f_0^2(\rho) d\rho = 1 \end{aligned} \quad (\text{B-20})$$

Finally, the form of χ_0 for $\rho < 0$ is obtained from Eq(B-6)

$$\chi_0(\rho) \approx \frac{1}{2} \begin{pmatrix} (f_0 - \dot{f}_0) e^{\frac{\nu}{2}\rho} \\ f_0 e^{-\frac{\nu}{2}\rho} \end{pmatrix} \quad (\rho < 0; \nu \rightarrow 2^-) \quad (\text{B-21})$$

Let us remark that the expression for χ_0 differs from the one for ξ_0 essentially because of the replacement $f_0 \rightarrow -f_0$ in the lower component. It must also be noticed that, since $f_0(0) \approx 0$ and $\dot{f}_0(\rho)$ is of order 1 for $\rho \approx L$, the term proportional to f_0 in the lower component of (B-19) is negligible compared to the other term for moderate values of ρ . On the other side, in this region the lower component of χ_0 can be estimated as follows (for $0 < \rho < L$)

$$\bar{g}_0 e^{-\frac{\nu}{2}\rho} = L_0 \frac{\partial \bar{\varphi}}{\partial \nu} \approx L_0 \frac{\bar{\varphi}}{\nu-2} \approx -L_0 \frac{\dot{\varphi} e^{-\nu\rho}}{2-\nu} \quad (\text{B-22})$$

Therefore, by comparing it with f_0 given by (B-9), we get

$$-\frac{L_0}{N_0} = \Omega = \frac{1}{2}(2-\nu) \quad (\text{B-23})$$

in agreement with the argument in the text.

The explicit expressions we have obtained for the vectors ξ_0 and χ_0 , together with the similar ones for $\xi_{\pm n}$ ($n \neq 0$) discussed in the text, allows us to prove the completeness relation :

$$\sum_{n=1}^{\infty} \left[\xi_n(\rho) \bar{\xi}_n(\rho') - \bar{\xi}_n(\rho) \xi_n(\rho') \right] + \quad (B-24)$$

$$+ \xi_0(\rho) \bar{\chi}_0(\rho') - \chi_0(\rho) \bar{\xi}_0(\rho') = \pm \delta(\rho - \rho')$$

In fact, by using the Eqs(V-8-9) of the text, we get for the left-hand side of (B-24) a matrix $M(\rho, \rho')$ which, for the case $\rho, \rho' > 0$ is given by

$$M_{11} = M_{22} = \sum_{n=0}^{\infty} e^{\frac{\sqrt{V}}{2}\rho} f_n(\rho) e^{-\frac{\sqrt{V}}{2}\rho'} f_n(\rho') \quad (B-25)$$

$$M_{12} = 0$$

$$M_{21} = \left\{ \sum_{n=1}^{\infty} e^{-\frac{\sqrt{V}}{2}\rho} g_n(\rho) e^{-\frac{\sqrt{V}}{2}\rho'} f_n(\rho') + e^{-\frac{\sqrt{V}}{2}\rho} \dot{f}_0(\rho) e^{-\frac{\sqrt{V}}{2}\rho'} f_0(\rho') - \right.$$

$$\left. - (\rho \leftrightarrow \rho') \right\}$$

It is clear that, by the completeness relation for the hermitian operator D_s given by Eq(V-19) in the text, we get

$$M_{11} = M_{22} = \delta(\rho - \rho') \quad (B-26)$$

because the similarity transformation $\exp(\frac{\sqrt{V}}{2}\rho)$ leaves the δ -function unchanged. On the other hand, from Eq(B-14) we get

$$\dot{f}_0(\rho) = - \sum_{n \neq 0} \frac{f_n(\rho)}{\omega_n} \langle 2\dot{\psi} \rangle_{n0} + \text{Constant} \cdot f_0(\rho) \quad (B-27)$$

However, only the first term contributes to M_{21} , the second are being canceled when the expression is antisymmetrized in ρ and ρ' . Then, using (B-27) and the expression (V-25) for $g_n(\rho)$, we get

$$M_{21} = \sum_{n,k=1}^{\infty} \frac{\langle 2\dot{\psi} \rangle_{nK}}{\omega_n + \omega_K} f_k(\rho) f_n(\rho') e^{-\frac{\sqrt{V}}{2}(\rho + \rho')} - \quad (B-28)$$

$$- (\rho \leftrightarrow \rho') \equiv 0$$

because the first term is symmetric in (ρ, ρ') . This completes the proof of Eq(B-24).

From the completeness relation we also get the spectral decomposition of the operator $\mathcal{K}_2$. Using the relations

$$\mathcal{K}_2 \chi_0 = - \frac{L_0}{N_0} \xi_0 \quad ; \quad \mathcal{K}_2 \xi_0 = 0 \quad (\text{B-29})$$

and acting with $\mathcal{K}_2$ on both sides of (B-24), it follows that

$$\mathcal{K}_2 = \sum_{n=1}^{\infty} \omega_n \left[\xi_n(\rho) \bar{\xi}_n(\rho') + \bar{\xi}_n(\rho) \xi_n(\rho') \right] - 2 \xi_0(\rho) \bar{\xi}_0(\rho') \quad (\text{B-30})$$

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FIGURE CAPTIONS

Fig. 1 - The phase-space trajectories of the Cardy-Sugar kink solutions.

Fig. 2 - The sign of $\dot{\varphi} \ddot{\varphi}$ in the $(\varphi, \ddot{\varphi})$ plane.

Fig. 3 - The phase-space trajectories of the family of kink solutions inside the triangle, with $-\infty < v < 2$.

Fig. 4 - The value of $\mathcal{H}(0)$ as a function of v for the kink solutions inside the triangle.

Fig. 5 - The phase-space (w, φ) trajectory for the kink solutions of the single-field equation. Full line: $v < 2$. Dotted line: $v = 2$.

Fig. 6 - The action per unit rapidity for the various families of classical kink solutions, as a function of

- a) _____ Solutions inside the triangle
- b) - - - - - Solutions outside the triangle
- c) - · - · - · - Single-field solutions

Fig. 7 - Shape of the potential W of Eq. (V-11) and of the classical fields. $\ddot{\varphi}$ is represented by a dashed line, and φ by a dotted-dashed one.

Fig. 8 - Time evolution of (a) the kink and (b) the antikink states.

Fig. 9 - The potential $V(\varphi)$ of the single field equation (A-15).

Fig. 10 - Phase space (w, φ) trajectories for the kink solutions of the single-field equation. (a) $v < 2$, (b) $v \geq 2$.

Fig. 11 - Different shapes of kink solutions of the single-field equation, for $v \geq 2$.

Fig. 12 - The way in which the solution of the diffusion equation (A-11) gets asymptotically trapped between kinks with $v > 2$ and $v < 2$.

FIG.4

FIG.5

FIG.6

FIG. 7

FIG. 8

FIG. 9

FIG. 10

FIG.12

FIG. 11