The CNOT Quantum Logic Gate Using q-Deformed Oscillators

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In Ref.[1] it was shown that the Hadamard and Phase Shift quantum logic gates can be realised with q-deformed oscillators. Here it is shown that the two qubit CNOT (controlled NOT) gate can also be realised with q-deformed oscillators. Thus all the three gates necessary for universality are realisable with q-deformed oscillators. So an alternative formalism for quantum computation is hereby established.

Keywords: universality of quantum logic gates; q-deformed oscillators; quantum computation

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1. Introduction

Recently it was shown that the single qubit quantum logic gates, viz., the Hadamard and Phase Shift gates can also be realised with two q-deformed oscillators [1]. q is the deformation parameter and \( q = e^{i\pi s} \); \( 0 < s < 1 \). The advantage over the usual formalism (obtained for \( q \rightarrow 1 \)) lies in that the alternate formalism allows the presence of an arbitrary function which may be exploited for experimental purposes. However, the formalism will be more meaningful if the realisation with q-deformed qubits is possible for all the gates required for universality. A set of gates is said to be universal for quantum computation if any unitary operation may be approximated to arbitrary accuracy by a quantum circuit involving those gates. In the case of usual quantum computation the Hadamard, Phase Shift and the CNOT (controlled NOT) gates constitute such a set [2]. Here I show that the 2-qubit CNOT gate is also realisable with q-deformed qubits. So all three gates required for universality are realisable with q-deformed qubits. First, a brief review of relevant facts.

2. Brief Review

Quantum logic gates are basically unitary operators [3-7]. The single qubit Hadamard and Phase Shift gates and the 2-qubit CNOT gate, are sufficient to construct any unitary operation on a single qubit [2]. This is the universality referred to above. These gates are constructed using the "spin up" and "spin down" states of SU(2) angular momentum i.e., the basis states of a qubit are represented by "spin up" and "spin down" states. In [1] the Hadamard and Phase Shift gates were realised with q-deformed qubits using the technique of harmonic oscillator realisation [10-15] of q-oscillators [8,9].

q-Oscillators: \( a_q, a_q^\dagger \) are creation and annihilation operators for q-oscillators. For ordinary oscillators these are \( a \) and \( a^\dagger \). \( q = e^{i\pi s} \); \( 0 \leq s \leq 1 \) and

\[ a_q a_q^\dagger - q a_q^\dagger a_q = q^{-N} ; \quad N^\dagger = N \quad (1a) \]

\[ [N, a_q] = -a_q ; \quad [N, a_q^\dagger] = a_q^\dagger ; \quad a_q a_q^\dagger = [N] ; \quad a_q a_q^\dagger = [N+1] \quad (1b) \]

\[ a_q f(N) = f(N+1)a_q ; \quad a_q^\dagger f(N) = f(N-1)a_q^\dagger \quad (1c) \]

\[ x = (q^{-x} - q^{-x})/(q - q^{-1}) ; \quad N \text{ is the number operator (eigenvalue) for the q-oscillators and } f(N) \text{ is any function of } N. \text{ We confine to real } q. \text{ From [10,11]:} \]

\[ a_q = a \sqrt{q^N \psi_1 - q^{-N} \psi_2} / \sqrt{N(q - q^{-1})} ; \quad a_q^\dagger = \sqrt{q^N \psi_1 - q^{-N} \psi_2} a_q^\dagger / \sqrt{N(q - q^{-1})} \quad (2a) \]

\[ N = \hat{N} - (1/s) \ln \psi_2 \quad (2b) \]

\( \hat{N} \) is the number operator for usual oscillators with eigenvalue \( \hat{n} \); \( \psi_1, \psi_2 \) are arbitrary functions of \( q \) only with \( \psi_{1,2}(q) = 1 \) for \( q = 1 \). If all the \( \psi \)'s are unity, then \( N = \hat{N} \). This is the realm of usual quantum computation. But (2a,b) are general if the \( \psi_i(q), i = 1, 2 \) are not all equal to unity. Let \( \psi_1 = \psi_2 = \psi(q) \). Now \( N = \hat{N} - (1/s) \ln \psi(q) \) (equation (2b)). This will show up in the Jordan-Schwinger construction of angular momentum states and the states in the two cases will be distinguishable through \( \psi(q) \). Further details are in Ref.[1].

Jordan-Schwinger construction: (a) States defined by the total angular momentum \( j \) and \( j_z \) i.e. m,

\[ |jm> = (a_j^\dagger)^j (a_{-j}^\dagger)^{-j+m} [(j+m)!/(j-m)! 1/2] \phi > \quad (3) \]

\[ |\phi \equiv |0> \equiv |0 \rangle > |0 \rangle > 2 \text{ is the ground state } (j = 0, m = 0), |0 \rangle >, i = 1, 2 \text{ are oscillator ground states. } j = (n_1 + n_2)/2 ; \quad m = (n_1 - n_2)/2 \text{ where } n_1, n_2 \text{ are the eigenvalues of the number operators of the two oscillators. For qubits }, (n_1 + n_2)/2 = 1/2. \quad j = 1/2 \text{ for both qubit states. q-deformed states are denoted as } |q,a> \text{ replaced by } a_q, \text{ n by } [n] \text{ etc. In terms of } n, \text{ states are} \]

\[ |n - 1/2> = (a_q^\dagger)^n (a_2^\dagger)^{-n} [(n)!/(1-n)! 1/2] |0> \quad (4) \]

q-deformed states are \( |n - 1/2> > q,a> \text{ replaced by } a_q \text{ etc.} \)

(b) Basis states are:

\[ |1 \equiv |1/2, 1/2> \equiv |1/2, 1/2> = a_2^\dagger |0> = a_q^\dagger |0> > 1 \equiv |0 > > \text{ and} \]

\[ |0 \equiv |1/2, -1/2> \equiv |1/2, -1/2> = a_2^\dagger |0> = a_q^\dagger |0> > 1 \equiv |0 > > \text{ and} \]

\[ |0 > > |0 > > 1 > > \text{ and} \]

\[ |0 > > |1 > > 1 > > \text{ and} \]

\[ |0 > > |1 > > 1 > > \text{ and} \]

\[ |0 > > |1 > > 1 > > \text{ and} \]
Physical meaning of notation: The $|1\rangle$ ($|0\rangle$) spin "up"("spin "down") state is constructed out of two oscillator states where the first oscillator state has occupation number 1 (0) while the other has occupation number 0 (1). So any qubit state $|x\rangle$ is:

$$|x\rangle = (a_1^\dagger)^x(a_0^\dagger)^{1-x}|\tilde{0}\rangle$$

(5)

$|0\rangle$ represents one of the two qubit states; $|\tilde{0}\rangle$ denotes oscillator ground state i.e. occupation number 0 etc.

The Hadamard transformation for $q$-deformed quibits:
The Hadamard transformation [4-6] on a single qubit state ($x = 0, 1$) is (modulo $1/\sqrt{2}$)

$$|x\rangle \rightarrow (-1)^x|x\rangle + |1-x\rangle$$

(6)

So the Hadamard transformation for q-deformed state is

$$|n_1 - 1/2\rangle \rightarrow (-1)^{n_1}|n_1 - 1/2\rangle + |1/2 - n_1\rangle$$

(7)

This simplifies to:

$$\begin{align*}
[F_1(\hat{N}_1, q)a_1^\dagger]|n_1[F_2(\hat{N}_2, q)a_2^\dagger]|n_1\phi \rightarrow \\
(-1)^{n_1}[F_1(\hat{N}_1, q)a_1^\dagger][F_2(\hat{N}_2, q)a_2^\dagger]|n_1\phi + \\
+F_1(\hat{N}_1, q)a_1^\dagger|n_2[F_2(\hat{N}_2, q)a_2^\dagger]|n_1\phi
\end{align*}$$

(8)

where

$$F_1(\hat{N}_1, q) = \sqrt{\frac{q^{N_1}\psi_3 - q^{-N_1}\psi_4}{N_1(q - q^{-1})}}$$

$$F_2(\hat{N}_2, q) = \sqrt{\frac{q^{N_2}\psi_3 - q^{-N_2}\psi_4}{N_2(q - q^{-1})}}$$

(9)

$n_1, n_2$ is always 0 or 1 so as to correspond to the qubit. Hence the q-numbers $|n_1\rangle, |n_2\rangle$ are always the usual numbers $n_1, n_2$. Same restrictions also apply to usual (i.e. undeformed) oscillators. So we restrict the hatted number operators, $\hat{N}_1$ and $\hat{N}_2$, by $\hat{N}_1 + \hat{N}_2 = I$ where $I$ is the identity operator.

The Phase Shift transformation for q-deformed quibits: The Phase Shift transformation of qubit states is:

$$|x\rangle \rightarrow e^{ix\theta}|x\rangle$$

which in our notation is $|n - \frac{1}{2}\rangle \rightarrow e^{in\theta}|n - \frac{1}{2}\rangle$ where $\theta$ is the phase shift. Details are in Ref.[1] where both the Hadamard and Phase Shift transformations were realised with q-deformed qubits. Below I show that the same is possible for both the NOT gate and the CNOT gate.

3. The NOT gate

The NOT gate is characterised by its action on a qubit as:

$$|x\rangle \rightarrow |1-x\rangle$$

where $x = 0, 1$. For q-deformed states this means $|n - \frac{1}{2}\rangle \rightarrow |\frac{1}{2} - n\rangle$. In terms of q-deformed oscillator states this becomes

$$\begin{align*}
(a_1)^{n_1}(a_0)^{n_2}
\end{align*}

(|n_1||n_2||1/2)|\phi\rangle \rightarrow \\
(a_0)^{n_1}(a_1)^{n_2}
\end{align*}

(9)

which rewritten in terms of the functions $F$ looks like

$$[F(\hat{N})]^n[F(1+n-\hat{N})]^{-n}(a_1^\dagger)^n(a_2^\dagger)^n|\phi\rangle$$

$$\rightarrow [F(\hat{N})]^n[F(2-n-\hat{N})]^n(a_1^\dagger)^n(a_2^\dagger)^n|\phi\rangle$$

(11)

where one has used $n_1 + n_2 = 1$, followed arguments of Ref.1, relabelled $n_2$ as $n$, and used (1c). With respect to the states $|\phi\rangle$, (11) would be indistinguishable from the usual "NOT" transformation if

$$[F(\hat{N})]^n[F(1+n-\hat{N})]^{-n} = [F(\hat{N})]^n[F(2-n-\hat{N})]^n$$

(12)

Writing (12) in terms of its eigenvalues, the solution is when $\psi_1(q) = \psi_2(q) = \psi(q)$ (say) for both $n = 0$ and $n = 1$.

Thus the NOT gate is realisable with deformed qubits.

4. The CNOT gate

The 2-qubit CNOT gate is defined by the following transformations:

$$|00\rangle \rightarrow |00\rangle; |01\rangle \rightarrow |01\rangle; |10\rangle \rightarrow |11\rangle; |11\rangle \rightarrow |10\rangle$$

(13)

where $|0\rangle \equiv |0\rangle \equiv |0\rangle \equiv |0\rangle \equiv |0\rangle$. This may be written as (modulo constants) as $|xy\rangle \rightarrow (1-x)|xy\rangle + x|y\rangle$ i.e. $|x|y\rangle \rightarrow (1-x)|x\rangle + y|x\rangle$.

Let the oscillators corresponding to the $|x\rangle$ qubit be denoted by $a, a^\dagger$ and those corresponding to the $|y\rangle$ qubit be $b, b^\dagger$. Then in terms oscillator states the CNOT transformation reads:

$$\begin{align*}
(a_1)^{n_1}(a_0)^{n_2}
\end{align*}

(|n_1||n_2||1/2)|\phi\rangle \rightarrow \\
(a_0)^{n_1}(a_1)^{n_2}
\end{align*}$$

(14)

$$\begin{align*}
(a_0)^{n_1}(a_1)^{n_2}
\end{align*}$$

(15)

where $n_1, n_2$ and $k_1, k_2$ are the eigenvalues of the number operators corresponding to the respective oscillators with $n_1 + n_2 = 1$, $k_1 + k_2 = 1$ and $|\phi\rangle > a$, $|\phi\rangle > b$ denote the ground states corresponding to oscillators $a_{1,2}$ and $b_{1,2}$ respectively. The CNOT transformation for deformed oscillator states is obtained simply by replacing the states $|\phi\rangle > a$, $|\phi\rangle > b$ etc. in (13). Following Ref.1, denote the harmonic oscillator realisations for the operators $a_{1,2}$ and $b_{1,2}$ respectively by

$$a_{1q} = F(\hat{\tilde{N}}, q)a_1^\dagger$$

$$b_{1q} = G(\hat{K}, q)b_1^\dagger$$

(16)

$\hat{\tilde{N}}$ and $\hat{K}$ are the respective number operators with eigenvalues $\tilde{n}$ and $K$ and

$$F(\hat{\tilde{N}}, q) = \sqrt{\frac{q^N\psi_1 - q^{-N}\psi_2}{N(q - q^{-1})}}$$

(17)
\[
G(\hat{K}, q) = \sqrt{\frac{q^K \beta_1 - q^{-K} \beta_2}{\hat{K}(q-q^{-1})}}
\]  

(15)

(14) and (15) follow from the general arguments given in Ref.[1]. Using these expressions in (13) (and relabeling \(n_1\) as \(n\) and \(k_1\) as \(k\) etc.) gives

\[
F^n(\hat{N}, q) F(1 - \hat{N} + n)^{1-n} (a_1^n) (a_2^n) |\phi >_{aq}
\]

\[
G^k(\hat{K}, q) G(1 - \hat{K} + k)^{1-k} (b_1^k) (b_2^k) |\phi >_{bq}
\]

\[
(1 - n) F^n(\hat{N}, q) F(1 - \hat{N} + n)^{1-n} (a_1^n) (a_2^n) |\phi >_{aq}
\]

\[
G^k(\hat{K}, q) G(1 - \hat{K} + k)^{1-k} (b_1^k) (b_2^k) |\phi >_{bq}
\]

\[
+n F^n(\hat{N}, q) F(1 - \hat{N} + n)^{1-n} (a_1^n) (a_2^n) |\phi >_{aq}
\]

\[
G^{1-k}(\hat{K}, q) G(1 - \hat{K} + k)^{1-k} (b_1^{1-k}) (b_2^{1-k}) |\phi >_{bq}
\]

Note that with respect to the states |\(\phi >_{aq}\) |\(\phi >_{bq}\), (16) will be indistinguishable from the usual CNOT transformation (13) if

\[
F^n(\hat{N}, q) F^{1-n} (1 - \hat{N} + n) = 1
\]

(17a)

\[
G^k(\hat{K}, q) G^{1-k} (1 - \hat{K} + k) = 1
\]

(17b)

\[
G^{1-k}(1 - \hat{K}, q) G^k(\hat{K} - 1 + k) = 1
\]

(17c)

Equation (17a,b,c) are true for both \(n = 0\) and \(n = 1\) if

\[
\psi_2(q) = q^2 \psi_1 + 1 - q^2 ; \quad \beta_2(q) = q^2 \beta_1 + 1 - q^2
\]

(18)

Thus when the arbitrary functions are chosen in this way, the two qubit CNOT gate can be realised with q-oscillators. Hence all the gates required for universality can also be realised with q-oscillators. So any quantum logic gate can be realised with q-oscillators. Hence quantum computation has an alternative formalism.

5. Two types of possible states

There are two possibilities as regards the arbitrary functions.

Case: I  All of them are unity and hence \(N = \hat{N}\) and similarly \(K = \hat{K}\). So (2a) just relates the operators \(a, a^\dagger\) with \(a_q, a_q^\dagger\). A similar argument holds for the operators \(b, b^\dagger\) and \(b_q, b_q^\dagger\). Also from (2b) we then have \(N = \hat{N}\) and \(K = \hat{K}\). This means that at the occupation number level the deformed states cannot be distinguished from the usual states and we are in the realm of usual quantum computation. Let the number operators for deformed oscillators in Case I be \(n, k\); the states in Case I be |\(>_{1}\). Then

\[
|n - 1/2 >_{1} |k - 1/2 >_{1}
\]

\[
= |n >_{Ia1} |1 - n >_{Ia2} |k >_{Ib1} |1 - k >_{Ib2}
\]

(19)

where \(n = 0, 1; k = 0, 1\) and \(n = \hat{n}; k = \hat{k}\).

Case: II  The arbitrary functions \(\psi_i(q), \beta_i(q), i = 1, 2\) are not all equal to unity. Then \(N = \hat{N} - (1/s) ln \psi(q); \quad K = \hat{K} - (1/s) \ln \beta(q), \quad [(2b)]\). Hence states labelled by the occupation number are different as the eigenvalues of the number operator of usual oscillator states (i.e. usual quantum computation) and the eigenvalues of the number operator of deformed oscillator states are now related by \(n = \hat{n} - (1/s) \ln \psi(q); \quad k = \hat{k} - (1/s) \ln \beta(q)\). This would show up in the Jordan-Schwinger construction.

In Case II denote the states |\(>_{1}\). So

\[
|n - 1/2 >_{II} |k - 1/2 >_{II}
\]

\[
= |n >_{IIa1} |1 - n >_{IIa2} |k >_{IIb1} |1 - k >_{IIb2}
\]

(20)

Consistency demands the following interpretations: The states \(|0 >_{IIa1} |1 >_{IIa2}\) corresponds to an usual oscillator occupation number \(\hat{n} > 0; |0 >_{IIb1} |1 >_{IIb2}\) corresponds to an usual oscillator occupation number \(\hat{k} > 0; |1 >_{IIa1} |0 >_{IIa2}\) corresponds to an usual oscillator occupation number \(\hat{k} > 1; |1 >_{IIa1} |0 >_{IIa2}\) corresponds to an usual oscillator occupation number \(\hat{n} > 1\). So we always have \(\hat{n} > n, \hat{k} > k, \psi(q), \beta(q)\) cannot be unity (i.e. \(\hat{n}, \hat{k}\) cannot be zero) because then we will have \(n = \hat{n}, k = \hat{k}\) i.e. Case I. So the deformed states in Case II can be related to any usual oscillator states with occupation numbers greater than zero.

Denote the \(F\) and \(G\) functions corresponding to the two possibilities by \(F_1, G_1\) and \(F_{II}, G_{II}\). Then

\[
F_1(\hat{N}, q) = \sqrt{\frac{q^N - q^{-N}}{N(q-q^{-1})}}, \quad G_1(\hat{K}, q) = \sqrt{\frac{q^K - q^{-K}}{K(q-q^{-1})}}
\]

(21)

\[
F_{II}(\hat{N}, q) = \sqrt{\frac{q^N \psi - q^{-N}(q^2 \psi + 1 - q^2)}{N(q-q^{-1})}}
\]

\[
G_{II}(\hat{K}, q) = \sqrt{\frac{q^K \beta - q^{-K}(q^2 \beta + 1 - q^2)}{K(q-q^{-1})}}
\]

(22)
where $\psi, \beta$ are the arbitrary functions. Now, properties of $F$ and $G$ have to be understood in terms of their eigenvalues. Then

$$\frac{\text{Eigenvalue of } F_{II}}{\text{Eigenvalue of } F_I} = \left( \frac{q^{2\hat{n}}\psi(q) - q^2\psi(q) + q^2 - 1}{q^{2\hat{n}} - 1} \right)^{1/2} \sim W_{\hat{n}}(\hat{n}, q) \quad (23a)$$

So we may write

$$F_{II} \equiv W_{\hat{n}}(\hat{n}, q)F_I \quad (23b)$$

Similarly

$$G_{II} \equiv X_{\hat{n}}(\hat{n}, q)G_I \quad (24a)$$

$$X(\hat{n}, q) = \left( \frac{q^{2\hat{k}}\beta(q) - q^2\beta(q) + q^2 - 1}{q^{2\hat{k}} - 1} \right) \quad (24b)$$

Thus

$$|n, k >_{II} = W(\hat{n}) W(1 - \hat{n})^{1/2} X(\hat{k})^{1/2} X(1 - \hat{k})^{1/2} |n, k >_I \quad (29)$$

Therefore

$$\frac{I}{II} < n, k |n, k >_{II} \quad \frac{I}{I} < n, k |n, k >_{I}$$

$$= W(\hat{n}, q)^{n} W(1 - \hat{n}, q)^{1-n} X(\hat{k}, q)^{k} X(1 - \hat{k}, q)^{1-k} \quad (30)$$

$n, k, \hat{n}, \hat{k}$ etc. are all numbers. So right hand side of (30) is a function of $q$ only. For $\psi(q) = \beta(q) = 1$, one has $W = X = 1$ and then the two cases are indistinguishable. However, if $\psi, \beta$ are not unity then Case I states are distinguishable from those of Case II at the level of experimental realisations or consequences.

**7. Conclusion**

The CNOT quantum logic gate has been realised with q-deformed oscillators. With this all three logic gates required for universality are now realisable with q-oscillators. Hence all quantum logic gates are realisable with q-deformed qubits. So quantum computation admits an alternative formalism. An advantage of this alternative formalism is the occurrence of arbitrary functions of $q = e^s$. The functions are of the form $q^{\hat{n}} = e^{s\hat{n}}$. So there are at least two parameters, (i) $s, 0 < s < 1$ and (ii)$\hat{n} > 0$. There exist states in this new scheme whose amplitudes can be distinguished from those in the usual one and so the two situations are experimentally comparable. Existence of additional parameters will enable comparison between different experimental scenarios using the usual scheme and the alternative one. These parameters may be utilised to determine whether observed experimental realisations of theoretical predictions obtained from the usual formalism are fully satisfactory or not. If not, then these parameters may provide a framework for computing corrections.

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