Tracking quintessence and \( k \)-essence in a general cosmological background

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We derive conditions for stable tracker solutions for both quintessence and \( k \)-essence in a general cosmological background, \( H^2 \propto f(\rho) \). We find that tracker solutions are possible only when \( \eta \equiv d\ln f/d\ln \rho \approx \text{constant} \), aside from a few special cases, which are enumerated. Expressions for the quintessence or \( k \)-essence equation of state are derived as a function of \( \eta \) and the equation of state of the dominant background component.

I. INTRODUCTION

The universe appears to consist of approximately 30% nonrelativistic matter, including both baryons and dark matter, and 70% dark energy (see Ref. \[1\] for a recent review, and references therein). The evolution of the dark energy density depends on its equation of state, which is usually parametrized in the form

\[ p_{\text{DE}} = w \rho_{\text{DE}}, \]

where \( p_{\text{DE}} \) and \( \rho_{\text{DE}} \) are the pressure and density of the dark energy. Then the density of the dark energy scales as

\[ \rho_{\text{DE}} \propto R^{-3(1+w)}. \]

The simplest model for the dark energy is a cosmological constant, for which \( w = -1 \) and \( \rho_{\text{DE}} = \text{constant} \). More complex models have been proposed, in which the dark energy arises from a scalar field \( \phi \); these are called quintessence models [2, 3]. These models generally give rise to a time-varying \( w_\phi \) and more complex behavior for \( \rho_{\text{DE}} \). One advantage of such models is that certain classes of quintessence potentials lead to tracker behavior, in which the evolution of the scalar field is independent of the initial conditions. The conditions for such tracking behavior have been worked out in detail by Steinhardt, et al. [6].

A second class of models generalizes quintessence to allow for a non-standard kinetic term. These models, dubbed \( k \)-essence, have also been explored in great detail [7, 8, 9, 10, 11, 12, 13, 14, 15]. These models can also lead to tracking behavior, and the conditions necessary for such behavior have been discussed by Chiba [12].

Both quintessence and \( k \)-essence can be generalized to modified versions of the Friedmann equation. In the standard Friedmann equation, the relation between the scale factor \( a \) (or, alternatively, the Hubble parameter \( H \)) and the density is

\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{\rho}{3}, \]

where we set \( 8\pi G = 1 \) throughout. However, various proposals have been put forward to modify this equation at high energy. In type II Randall-Sundrum models, for example, one has [16, 17]

\[ H^2 \propto \rho^2, \]

which is the Cardassian model [19] assumes an expansion law of the form

\[ H^2 = \frac{\rho}{3} + B\rho^n \]

with \( n < 2/3 \). Motivated by these examples, numerous authors have examined the evolution of various dark energy models in the context of non-standard expansion laws [20, 21, 22, 23, 24, 25]. The most general treatments are given in Refs. [23, 24].

Copeland et al. [24] discuss “scaling” quintessence models, i.e., models for which \( w_\phi = w_B \) with an arbitrary expansion law, \( H^2 \propto f(\rho) \). Here \( w_B \) is the ratio of pressure to density for the dominant, “background” fluid, e.g., \( w_B = 0 \) for a matter-dominated universe, and \( w_B = 1/3 \) for a radiation-dominated universe. Tsujikawa and Sami [25] examine arbitrary scalar field models (including both quintessence and \( k \)-essence) with scaling behavior \( (w_\phi = w_B) \) in models with a power-law modification to the Friedmann equation, \( H^2 \propto \rho^{1/3} \).

Here we generalize this earlier work by examining tracking solutions for both quintessence and \( k \)-essence in a general cosmological background characterized by \( H^2 \propto f(\rho) \). Although we adopt the approach of Steinhardt et al. [6] for quintessence and Chiba [12] for \( k \)-essence, our formalism encompasses tracking solutions not only for a wide range of potentials but also for a wide range of \( f(\rho) \). We derive sufficient conditions for both \( V(\phi) \) and \( f(\rho) \) to obtain tracking solutions with a constant \( w_\phi \). This formalism provides us with a generic method to study these solutions for a wide variety of scalar field models such as quintessence, tachyon, \( k \)-essence, and phantom models.
II. QUINTESENCE

A. Tracking solutions

The equation of motion for the $\phi$-field is

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0,$$

where

$$V_\phi \equiv dV/d\phi,$$

and

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = f(\rho).$$

(9)

Here $a$ is the Robertson-Walker scale factor, and $\rho$ is the total density, given by

$$\rho = \rho_B + \rho_\phi,$$

where $\rho_B$ is the background (radiation + matter) density, and $\rho_\phi$ is the scalar field energy density. The standard Hubble expansion law corresponds to equation (9) with $f(\rho) = \rho$; in this paper we allow $f(\rho)$ to have an arbitrary functional form.

By definition, the tracking solutions are the solutions to which the evolution of the scalar field $\phi$ converges for a wide range of initial conditions for $\phi$ and $\dot{\phi}$. We follow the approach prescribed by Steinhardt et al. [6] for quintessence, but now generalize it to the arbitrary expansion law given by equation (9). For tracking solutions, $w_\phi$ is nearly constant [6], where $w_\phi$ is given by

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{1}{2}\dot{\phi}^2 - V.$$

(10)

It follows from equation (9) that

$$H \approx \frac{3H^2}{2} \eta[(w_\phi - w_B)(1 - \Omega_\phi) - (1 + w_\phi)],$$

(12)

where $\eta$ encodes the information on the generalized expansion law in equation (9):

$$\eta = \frac{d\ln f(\rho)}{d\ln(\rho)}.$$

(13)

For the standard Hubble expansion, $\eta = 1$. In this paper, we will confine our attention to the case $\eta > 0$, and our conclusions will be valid only for this case. However, we note that $\eta < 0$ can lead to interesting types of behavior (e.g., a phantom-like future singularity in a matter-dominated universe). Several specific models of this type are mentioned in Ref. [20].

By combining these relations, it is useful to cast the equation of motion into the following form:

$$\frac{V_\phi}{\sqrt{V}} = \pm 3H\sqrt{\frac{1 - w_\phi^2}{2}(1 + x')},$$

(14)

where $x = (1 + w_\phi)/(1 - w_\phi) = \frac{1}{2}\dot{\phi}^2/V$ is the ratio of the kinetic to potential energy for $\phi$, and $x' \equiv d\ln x/d\ln a$. The $\pm$ sign depends on whether $V_\phi > 0$ or $V_\phi < 0$, respectively. It follows from equation (14) that the tracker condition ($\dot{w}_\phi \approx 0$) becomes

$$V_\phi \sqrt{V} \approx \frac{1}{2} \left(\frac{\rho_\phi}{\rho_B}\right)^{\frac{3}{2}}.$$

(15)

This is the generalization of the Steinhardt et al. [6] tracking condition to an arbitrary expansion law.

As in Ref. [6], we define the function

$$\Gamma_V \equiv V_\phi V/(V_\phi)^2,$$

(16)

whose properties determine whether tracking solutions exist. By taking the time derivative of equation (14) and combining with the equation (12) and (14) itself, we obtain the following equation:

$$\Gamma_V - \frac{1 + \eta}{2} \frac{\eta(w_B - w_\phi)\Omega_B}{2(1 + w_\phi)} - \frac{\eta(w_B - w_\phi)\Omega_B + \eta + (\eta - 2)w_\phi}{2(1 + w_\phi)} \frac{x'}{6 + x'} - \frac{2}{1 + w_\phi} x'' \left(6 + x''\right)^2,$$

(17)

where $x'' \equiv d^2\ln x/d\ln a^2$. As expected, equation (17) reduces to the corresponding equation in Ref. [6] for $\eta = 1$. In a universe dominated by a background fluid ($\Omega_B \approx 1$) with $w_\phi \approx$ constant and nearly constant $\Gamma_V$, the above equation becomes

$$\Gamma_V \approx \frac{\eta + 1}{2} + \frac{\eta(w_B - w_\phi)}{2(1 + w_\phi)} \left(1 + \frac{\rho_B}{\Omega_B}\right).$$

(18)

In deriving the above equation, the plausibility of the condition that $\Gamma_V \approx$ constant has been discussed in detail in Ref. [6]. The crucial point is that this condition encompasses a wide range of potentials including inverse power law potentials and combinations of inverse power law terms to give rise to tracking solutions.

We must know the appropriate restrictions on $\eta$, i.e., on $f(\rho)$ to extract the tracking solutions from equation (17). Since the left-hand side of equation (18) is nearly constant, it follows that $\eta$ must be nearly constant during background domination, i.e., the function $f(\rho)$ must satisfy (18) for a nearly constant $\eta$. Thus we require an extra condition, in addition to the conditions on $\Gamma_V$, to derive tracking solutions for both quintessence and k-essence. It is obvious that this extra condition arises from the extra “degree of freedom” in choosing a different cosmological background. The only case for which $\eta$ is exactly constant is $f(\rho) \propto \rho^n$ for a constant $n$. This power-law behavior includes both the Randall-Sundrum and Gauss-Bonnet models as special cases, and it was studied in detail in
Ref. [23]. Of course, more general conditions can produce an expression for $f(\rho)$ that is roughly constant over a wide range in the scale factor. For instance, a sum of power laws, e.g., as in equation (19), gives a value for $\eta$ that is nearly constant over most of the evolution of the universe, i.e., at all times except for the epoch when the two contributions to $f(\rho)$ are roughly equal.

Note that there are a few trivial special cases for which this argument breaks down. In particular, if $V$ is a constant, the right hand side of equation (14) must be zero; this can be achieved by taking $w_\phi = \pm 1$. The case $w_\phi = -1$ corresponds to a non-zero constant potential, while $w_\phi = 1$ is the solution for $V = 0$. Both of these results are independent of the value of $H$ on the right-hand side of equation (14) and are therefore independent of $\eta$.

The validity of equation (18) may be checked by comparing with the results obtained by Sami et al. [23]. For scaling solutions with a constant $w_\phi$ in a background dominated universe, the potential function takes the following form [23]

$$ V(\phi) \propto \phi^{-\alpha}, $$

where $\alpha$ is constant. Then we obtain from equation (18)

$$ 1 + w_\phi \approx \eta(1 + w_B) \frac{\alpha}{\alpha + 2}. $$

This solution agrees with the result obtained in Ref. [23].

B. Stability of the tracking solutions

So far, we have derived solutions with constant $w_\phi$ in a general cosmological background; now we want to check the stability of these solutions with constant $w_\phi$. In order to check the stability, we perturb the tracker value of $w_\phi$, which we will call $w_0$, by an amount $\delta$. Then we expand equation (17) to lowest order in $\delta$ and its derivatives to obtain

$$ 2\delta'' + 3[\eta(1 + w_B) - 2w_0]\delta' 
+ 9\eta(1 + w_B)(1 - w_0)\delta = 0, $$

where the prime means $d/d\ln a$ and $w_0$ is the value of $w_\phi$ derived from equation (18). The solution of this equation is

$$ \delta \propto a^\gamma, $$

where

$$ \gamma = -\frac{3}{4} \left[ \eta(1 + w_B) - 2w_0 \right] 
\pm \frac{3i}{4} \sqrt{8\eta(1 + w_B)(1 - w_0) - [\eta(1 + w_B) - 2w_0]^2}. $$

In the derivation of this equation, $\Gamma_V$ and $\eta$ are assumed to be constant.

In order to have $\delta$ decay, the real part of $\gamma$ has to be negative. Hence, it follows that

$$ w_0 < \frac{\eta(1 + w_B)}{2}, $$

provided the quantity under the square root is positive. If the quantity under the square root is negative (so that both values are real), then the above equation is also a necessary condition since the first term under the square root is always positive, provided $\eta > 0$ and $w_0 < 1$. Using equation (18), the above inequality can be written in terms of $\Gamma_V$ as

$$ \Gamma_V > \frac{3\eta(1 + w_B) + 2}{2\eta(1 + w_B) + 4}. $$

Therefore, for a nearly constant $\Gamma_V$, $\eta$ and $w_\phi$, the tracker condition, i.e., equation (15) gives the following possibilities:

a. If $w_\phi < w_B$, then $\Omega_\phi$ increases with time. Then we conclude from equation (17) that $|V_\phi/V^{1/2}|$ decreases for a tracker solution. However, taking the time derivative of $V_\phi/V^{1/2}$, we obtain

$$ \frac{d}{dt} \left( \frac{V_\phi}{V^{1/2}} \right) = \frac{V_\phi^2}{V^{3/2} \phi} \left( \Gamma_V - \eta + \frac{1}{2} \right). $$

Hence, $|V_\phi/V^{1/2}|$ decreases if $\Gamma_V > \frac{1 + \eta}{2}$. Thus, $w_\phi < w_B$ is observed for

$$ \Gamma_V > \frac{1 + \eta}{2}. $$

Combining this with the condition for stable tracking behavior (equation (25)), we obtain

$$ \Gamma_V > \max \left[ \frac{3\eta(1 + w_B) + 2}{2\eta(1 + w_B) + 4} \right]. $$

This is the most interesting case, as it gives viable models for an accelerating universe. These conditions encompass more solutions than the ones derived in Refs. [23, 24]. For example, for the exponential potential, we have $\Gamma_V = 1$, and the above conditions are satisfied as long as $\eta < 1$ (including, for example, the Gauss-Bonnet expansion law).

b. If $w_\phi > w_B$, then tracking behavior is observed for

$$ \frac{3\eta(1 + w_B) + 2}{2\eta(1 + w_B) + 4} < \Gamma_V < \frac{1 + \eta}{2}. $$

c. If $\Gamma_V = (1 + \eta)/2$, then $w_\phi = w_B$. This is one of the main results (using somewhat different notation) derived in Ref. [24].

III. k-essence

A. Tracking solutions

In general, $k$-essence can be defined as any scalar field with non-canonical kinetic terms, but in practice such
models are usually taken to have a Lagrangian of the form:

$$\mathcal{L} = V(\phi)F(X), \quad (30)$$

where $\phi$ is the scalar field, and $X$ is defined by

$$X = \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi. \quad (31)$$

The pressure in these models is given by

$$p_\phi = \mathcal{L}, \quad (32)$$

where $\mathcal{L}$ is given by equation (30), while the energy density is

$$\rho_\phi = V(\phi)[2XF_X - F], \quad (33)$$

where $F_X \equiv dF/dX$. Therefore, the equation of state parameter, $w_\phi \equiv p_\phi/\rho_\phi$, is just

$$w_\phi = \frac{F}{2XF_X - F}. \quad (34)$$

In defining the sound speed, we follow the convention of Garriga and Mukhanov [8], who argued that the relevant quantity for the growth of density perturbations is

$$c_s^2 = \frac{(\partial p/\partial X)}{(\partial p/\partial X)} = \frac{F_X}{F_X + 2XF_{XX}}, \quad (35)$$

with $F_{XX} \equiv d^2F/dX^2$.

In a flat Robertson-Walker metric, the equation of motion for the $k$-essence field takes the form:

$$(F_X + 2XF_{XX})\ddot{\phi} + 3HF_X \dot{\phi} + (2XF_X - F)\frac{V_\phi}{V} = 0. \quad (36)$$

We can express the equation of motion for $\phi$ in an alternative form which will be useful for subsequent analysis:

$$\pm \frac{V_\phi}{V} \sqrt{2X} = H \left(1 + \frac{w_\phi}{2}\right)(6 + Ay'), \quad (37)$$

where

$$A = \frac{(XF_X - F)(2XF_{XX} + F_X)}{XF_X^2 - FF_X - XF_{XX}} = \frac{1 - w_\phi}{c_s^2 - w_\phi}. \quad (38)$$

$y = (1 + w_\phi)/(1 - w_\phi)$ and $y' = d\ln y/d\ln a$, and plus (minus) sign corresponds to $\phi < 0$ ($\phi > 0$), respectively. The tracker condition ($w_\phi \approx \text{constant}$) becomes

$$\pm \frac{V_\phi}{V(n+2)/2} \approx \left(\frac{F}{\Omega_\phi}\right)^{n/2} \frac{1}{\sqrt{2X}}. \quad (39)$$

It is not surprising to see that the tracker condition for $k$-essence has an extra “degree of freedom” in $F(X)$. The functional form of $F(X)$ plays a crucial role in determining the tracking conditions for $k$-essence and we shall consider it in the next section.

After taking the time derivative of equation (37) and using equation (32), we obtain

$$\Gamma_V - (1 + \frac{\eta}{2}) = \frac{\eta(w_B - w_\phi)\Omega_B}{2(1 + w_\phi)} - \frac{[\eta(w_B - w_\phi)\Omega_B + \eta + (\eta - 2)w_\phi]Ay'}{2(1 + w_\phi)(6 + Ay')} - \frac{2(1 - w_\phi)Ay''}{(1 + w_\phi)(6 + Ay')^2(c_s^2 - w_\phi)^2} + \frac{2(\dot{\phi}^2 - (dc_s^2/dt)(1 - w_\phi))y'/H}{(1 + w_\phi)(6 + Ay')^2(c_s^2 - w_\phi)^2}, \quad (40)$$

where $y'' = d^2\ln y/d\ln a^2$. We note that for $\eta = 1$, equation (40) reduces to the one derived in Ref. [12].

For a background-dominated universe with a constant $w_\phi$ and almost constant $\Gamma_V$, the tracker equation (40) reduces to

$$\Gamma_V \approx \frac{\eta + 2}{2} \frac{\eta(w_B - w_\phi)}{2(1 + w_\phi)}, \quad (41)$$

Note that equation (41) for $k$-essence closely resembles equation (13) for quintessence; the only difference is the constant appearing in the first term. For the standard Hubble expansion law ($\eta = 1$), we obtain

$$\Gamma_V \approx 1 + \frac{1}{2} \frac{1 + w_B}{1 + w_\phi}, \quad (42)$$

in agreement with the results of Ref. [12].

**B. Stability of the tracking solutions**

To determine the stability of the tracking solution, we repeat the calculation of Sec. II.B. for the case of $k$-essence. We assume a $k$-essence field with equation of state parameter $w_0$ and perturb $w_0$ by an amount $\delta$. Then we expand equation (40) to lowest order in $\delta$ and its derivatives to obtain

$$2\delta'' + 3[\eta(1 + w_B) - 2w_0]\delta' + 9\eta(1 + w_B)(c_s^2 - w_0)\delta = 0, \quad (43)$$

where the prime means $d/d\ln a$. The solution of this equation is

$$\delta \propto a^\gamma, \quad (44)$$

where

$$\gamma = -\frac{3}{4} [\eta(1 + w_B) - 2w_0] \quad \frac{3i}{4} \sqrt{8\eta(1 + w_B)(c_s^2 - w_0) - [\eta(1 + w_B) - 2w_0]^2}. \quad (45)$$
Again, in order to have $\delta$ decay, the real part of $\gamma$ has to be negative. Hence, it follows that
\begin{equation}
w_0 < \frac{\eta(1 + w_B)}{2},
\end{equation}
and
\begin{equation}
w_0 < c_s^2.
\end{equation}

At this point, the above conditions cannot be translated into relations in terms of $\Gamma_V$ without considering the functional form of $F(X)$, since $w_\phi$ and $c_s^2$ both depend on $F(X)$. Now we discuss the restrictions on the form of $F(X)$ for constant $w_\phi$.

A variety of functional forms for $F(X)$ and $V(\phi)$ have been considered in $k$-essence models (see, e.g., Refs. [10, 11]). However, we will focus on the form of $F(X)$ responsible for stable tracking solutions for a constant equation of state. In order to find the functional form of $F(X)$ for stable tracking solutions with constant $w_\phi$, we note that equation (34) can be written as
\begin{equation}
\frac{\partial \ln F(X)}{\partial \ln X} = \frac{1 + w_\phi}{2w_\phi},
\end{equation}

### Case 1
The first possibility emerges if we treat equation (39) as a differential equation and derive the general solution, which is
\begin{equation}
F(X) = X^\beta,
\end{equation}
where $\beta$ is a constant, and $w_\phi$ is then
\begin{equation}
w_\phi = \frac{1}{2\beta - 1}.
\end{equation}

By inserting equation (40) into equation (35), we obtain
\begin{equation}
c_s^2 = \frac{1}{2\beta - 1},
\end{equation}
so that
\begin{equation}
c_s^2 = w_\phi.
\end{equation}

These solutions were previously derived in Ref. [27]; we note here that they are independent of $\eta$, and therefore of the expansion law. These solutions also do not depend on the form of $V(\phi)$.

It is obvious from equation (52) that $c_s^2 < 0$ for any of these models with negative pressure ($w_\phi < 0$). If $c_s^2 < 0$, then the $k$-essence fluid is unstable against perturbation. Moreover, equation (40) describes a phantom field for $0 < \beta < 1/2$.

### Case 2
A second class of solutions arises if the field evolves to a state for which $X = X_0$, where $X_0$ is a constant [2, 13]. In this case, we have
\begin{equation}
\frac{\partial \ln F(X)}{\partial \ln X} \bigg|_{X=X_0} = \frac{1 + w_\phi}{2w_\phi},
\end{equation}
Again, we see that equation (53) is independent of $\eta$ and hence, independent of the expansion law. However, the condition for a stable solution of the form $X = X_0$ does depend on $\eta$, as we now show.

From equation (41), the tracking conditions, equations (40)-(47), take the following form in terms of $\Gamma_V$:
\begin{equation}
\Gamma_V > \frac{2\eta(1 + w_B) + 2}{\eta(1 + w_B) + 2}.
\end{equation}

Therefore, for a nearly constant $\Gamma_V$, $\eta$, and $w_\phi$, equation (39) gives the following possibilities:

a. If $w_\phi < w_B$, then $\Omega_\phi$ increases with time. Then we conclude from equation (39) that $\sqrt{2X_w/\Gamma_{V}^2 /\Gamma_{V}^{2+2}}$ decreases for a tracker solution. However, taking the time derivative of $\left(\sqrt{2X_w/\Gamma_{V}^2 /\Gamma_{V}^{2+2}}\right)$, we obtain
\begin{equation}
\frac{d}{dt} \left( \frac{\sqrt{2X_w}}{\Gamma_{V}^{2+2}} \right) = 2X_{w} \left( \frac{2X_{w}^{2}}{\Gamma_{V}^{2+2}} \right) \left( \Gamma_{V} - \frac{\eta + 2}{2} \right).
\end{equation}

In the derivation of this equation, we have used the condition that $X = X_0$. Hence, $\sqrt{2X_w/\Gamma_{V}^2 /\Gamma_{V}^{2+2}}$ decreases if $\Gamma_V > (\eta + 2)/2$. Thus, $w_\phi < w_B$ for
\begin{equation}
\Gamma_V > \frac{\eta + 2}{2}.
\end{equation}

Combining this with the conditions for stable tracking behavior (equations (54-55)), we obtain
\begin{equation}
\Gamma_V > \max \left[ \frac{\eta + 2}{2}, \frac{2\eta(1 + w_B) + 2}{\eta(1 + w_B) + 2} \right] > \frac{\eta(1 + w_B)}{2(1 + c_s^2)}.
\end{equation}

b. If $w_\phi > w_B$, then tracking behavior is observed for
\begin{equation}
\max \left[ \frac{2\eta(1 + w_B) + 2}{\eta(1 + w_B) + 2}, \frac{\eta(1 + w_B)}{2(1 + c_s^2)} \right] < \Gamma_V < \frac{\eta + 2}{2}.
\end{equation}

c. If $\Gamma_V = (\eta + 2)/2$, then $w_\phi = w_B$. This case encompasses the solutions presented in Ref. [27].

### IV. DISCUSSION

We have extended the formalism in Refs. [6] and [12] to derive the tracker conditions for quintessence and $k$-essence, respectively, for an arbitrary cosmological expansion law, $H^2 = f(\rho)$, when the universe is dominated by a background fluid. Our main new result is that, with the exception of the special cases discussed above, tracking solutions for either quintessence or $k$-essence are possible only for $\eta = d\ln f/d\ln \rho \approx \text{constant}$, which is the case only when $f(\rho)$ is well-approximated as a power-law. In fact, such power-law behavior corresponds to most of
the models previously considered for non-standard expansion laws.

We note further that the expressions for \( w_\phi \) for both quintessence and \( k \)-essence, and the conditions for stable tracking behavior, can be derived by replacing \( 1 + w_B \) with \( \eta(1 + w_B) \) in all of the corresponding equations for the standard expansion law. This is not surprising, since a given value of \( w_B \) corresponds to a background density scaling as \( \rho_B \propto a^{-3(1+w_B)} \). Taking a constant value of \( \eta \) in equation (9) then gives \( H^2 \propto a^{-3\eta(1+w_B)} \), so \( 1 + w_B \) is replaced by \( \eta(1 + w_B) \) in the expression for \( H^2 \) (see also the discussion in Ref. [28]).

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