A New Approach to Nonrenormalizable Models

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Abstract

Nonrenormalizable quantum field theories require counterterms; and based on the hard-core interpretation of such interactions, it is initially argued, contrary to the standard view, that counterterms suggested by renormalized perturbation theory are in fact inappropriate for this purpose. Guided by the potential underlying causes of triviality of such models, as obtained by alternative analyses, we focus attention on the ground-state distribution function, and suggest a formulation of such distributions that exhibits nontriviality from the start. Primary discussion is focused on self-interacting scalar fields. Conditions for bounds on general correlation functions are derived, and there is some discussion of the issues involved with the continuum limit.

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1 Introduction

Nonrenormalizable quantum field theories have long been the “black sheep of the family”.\footnote{Common meaning: “A worthless or disgraced member of the family”. See, for example, \url{http://www.phrases.org.uk/meanings/66250.html}.} While super-renormalizable and strictly renormalizable quantum field theories (especially asymptotically-free theories) have been well served by renormalized perturbation theory, this has not been the case for the usual nonrenormalizable family members. For them – so the perturbation story goes – an infinite set of distinct counter-terms are necessary, requiring thereby an infinite number of experiments to establish the needed coefficients of the counterterms, an endeavor which nullifies any predictive power the theory may have had. A key concept in the preceding sentence relates to the perturbative counterterms, and we shall stress the shortcomings of this concept below.

Nonrenormalizable models may also be approached nonperturbatively. Here, we have in mind the analysis of $\varphi^4_n$ models for spacetime dimension $n \geq 5$ made by Aizenman [1] and Fröhlich [2]. In their approach, they formulated the models as Euclidean-space lattice theories with arbitrary coefficients for the mass term and the coupling constant, both taken as functions of the cutoff parameter, namely, the lattice spacing $a$; of course, the coupling constant was required to be nonnegative. They were rigorously able to show, independent of the choice of the indicated parameters, that the continuum limit (as combined with the infinite volume limit) of the theory led to the result of a (possibly generalized) free field. In other words, the manifestly non-Gaussian nature of the lattice field distribution has, in the continuum limit, become a strictly Gaussian distribution. Since Monte Carlo calculations suggest that triviality holds for $\varphi^4_4$ as well, this overall story has given rise to the widely held view of the “triviality of the $\varphi^4$ theories”.

In the author’s judgement, the infinite number of distinct counter-terms as well as the triviality result, while correct conclusions given their respective underlying assumptions, are both unacceptable as physically correct quantum statements for the problem at hand. This conclusion is based on the following assertions: (1) For a nonrenormalizable theory, results based on regularization and the introduction of counterterms given by a perturbation analysis about the free theory are unacceptable for the simple reason, as
we shall argue, that the interacting nonrenormalizable theories are not even continuously connected to the free theory as the coupling constant is reduced to zero; (2) The classical $\varphi^4_n, n \geq 5$, models are manifestly nontrivial theories exhibiting nonvanishing scattering, etc. If the quantum formulation of such models is taken to be the trivial one, then it follows that the classical limit of that quantum theory leads to a trivial classical theory which is not equivalent to the original classical theory one started with. Such a dilemma can only mean that the quantum formulation leading to triviality cannot be the physically correct quantum formulation for nonrenormalizable $\varphi^4_n, n \geq 5$, models.

We claim there is an alternative formulation of the quantum theory for such models that can explain the unacceptable results of infinitely many counterterms as well as triviality, and which furthermore offers alternative calculational possibilities that we believe may lead to satisfactory results. The purpose of the present paper is to lay out the theoretical arguments supporting the point of view under present consideration. We start by reexamining the limiting Gaussian behavior of the $\varphi^4_n, n \geq 5$, models.

In probability theory, the Central Limit Theorem (CLT) leads to Gaussian distributions for a very large class of independent, identically distributed random variables. By concentrating attention on random variables that are independent and identically distributed, it is of course possible to demonstrate the CLT quite generally. However, it stands to reason that there exist distributions of random variables that are neither identical nor independent, but which are close to such behavior in some unspecified manner, yet the combination nevertheless tends to a Gaussian distribution as the number of variables increases without limit. General theorems for such cases are naturally most unlikely since a Gaussian or non-Gaussian limit depends explicitly on the particular details at hand. Nevertheless, based on the evidence, it is plausible to regard $\varphi^4_n, n \geq 5$, models as satisfying the criteria to lie within the basin of attraction of the CLT. Our further development will be partially influenced by this viewpoint.

In Sec. 2 we outline the argument why nonrenormalizable interactions act as discontinuous perturbations, and as such, counterterms suggested by regularized and renormalized perturbation theory are unsuitable. Attention shifts from the lattice action and the field distribution it engenders in Sec. 3, and instead focusses on the sharp-time, ground-state distribution. Arguing from the CLT, we propose that this distribution should be a generalized
Poisson distribution, a subject discussed at length in Sec. 4. A generalization to a linear superposition of Poisson distributions is the subject of Sec. 5. It is shown in Sec. 6 that general spacetime correlation functions can be expressed as, or are at least bounded by, properties of the Poisson distribution, while Sec. 7 offers a useful reformulation of some of the expressions developed in Sec. 6 as well as a brief discussion of the continuum limit. Section 8 is devoted to a conclusion, and some details about specific properties of a given generalized Poisson distribution are discussed in the Appendix.

It is not without interest that in an archived (but unpublished) paper [3], the author already made the proposal that the ground-state distribution for nonrenormalizable scalar models should be a generalized Poisson distribution. Unfortunately, how that proposal was used in that earlier paper to suggest an auxiliary potential and thus a lattice action was incorrect. Hopefully that situation has been advanced in the present paper.

2 Inadequacy of Regularized, Renormalized Perturbation Theory

2.1 The hard-core picture

Chapter 8 in [4] describes in some detail the hard-core picture of nonrenormalizable interactions, and for a detailed account of that approach we can do no better than to encourage the reader to examine that presentation, which, importantly, is for all intents and purposes an independently readable chapter. However, for the sake of the reader who does not need a detailed account of the hard-core picture of nonrenormalizability, we offer the following brief synopsis.

In general terms, let the theory in question be described by a formal (Euclidean-space) functional integral commonly called a generating functional, which for the “free” theory is given by

\[ S_0\{h\} \equiv N_0 \int e^{\int h \phi d^nx - W_0(\phi)} D\phi , \]

where \( h \) is a smooth “source” function, and \( W_0(\phi) \geq 0 \) denotes the free field (Euclidean) action, an expression which is often quadratic in the fields
but need not be so. Likewise we have the generating functional for the “interacting” theory

\[ S_\lambda \{ h \} \equiv N_\lambda \int e^{\int h \phi \, d^x - W_0(\phi) - \lambda V(\phi)} \, D\phi , \]

where the coupling constant \( \lambda > 0 \), and \( V(\phi) \geq 0 \) denotes the (Euclidean) interaction action. In each case, the formal normalization is chosen so that \( S_0 \{ 0 \} = S_\lambda \{ 0 \} = 1 \).

We say that \( V \) is a continuous perturbation of \( W_0 \) if

\[ \lim_{\lambda \to 0} S_\lambda \{ h \} = S_0 \{ h \} , \]

for all \( h \) (say \( C_0^\infty \)); in addition, we say that \( V \) is a discontinuous perturbation of \( W_0 \) if, instead,

\[ \lim_{\lambda \to 0} S_\lambda \{ h \} = S'_0 \{ h \} \neq S_0 \{ h \} , \]

for all \( h \), where \( S'_0 \{ h \} \) is referred to as the generating functional for the “pseudofree” theory.

At first glance, it seems self evident from the given expressions for the generating functionals that the limit of \( S_\lambda \{ h \} \) as \( \lambda \to 0 \) must be \( S_0 \{ h \} \). Moreover, the unquestioned belief that this must be so would appear to be a tacit assumption behind any perturbative analysis of \( S_\lambda \{ h \} \) in a series expansion in \( \lambda \) about \( \lambda = 0 \). However formal (i.e., nonconvergent) that series may be, the introduction and use of such a series expansion presumes that \( V \) is a continuous perturbation of \( W_0 \). If \( V \) is not a continuous perturbation, i.e., if \( V \) is in fact a discontinuous perturbation of \( W_0 \), then there is no way in which one could legitimately claim that “\( S_\lambda \{ h \} \) has a perturbation series expansion about \( S_0 \{ h \} \)”.

That statement is as likely to be true as the statement that properties of the (discontinuous!) function

\[ F(\lambda) \equiv e^\lambda, \quad \lambda > 0, \]
\[ F(0) \equiv \pi \]

can be determined by a power series expansion about \( \lambda = 0 \).

Qualitatively speaking, a continuous perturbation is one for which

\[ W_0(\phi) < \infty \implies V(\phi) < \infty . \]
Similarly, a discontinuous perturbation is one for which

\[ W_0(\phi) < \infty \implies V(\phi) < \infty, \]

or in other words, there are fields (comprising a set of nonzero measure) for which

\[ W_0(\phi) < \infty, \quad V(\phi) = \infty. \]

For a discontinuous perturbation, the contributions such fields would have made if the interaction term \( V \) had never been present are now missing altogether because those fields are effectively projected out by the factor \( e^{-\lambda V} \), for any \( \lambda > 0 \), however small. If we introduce

\[ X(\phi) \equiv 1, \quad \text{if} \quad W(\phi) < \infty, \quad V(\phi) < \infty, \]

\[ X(\phi) \equiv 0, \quad \text{if} \quad W(\phi) < \infty, \quad V(\phi) = \infty, \]

then we see that we can freely redefine \( S_\lambda\{h\} \) as

\[ S_\lambda\{h\} = N_\lambda \int e^{\int h(\phi) d^n x - W_0(\phi) - \lambda V(\phi)} X(\phi) D\phi. \]

Now, as \( \lambda \to 0 \), we can readily see that we obtain

\[ S_0'\{h\} = N_0' \int e^{\int h(\phi) d^n x - W_0(\phi)} X(\phi) D\phi. \]

In brief, the interaction term has acted partially as a hard core projecting out – thanks to \( X(\phi) \) – certain contributions that would have been included for the free generating functional!

To show the relevance of this discussion to \( \phi_n^4 \) models, we need only recall for \( n \leq 4 \) (the renormalizable cases) that

\[ \left\{ \int \phi^4(x) d^n x \right\}^{1/2} \leq (4/3) \int \{ [\nabla \phi(x)]^2 + m^2 \phi^2(x) \} d^n x \]

for any \( m^2 > 0 \), while for \( n \geq 5 \) (the nonrenormalizable cases) there are singular fields, such as

\[ \phi_s(x) = e^{-x^2} \frac{1}{|x|^p}, \quad \frac{n}{4} < p < \frac{n}{2} - 1, \]
for which
\[ \int \{ [\nabla \phi_s(x)]^2 + m^2 \phi_s^2(x) \} \, d^n x < \infty , \]
while
\[ \int \phi_s^4(x) \, d^n x = \infty , \]
fulfilling the requirement for the interaction to act as a hard core.

A derivation of the \((4/3)\) bound given above when \(n \leq 4\) is part of the discussion in Chapter 8 of [4]. The factor \((4/3)\) also provides a bound for the more general \(\phi_p^p\) models when \(p \leq 2n/(n-2)\) (the renormalizable cases), while there is no finite bound when \(p > 2n/(n-2)\) (the nonrenormalizable cases). When the gravitational action is divided into quadratic and non-quadratic terms, the latter term has the properties to act as a discontinuous potential consistent with the well-known nonrenormalizability of gravity. In fact, as also detailed in Chapter 8 of [4], discontinuous perturbations of the harmonic oscillator arise even in conventional quantum mechanics. If hard-core potentials exist in quantum mechanics, it should be no surprise that they arise in quantum field theory!

We are about to leave the general description of hard-core interactions hoping that the reader has accepted that hard-core interactions may indeed exist and that nonrenormalizable quantum field models are likely candidates. Harder to accept, but nevertheless a direct consequence of the existence of hard-core interactions, is the concept that in such cases counterterms suggested by a (regularized) perturbative power series expansion about the original free theory are not relevant – indeed, harsh as it may seem, we can even say they are completely irrelevant! In fact, such terms are entirely misleading because regularized, perturbative counterterms are designed to preserve a continuity of the interacting theory with the original free theory, a continuity which in fact never existed!

This statement about perturbation theory is no doubt difficult to accept for some readers because it certainly goes against – even contradicts – the “general wisdom” of how nonrenormalizable models are normally presented. If the reader’s prejudice in favor of perturbation theory and its unquestioned, universal applicability prevent you from accepting the statement that in genuine hard-core cases perturbation theory about the original free theory is irrelevant, then this article is not for you. By all means, stop reading here and
now! However, if you can accept the fact that hard-core interactions have the power to change almost everything regarding perturbation theory about the free theory, then we invite you to read on. Indeed, we can offer you the comforting possibility that the interacting theory may indeed have a meaningful perturbation expansion – not about the free theory – but about the \textit{pseudofree} theory, which after all is the theory to which it is continuously connected as $\lambda \to 0$! As an elementary illustration of that very thought, let us revisit the discontinuous function $F(\lambda)$ introduced earlier. This time, however, we introduce an analogue of the pseudofree theory by letting

$$
F'(\lambda) \equiv F(\lambda) = e^\lambda, \quad \lambda > 0,
$$

$$
F'(0) \equiv \lim_{\lambda \to 0} F(\lambda) = 1.
$$

Clearly – and unlike the original function $F(\lambda)$ – the new function $F'(\lambda)$ does have a perturbative power series representation about the “pseudofree” value, $F'(0) = 1$, namely,

$$
F'(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!},
$$

which is valid in the present case for all $\lambda \geq 0$. More importantly, highly specialized, soluble nonrenormalizable models (see Chapters 9 & 10 in [4]) possess solutions that \textit{do} admit entirely reasonable perturbative treatments of their interactions about the appropriate pseudofree solution. We believe valid perturbative analyses about suitable pseudofree solutions hold widely, and we will make the assumption in this paper that such perturbation formulations exist.

Accepting the concept that the nonlinear interaction for $\varphi^4_n$, $n \geq 5$, models acts partially like a hard core leads immediately to the conclusion that some form of counterterm is needed, at the very least, to represent the irremovable effects of the hard core. The triviality argument also supports the need for counterterms since mass, coupling constant, and field-strength renormalization alone are insufficient to escape triviality. Our previous discussion has been necessary to open the way to consider counterterms other than those suggested by renormalized perturbation theory. Our task is to find suitable counterterms that will avoid triviality – a highly nontrivial task to be sure!
3 Focus on the Sharp-Time, Ground-State Distribution

3.1 Triviality also holds at sharp time

A Euclidean spacetime distribution that exhibits triviality is necessarily Gaussian, and has vanishing truncated correlation functions beyond second order. Such correlation functions \textit{a fortiori} vanish at equal times, a fact that implies that the associated ground-state distribution for the limiting Hamiltonian has a Gaussian distribution itself. Stated otherwise, the lattice-space ground-state distribution – the square of the lattice-space ground-state wave function itself – which because of the nonlinear interaction is manifestly non-Gaussian, has, in the continuum limit, become Gaussian. Just as the full-time distribution has become Gaussian because it lies in the appropriate basin of attraction for the CLT, it is equally plausible to conclude that the sharp-time field distribution for the nonlinear theory also lies in the associated basin of attraction of the CLT. Conversely, if we can ensure that the ground-state distribution remains non-Gaussian, then the non-Gaussian character of the full-time distribution will be assured. This remark leads us to focus some attention on the putative ground state.

3.2 Lattice space version of the ground-state wave function

We let the even, positive function \( \Psi(\phi) = \Psi(-\phi) \), denote the ground-state wave function on the Euclidean-space lattice, and we adjust the potential by a constant so that this ground state has zero energy. Consequently, the nonnegative Hamiltonian operator for the lattice is given by

\[
H = -\frac{1}{2} a^{-s} \sum_k' \frac{\partial^2}{\partial \phi_k^2} + \mathcal{V}(\phi),
\]

where the potential is given by

\[
\mathcal{V}(\phi) \equiv \frac{1}{2} a^{-s} \sum_k' \frac{1}{\Psi(\phi)} \frac{\partial^2 \Psi(\phi)}{\partial \phi_k^2}.
\]
Let us assume that the lattice has lattice spacing $a$, is hypercubic with periodic boundary conditions in all coordinate-axis directions, and the number of lattice sites $k = (k_0, k_1, k_2, \ldots, k_s)$, $k_j \in \mathbb{Z}$, $s = n - 1$, in each axial direction is $L$. Here the primed sum $\Sigma'_k$ signifies a sum over spatial values of $k$, i.e., $k_1, k_2, \ldots, k_s$, at some specific, but generally implicit, fixed, temporal value of $k$, i.e., $k_0$. The argument $\phi$ stands for the collection of lattice fields $\{\phi_k\}$ at an equal lattice time. Thus if $k = (k_0, k_1, k_2, \ldots, k_s)$, we select $k_0$ as the future time direction (under a possible Wick rotation), and so $\phi = \{\phi_k : \text{all } k \text{ with } k_0 \text{ fixed}\}$.

The probability distribution constructed as the square of the ground-state wave function may alternatively be characterized by its characteristic function

$$C(h) \equiv \int e^{i \Sigma'_k h_k \phi_k a^s} \Psi(\phi)^2 \Pi'_k d\phi_k;$$

the symbol $\Pi'_k$ denotes a product over spatial values of $k$ holding the temporal value fixed. We define the continuum limit of such expressions to mean: (i) the lattice spacing $a \to 0$ as well as (ii) the spatial volume $(La)^s \to \infty$ along with the spacetime volume $(La)^n \to \infty$ in an appropriate manner and combination. As noted above, if we choose the naive lattice form (including arbitrary cutoff dependent parameters) for the classical action of a $\varphi^4_n$, $n \geq 5$, model, the ground-state distribution becomes Gaussian, or equivalently that

$$C(h) \to E_{\text{Gaussian}}(h) \equiv \exp[-\frac{1}{2} \int h(x) U(x - y) h(y) \, d^x d^y],$$

for some covariance function $U(x - y)$. Our primary goal is to avoid a Gaussian limiting behavior in the continuum limit for the ground-state distribution.

### 3.3 Some partial matrix elements

The lattice-space Hamiltonian discussed above admits an alternative and useful form. Let us introduce the differential operator

$$A_k \equiv -a^{-s} \frac{\partial}{\partial \phi_k} + a^{-s} \frac{1}{\Psi(\phi)} \frac{\partial \Psi(\phi)}{\partial \phi_k},$$
and its adjoint operator
\[ A_k^\dagger \equiv a^{-s} \frac{\partial}{\partial \phi_k} + a^{-s} \frac{1}{\Psi(\phi)} \frac{\partial \Psi(\phi)}{\partial \phi_k} , \]
for each \( k \) in a given spatial lattice slice. It is straightforward to verify that the lattice Hamiltonian is also given by
\[ \mathcal{H} = \frac{1}{2} \sum_k A_k^\dagger A_k a^s , \]
Next consider states of the form
\[ \Psi_h(\phi) \equiv e^{i \sum_k h_k \phi_k a^s} \Psi(\phi) , \]
and note that
\[ A_k \Psi_h(\phi) = -i h_k \Psi_h(\phi) . \]
Consequently, it follows that
\[ \int \Psi_{h'}(\phi)^* \mathcal{H} \Psi_h(\phi) \Pi'_k d\phi_k = \frac{1}{2} \sum_k h'_k h_k a^s C(h - h') . \]
Additionally, we observe that
\[ \frac{-i}{2} (A^\dagger_l - A_l) = -i a^{-s} \frac{\partial}{\partial \phi_l} \equiv \pi_l , \]
the canonical momentum conjugate to the field \( \phi_l \). It follows that
\[ \int \Psi_{h'}(\phi)^* \pi_l \Psi_h(\phi) \Pi'_k d\phi_k = \frac{1}{2}(h'_l + h_l) C(h - h') . \]
Equivalently, if we use the connection
\[ \pi_l = -i a^{-s} \frac{\partial}{\partial \phi_l} = i[\mathcal{H}, \phi_l] \]
relating the canonical momentum with the field and Hamiltonian, then it follows that
\[ \int \Psi_{h'}(\phi)^* \pi_l \Psi_h(\phi) \Pi'_k d\phi_k = a^{-s} \left( \frac{\partial}{\partial h'_k} + \frac{\partial}{\partial h_k} \right) \int \Psi_{h'}(\phi)^* \mathcal{H} \Psi_h(\phi) \Pi'_k d\phi_k \\
= \frac{1}{2}(h'_l + h_l) C(h - h') . \]
as before. (Remark: There are now THREE different uses of the prime: one refers to the pseudofree theory; another refers to properties on a fixed time slice of the lattice; the last usage refers to labels of Hilbert space functions. The context suffices to tell which usage is meant.)

Observe that this calculation has yielded matrix elements of both the lattice Hamiltonian $H$ and the lattice momentum $\pi_l$ at site $l$. It is highly probable that these matrix elements actually determine the operators in question, albeit in an indirect manner.

3.4 Features of the continuum theory

The analysis given above for the lattice has a natural analog in the continuum theory [5]. The expressions given below should indeed follow directly as the continuum limit of the lattice expressions, despite the fact that we have changed notation to emphasize the fact that we are dealing with the putative continuum theory.

The continuum limit of the ground state may be formally identified with the abstract unit vector $|0\rangle$, a member of the abstract Hilbert space $\mathcal{H}$. Additionally, we can introduce the unit vectors

$$|h\rangle \equiv e^{i \int h(x) \hat{\phi}(x) \, dx} |0\rangle,$$

where $s = n - 1$, $\hat{\phi}(x)$ denotes the sharp-time (e.g., at $t = 0$), formally self-adjoint field operator, and $h(x)$ denotes a smooth, real test function. Additionally, the vector $|h\rangle$ is strongly continuous in a suitable topology [6]. The overlap of two such vectors,

$$E(h - h') \equiv \langle h'|h\rangle,$$

defines the important expectation functional $E(h)$. As a continuous function of positive type, $E(h)$ serves as a reproducing kernel for a reproducing kernel Hilbert space that provides a representation of $\mathcal{H}$ by continuous functionals. Furthermore, if we define the local operator $\hat{\pi}(x)$ as the canonical conjugate of the field operator $\hat{\phi}(x)$, then, under mild conditions, it follows [5] that

$$\langle h'|\hat{\pi}(x)|h\rangle = \frac{1}{2}[h'(x) + h(x)] E(h - h') .$$

Since we may assume that the vectors $|h\rangle$ span the Hilbert space, it is plausible that the given matrix elements of $\hat{\pi}(x)$ determine the local self-adjoint
operator uniquely. Moreover, we can also assert [5] that
\[ \langle h' | \mathcal{H} | h \rangle = \frac{1}{2} (h', h) E(h - h') , \]
where \( \mathcal{H} \) denotes the nonnegative, self-adjoint Hamiltonian operator for which \( \mathcal{H} | 0 \rangle = 0 \), provided we assume that
\[ \hat{\pi}(x) = i[\mathcal{H}, \hat{\phi}(x)] . \]

Note well, that these relations which hold in the continuum are simply natural continuum analogs of the corresponding lattice-space expressions derived above, and confirm that a smooth continuum limit of the function \( C(h) \) to \( E(h) \) is a primary ingredient in developing a satisfactory continuum theory.

4 The Role of Generalized Poisson Distributions

In probability theory, alternative distributions of the sum of an ever-increasing number of independent, identically distributed random variables that compete successfully in the struggle for convergence, such as in a continuum limit, generally lead to Poisson distributions, and in doing so they give rise to non-Gaussian final distributions. Significantly, a sequence of lattice-space ground state distributions with a limiting behavior that leads to a Poisson distribution necessarily has a qualitatively different dependence on the individual variables and various parameters than does a sequence of lattice-space ground state distributions destined to end up as a Gaussian distribution. Just as for the Gaussian limit, it is plausible that there are distributions for alternative sets of random variables, which are neither independent nor identically distributed, yet lie in suitable basins of attraction so that they have limiting Poisson distributions. If we ensure that the terms in the lattice action along with appropriate counterterms conform with the needed different dependence on the individual variables and parameters, then we can ensure that the continuum limit will avoid being Gaussian.

As stressed previously, the traditional free theory is Gaussian and therefore corresponds to an infinitely divisible distribution (defined below). There is some logic in the fact that a free theory is infinitely divisible in the sense
that the ability that each field variable may be expressed as a sum of arbitrarily many independent, identically distributed variables is, let us say, one hallmark of being “free”. Likewise, when discussing a pseudofree theory in which all interactions have been reduced to zero, it is not unrealistic to believe that the hallmark of infinite divisibility also applies to the resultant pseudofree distribution. While arguments of any weight are hard to advance, we shall nevertheless make the assumption that it is possible that this is the case and, as a consequence, we shall seek pseudofree solutions among the family of infinitely divisible distributions. Even so, we shall also hedge our bets by suggesting that the class of ground-state distributions which are given by rather general linear combinations of infinitely divisible distributions (and are therefore, in general, not infinitely divisible themselves) may be the proper place to find pseudofree theories.

Before studying particular examples of infinitely divisible distributions, however, it is appropriate to give a general outline of how such distributions may be characterized.

Successful limiting behavior of suitable sets of countable numbers of independent, identically distributed random variables may be formalized in a natural way in terms of their characteristic functions. In particular, the appropriate Gaussian and Poisson distributions share the property of infinite divisibility, which may be described as follows: If $C(h)$ denotes an infinitely divisible characteristic function for an appropriate distribution, then it follows that the $J^{th}$ root of that function, i.e., $C(h)^{1/J}$, $J \in \{2, 3, 4, \ldots\}$, is also a characteristic function. This property follows because we can always linearly decompose the set of random variables $\{\phi_k\}$ into $J$ components composed of independent, identically distributed random variables for any positive integer $J$. Furthermore, it is always the case that any integer power $K$ of a characteristic function is again a characteristic function, i.e., if $C(h)$ is a characteristic function, then $C(h)^K$, $K \in \{2, 3, 4, \ldots\}$ is also a characteristic function. Therefore, an infinitely divisible characteristic function $C(h)$ retains the property of being a characteristic function if we raise it to an arbitrary rational power such as $C(h)^{K/J}$. Finally, since characteristic functions are always continuous functions of their arguments, we can take a limit as the rational ratios converge to an arbitrary, positive real, $K/J \rightarrow r$. This limit converges, and the limit $C(h)^r$ is again a characteristic function for an arbitrary real number $r > 0$. This feature is a defining as well as a useful property of infinitely divisible characteristic functions and thereby of
The only categories of infinitely divisible distributions are either Gaussian or Poisson, or a combination of the two. And since a Gaussian distribution can be obtained as the limit of a sequence of Poisson distributions (but not vice versa!) it is appropriate to say that all infinitely divisible distributions are Poisson distributions or limits thereof.

In what follows we shall assume that the ground-state distribution function has the form of a fairly straightforward, basic Poisson distribution. We conjecture that some of the basic Poisson distributions on which we focus could possibly serve as appropriate pseudofree scalar field models.

In Sec. 5, ground-state distributions that are given by linear superpositions of basic Poisson distributions are treated as possibly interesting generalizations of the case of simple Poisson distributions.

In seeking models within the class of strictly infinitely divisible distributions, or within the class of linear superpositions of such distributions, it is important to keep in mind that some specific model, such as \( \varphi_n^4 \), \( n \geq 5 \), may not be among those contained in the class under consideration. Indeed, one should not look for any specific model, but, at the present stage, accept any reasonable model that might be present. This view is necessitated because we are faced with a highly nontrivial inverse problem; namely, we can access a limited amount of information about the model under consideration, i.e., some features of the ground-state distribution, and from that we would like to determine the lattice action that gave rise to those properties. In summary, we ask the reader to keep an open mind as to just what class of models of interest – if any – can be described by the procedures we have in mind.

### 4.1 Basic Poisson distributions

To describe a Poisson distribution for scalar field models in terms of the ground-state distribution \( \Psi(\phi)^2 \) is difficult because it generally does not involve known functions. On the other hand, the characteristic function for a general Poisson distribution can be described in a relatively simple form. In particular, the characteristic function of an even Poisson distribution has the form given by

\[
\int \cos(\Sigma_k h_k \phi_k a^s) \Psi(\phi)^2 \Pi_k' d\phi_k = \exp\{-\int [1 - \cos(\Sigma_k h_k \phi_k a^s)] \rho(\phi, a) \Pi_k' d\phi_k \} ,
\]
where the weight function \( \rho(\phi, a) \geq 0 \).

To ensure absolute continuity of the ground-state distribution, it is necessary and sufficient that

\[
\int \rho(\phi, a) \Pi'_k d\phi_k = \infty,
\]

a criterion that establishes our choice of Poisson distributions as so-called generalized Poisson distributions [7]. And to ensure proper meaning of the right-hand side of the equation for the generalized Poisson characteristic function above, it is necessary that the exponent – frequently referred to as the “second characteristic” [8] – satisfies

\[
0 \leq \int [1 - \cos(\Sigma'_{k} h_k \phi_k a^s)] \rho(\phi, a) \Pi'_k d\phi_k < \infty
\]

for all suitable \( \{h_k\} \). By symmetry, the weight function \( \rho(-\phi, a) = \rho(\phi, a) \), and we assume that all even “moments” exist in the sense that

\[
\int \phi_{k_1} \phi_{k_2} \cdots \phi_{k_{2q}} \rho(\phi, a) \Pi'_k d\phi_k < \infty,
\]

for all \( q \geq 1 \), where \( k^m \equiv (k_0, k_1^m, k_2^m, \ldots, k_s^m) \), and the time \( k_0 \) is the same for all \( k^m \), \( 1 \leq m \leq 2q \). By assumption all odd “moments” vanish. Note that such expressions are not moments since the weight function \( \rho(\phi, a) \) is not – and cannot be – normalized. To emphasize that distinction we shall generally refer to such expressions as n\( \text{oment} \)s (rhymes with moments, and is derived from not moments, a phrase that determines the meaning of moments).

In order that \( \rho(\phi, a) \) have the indicated properties, it must be singular near \( \phi = 0 \) in a suitable manner. For our purposes we shall focus on \( \rho(\phi, a) \) of the form

\[
\rho(\phi, a) \equiv \frac{R(a) e^{-U(\phi, a)}}{\Pi'_k [\Sigma'_{j} \phi_k a^s]^2}. \]

Let us explain the separate terms that enter this expression.

The numerator contains a factor \( R(a) \) to be fixed later and an unspecified function \( U(\phi, a) \), which is limited at present, let us say, by the requirements that \( U(0, a) = 0 \) and \( U(\phi, a) \geq c \Sigma'_{k} \phi_k a^s \), for some \( c > 0 \), and which is defined for field variables at a fixed time \( k_0 \); at the present moment, the only purpose of \( U(\phi, a) \) is to ensure that all moments of \( \rho(\phi, a) \) are finite. Below, we shall further specialize \( U(\phi, a) \) to the quadratic form

\[
U(\phi, a) = \Sigma'_{k,l} \phi_k A_{k,l} \phi_l a^{2s},
\]

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for suitable (but unspecified) matrices $A_{k,l}$ for closer examination as candidates for pseudofree models; we shall even take the liberty of calling models with this form “pseudofree models”. As will become clear, one argument for this designation is the fact that in its final form, this particular version of the weight function $\rho(\phi, a)$ has the same number of free, dimensional parameters as a traditional free theory. Additional arguments in favor of such a designation will arise later.

As shown in the Appendix, the exponent $\gamma$ in the denominator of the expression for $\rho(\phi, a)$ satisfies the bounds

$$\frac{1}{2} \leq \gamma < \frac{1}{2} + \frac{1}{N'},$$

where $N' = L^s = L^{n-1}$ denotes the number of lattice sites in a spatial slice. Since we eventually need to take $N' \to \infty$, we accommodate that limit already by choosing

$$\gamma = \frac{1}{2},$$

the only exception being in the Appendix where we establishing the bounds on $\gamma$ noted above. Since the parameters $\{\beta_k\}$ will turn out to be dimensionless, then, when $\gamma = 1/2$, it follows that $R(a)$ is dimensionless. Consequently, all the dimensional parameters in $\rho(\phi, a)$ reside in $U(\phi, a)$.

The factors $\beta_k \equiv \beta_{-k}$ are all nonnegative, i.e., $\beta_k \geq 0$, and satisfy the condition that

$$\Sigma'_k \beta_k = 1.$$

Various choices of the set $\{\beta_k\}$ are possible, but we shall concentrate on choosing a fixed, $N'$-independent number of $\beta_k$ terms as nonzero. In particular, we specialize to the specific choice of $2s + 1$ nonvanishing terms given by

$$\beta_k = J_k \equiv \frac{1}{2s + 1} \delta_{k,\{k \cup k_{nn}\}}.$$

This notation means that an equal weight of $1/(2s + 1)$ is given to the $2s + 1$ points in the set composed of $k$ and its $2s$ nearest neighbors in the spatial sense only. Specifically, we define $J_k = 1/(2s + 1)$ for the points $k =$
The suitability of this choice is addressed in the Appendix. For notational purposes we shall refer to the set of lattice sites \( \{ k \cup k_{nn} \} \) as the set “\( k^{\text{plus}} \)”. The name \( k^{\text{plus}} \) is chosen because for the graphically simple case for which \( s = 2 \), the set of nonvanishing \( \{ J_k \} \) points are arranged in the form of a “+” sign; indeed, see Fig. 1 in the Appendix in this regard. (Remark: Of course, one could give an unequal, positive weighting to each of the \( k^{\text{plus}} \) points in contrast to the set of equal weights we have chosen. While this choice would lead to a different sequence of results for finite lattice spacings, any difference due to unequal weighting would disappear in the continuum limit. Such an argument applies to any set \( \{ \beta_k \} \) that contracts to a single point in the continuum limit.)

Clearly, not all ground-state distributions \( \Psi(\phi)^2 \) correspond to infinitely divisible distributions. However, for every acceptable choice of \( U(\phi,a) \), the prescription that defines \( C(h) \) automatically defines – albeit indirectly – a ground-state distribution \( \Psi(\phi)^2 \) that is infinitely divisible.

### 4.2 What is the pseudofree ground-state distribution?

Given a characteristic function for a generalized Poisson measure, the associated distribution is given be an inverse Fourier transform, namely,

\[
\Psi(\phi)^2 \equiv \left( \frac{a^s}{2\pi} \right)^N \int \cos(\Sigma_k' \phi_k h_k a^s) \\
\times \exp\left\{ - \int [1 - \cos(\Sigma_k' \phi_k h_k a^s)] \rho(u, a) \prod_k' du_k \prod_k dh_k \right\}.
\]

In general, this integral cannot be evaluated analytically; however, there is a partial result which is illuminating. It follows from the property of infinite divisibility that \( C(h)^{1/M} \) is again a characteristic function for all \( M \in \{ 2, 3, 4, \ldots \} \), and we can put that property to use for us now.

Consider the expression

\[
\Psi_{1/M}(\phi)^2 \equiv \left( \frac{a^s}{2\pi} \right)^N \int \cos(\Sigma_k' \phi_k h_k a^s)
\]

\(^2\text{Originally [3], it was believed that it was necessary that all } \beta_k > 0 \text{ was required. We thank Erik Deumens for raising the question of whether finitely many positive } \beta_k \text{ terms might work just as well as requiring that all } \beta_k > 0. \text{ Happily, that has turned out to be the case.}\)
\[\times \exp\{- (1/M) \int [1 - \cos(\Sigma_k' h_k a^s)] \rho(u, a) \Pi_k' du_k \} \Pi_k' dh_k,\]

which yields the distribution belonging to the characteristic function \(C(h)^{1/M}\). For very large \(M\) it stands to reason that the second characteristic is small, which allows us to approximate the foregoing integral up to order \(O(1/M^2)\) as follows:

\[\Psi_{1/M}^2(\phi) \equiv \left( \frac{a^s}{2\pi} \right)^N' \int \cos(\Sigma_k' \phi_k h_k a^s) \times \left[ 1 - (1/M) \int [1 - \cos(\Sigma_k' h_k a^s)] \rho(u, a) \Pi_k' du_k \right] \Pi_k' dh_k,\]

Now we make use of the specifically assumed form for \(\rho(\phi, a)\) given by

\[\rho(\phi, a) = \frac{R(a) \exp[-U(\phi, a)]}{\Pi_k'[\Sigma_l' J_{k-l} \phi_l^2 + F(M)]^{1/2}},\]

and remark, since we are working to order \(O(1/M)\) that we can replace this weight function by (say)

\[\rho_{1/M}(\phi, a) = \frac{R(a) \exp[-U(\phi, a)]}{\Pi_k'[\Sigma_l' J_{k-l} \phi_l^2 + F(M)]^{1/2}},\]

where the additional term in the denominator, \(F(M)\), satisfies the bounds

\[0 < F(M) \ll 1.\]

With this replacement in place, we now claim, with the same accuracy as before, that

\[\Psi_{1/M}^2(\phi) = \left( \frac{a^s}{2\pi} \right)^N' \int \cos(\Sigma_k' \phi_k h_k a^s) \times \left[ 1 - (1/M) \int [1 - \cos(\Sigma_k' h_k a^s)] \rho_{1/M}(u, a) \Pi_k' du_k \right] \Pi_k' dh_k.\]

We are permitted to make this small change of introducing \(F(M)\) because our normal integrand for the second characteristic is “protected” from diverging for very small values of \(\{\phi_k\}\) by the factor

\[1 - \cos(\Sigma_k' h_k \phi_k a^s)\]
in the numerator. However, – and unlike \( \rho(\phi, a) \) – the weight function \( \rho_{1/M}(\phi, a) \) is integrable, and we now adjust \( F(M) \) so that the integral is \( M \) itself; i.e., we now assume that \( F(M) \) is chosen so that

\[
\int \rho_{1/M}(\phi, a) \Pi_k'\,d\phi_k = M.
\]

As \( M \to \infty \), it is clear that \( F(M) \to 0 \); however, any further properties of the function \( F(M) \) are not of particular interest to us. With the normalization of \( \rho_{1/M}(\phi, a) \) fixed, we may recast the expression for \( \Psi_{1/M}(\phi) \) into the form

\[
\Psi_{1/M}(\phi)^2 \equiv \left( \frac{a^s}{2\pi} \right)^N \int \cos(\Sigma_k' \phi_k h_k a^s) \times \left[ (1/M) \int \cos(\Sigma_k' \phi_k h_k a^s) \rho_{1/M}(u, a) \Pi_k'\,du \right] \Pi_k'\,dh_k,
\]

an equation which permits us to identify

\[
\Psi_{1/M}(\phi)^2 = \frac{1}{M} \frac{R(a) \exp[-U(\phi, a)]}{\Pi_k' [\Sigma_{k,\ell} J_{k-\ell} \phi_k^2 + F(M)]^{1/2}},
\]
correct to order \( O(1/M^2) \).

This latter equation makes clear, for large \( M \), that the ground-state distribution for this special case is just the weight function \( \rho(\phi, a) \) with its singularity regularized and then rescaled to have integral one. Apart from the denominator factor, the ground-state distribution \( \Psi_{1/M}(\phi)^2 \) is proportional to \( \exp[-U(\phi, a)] \), which for the particular case in which \( U(\phi, a) \) is quadratic in the fields is highly suggestive of a free model. Here is further evidence supporting our designation of weight functions of the form

\[
\rho(\phi, a) = \frac{R(a) \exp[-\Sigma_{k,\ell} \phi_k A_{k,\ell} \phi_\ell' a^{2s}]}{\Pi_k' [\Sigma_{k,\ell} J_{k-\ell} \phi_k^2 + F(M)]^{1/2}}
\]
as candidates for pseudofree models. (Remark: It is noteworthy that certain highly specialized, soluble, nonrenormalizable models (see Chapters 9 & 10 in [4]) also exhibit ground-state distributions of a qualitatively similar form, including certain denominators not unlike our featured expression \( \Pi_k' [\Sigma_{k,\ell} J_{k-\ell} \phi_k^2 + F(M)]^{1/2} \). In those cases, the analog of our function \( U(\phi, a) \) was necessarily quadratic for the associated pseudofree models.)
While the characteristic function of interest is easily regained as the $M$th power of the characteristic function $C(h)^{1/M}$, it is not nearly so easy to regain the ground-state distribution of interest since $\Psi(\phi)^2$ is given by the $M$-fold convolution of the distribution $\Psi_{1/M}(\phi)^2$, followed, ultimately, by the limit $M \to \infty$ to eliminate the error made in expanding the exponent. If we let the symbol $\ast$ denote convolution, then the computation just outlined is given by

$$
\Psi(\phi)^2 = \lim_{M \to \infty} \Psi_{1/M}(\phi)^2 \ast \Psi_{1/M}(\phi)^2 \ast \cdots \ast \Psi_{1/M}(\phi)^2,
$$

where the multiple convolution involves $M$ factors.

After multiple convolutions, what happens to the original form of the expression for the ground-state distribution $\Psi_{1/M}(\phi)^2$ is hard to predict. If $U(\phi, a)$ is chosen as a quadratic function, which we have advocated for our pseudofree models, and if we momentarily assumed that the denominator in $\rho(\phi, a)$ were absent, then the original distribution is a pure Gaussian, and under repeated convolutions it would still remain Gaussian; this is the usual situation for a strictly free model. However, the denominator of $\rho(\phi, a)$ is exactly what separates the Gaussian and the generalized Poisson distributions, and therefore the denominator factor is of the utmost importance.

4.3 A glimpse at the potential

We are unable to offer an analytic form for the ground state $\Psi(\phi)$, but we are able to offer an analytic form for the ground state $\Psi_{1/M}(\phi)$ valid for large $M$. Of course, the latter ground state is not the one of ultimate interest, but for someone starved for a look at any form of a ground state, it may offer a glimpse into the problem that is unavailable elsewhere. The potential associated with the ground state $\Psi_{1/M}(\phi)$ is given by the general formula, namely

$$
V_{1/M}(\phi) = \frac{1}{2} \sum_k' \frac{a^{-s}}{\Psi_{1/M}(\phi)} \frac{\partial^2 \Psi_{1/M}(\phi)}{\partial \phi_k^2},
$$

which leads to the expression

$$
V_{1/M}(\phi) = \frac{1}{8} a^{-s} \sum_{k,s,t}' \frac{J_{s-k} J_{t-k} \phi_k^2}{[\sum_m J_{s-m} \phi_m^2 + F(M)] [\sum_n J_{t-n} \phi_n^2 + F(M)]}.
$$
where $\Tilde{\mathcal{V}}_{1/M}(\phi)$ follows from terms which include either one or two derivatives of the unspecified term $U(\phi, a)$. If we choose

$$U(\phi, a) = \Sigma'_{k,l} \phi_k A_{k,l} \phi_l a^{2s},$$

it follows that the remainder potential $\Tilde{\mathcal{V}}_{1/M}(\phi)$ contains a quadratic potential term plus a term that arises from one derivative of $U$ and one of the denominator. Observe well, that when factors of $\hbar$ are taken into account, the quadratic term is $O(\hbar^0)$, the cross term is $O(\hbar^1)$, while the term displayed in $\Tilde{\mathcal{V}}_{1/M}(\phi)$ above – coming from two derivatives of the denominator factor – is $O(\hbar^2)$. In the classical limit, therefore, the only term that survives is the quadratic term; the other two terms are quantum corrections to the potential that arise strictly from the denominator factor present in the weight function $\rho_{1/M}(\phi, a)$, and they make no contribution in the classical limit. Since the denominator carries no dimensional parameters, there is no possible way for it to contribute to the classical limit!

Of course, we must not forget that we are only looking at the potential associated with $\Psi_{1/M}(\phi)$ and not that associated with $\Psi(\phi)$, the one of real interest. There is no simple way to tell how the potential $\mathcal{V}(\phi)$ will appear after the multiple convolutions that $\Psi_{1/M}(\phi)^2$ must undergo to become $\Psi(\phi)^2$; nonetheless, there is no denying the feeling that something like the contribution displayed in the expression for $\Tilde{\mathcal{V}}_{1/M}(\phi)$ will survive, if for no other reason, than on dimensional grounds alone. However, that is only a guess.

Understanding repeated convolutions of the distribution $\Psi_{1/M}(\phi)^2$ for large $M$ is one key to getting a handle on the ground-state distribution $\Psi(\phi)^2$ of ultimate interest. Nevertheless, there may be another way to gain some information about $\Psi(\phi)^2$, and that is to proceed numerically.

### 4.4 Possible numerical studies

Whatever expression one chooses for the characteristic function $C(h)$ of the ground-state distribution $\Psi(\phi)^2$, it follows that the ground-state distribution
itself is given simply by the inverse Fourier transform, i.e.,

\[ \Psi(\phi)^2 \equiv \left( \frac{a^s}{2\pi} \right)^{N'} \int \cos(\Sigma_k' \phi_k h_k a^s) C(h_k) \Pi_{k} \, dh_k . \]

If this expression could be numerically computed, at least for a modest lattice size, we could then numerically determine \( \Psi(\phi) \), the potential \( V(\phi) \), the lattice Hamiltonian \( H \), and finally the lattice action with which to perform Monte Carlo calculations for the various correlation functions of interest.

Admittedly, all that appears to be a rather formidable task. Therefore, one of the aims of this paper is to make this task appear more attractive.

5 A Linear Superposition of Poisson Distributions

Up to this point, we have placed all our bets on studying generalized Poisson distributions as candidates for the ground-state distribution. We now broaden the base of our inquiry to include a much larger class of ground-state distributions based on linear superpositions of the very class of distributions we have featured so far.

In one-dimensional probability theory, it is noteworthy that a linear superposition of even Gaussian distributions over their variance parameter is dense in all possible even probability distributions. This is likewise true for their characteristic functions; namely, one may study the set of even characteristic functions \( C(-s) = C(s) \) that may be obtained from the relation

\[ C(s) = \int_{0}^{\infty} f(\tau) e^{-\tau s^2} d\tau \]

as \( f \) ranges over (signed) generalized functions. It is easy to see that all even characteristic functions and thereby all even probability distributions may be obtained this way.\(^3\) A similar question can be raised about linear sums of even Poisson distributions, or equivalently about their even characteristic

\(^3\)For example, let \( f(\tau) \) equal \( \delta^{(m)}(\tau-1) \) to generate the elements \( s^{2m} e^{-s^2} \), \( m = 0, 1, 2, \ldots \), take linear sums to construct the even order Hermite functions, and then note that sums of such functions are dense in all even functions, and hence in the subset of such functions that are characteristic functions.
functions. In equation form, this latter question reduces to the class of characteristic functions that can be reached by linear superposition (say) of the limited set of Poisson characteristic functions

\[ C(s) = \int_0^\infty \int_0^\infty f(\tau, \omega) \exp\{-\tau \int [1 - \cos(\omega u)] e^{-\omega u^2} du/u^2\} d\tau d\omega \]

again as \( f \) ranges over (signed) generalized functions. Since we can reconstruct the Gaussian case from the Poisson case, it is clear that once again such linear combinations are dense in the set of all even characteristic functions and thereby all even probability distributions.

In this section we study expressions for the lattice-space characteristic function associated with the (even) ground-state distribution in the form of a linear superposition of generalized Poisson characteristic functions given by

\[ C_K(h) \equiv \int \cos(\Sigma_k h_k \phi_k a^s) \Psi_K(\phi) \Pi_k' d\phi_k \]

\[ \equiv \int d\tau K(\tau) \exp\{-\int [1 - \cos(\Sigma_k h_k \phi_k a^s)] \rho_\tau(\phi, a) \Pi_k' d\phi_k\} . \]

The basic ingredient in this superposition is the expression

\[ C_\rho(h) \equiv \exp\{-\int [1 - \cos(\Sigma_k h_k \phi_k a^s)] \rho_\tau(\phi, a) \Pi_k' d\phi_k\} , \]

which, as discussed in Sec. 4, is the characteristic function for a generalized Poisson distribution, an infinitely divisible characteristic function. Note well that we do not require that \( C_K(h) \) is an infinitely divisible distribution itself; instead, we use a suitable set of infinitely divisible distributions as “building blocks” for \( C_K(h) \).

Since a limit of Poisson distributions exists that converge to a Gaussian distribution, we may allow for such a possibility in the continuum limit as \( a \to 0 \). In this sense, the expression given for our basic element incorporates both Poisson and Gaussian distributions, and therefore potentially includes a wide class of infinitely divisible distributions. With regard to the superposition by an integral over \( \tau \in \mathbb{R}^p \), for some \( p \), one may focus initially on positive measures \( K(\tau) d\tau \) (thus including \( \delta \)-functions) such that \( \int K(\tau) d\tau = 1 \). However, it is also possible to extend such examples to suitable signed measures.
$K(\tau)\,d\tau$ such that $\int |K(\tau)|\,d\tau < \infty$, $\int K(\tau)\,d\tau = 1$, and $\Psi(\phi)^2 > 0$; we content ourselves with a dense set of results and do not pursue examples where $K(\tau)$ is a generalized function. Our purpose in this exercise is to increase, broadly, we hope, the scope of characteristic functions and thereby of ground state distributions that may be considered. We make no claim that our limited family of superpositions is exhaustive of all such distributions (unlike the one-dimensional case).

By assumption, the continuum limit of our lattice-space formulation $C_{(K)}(h)$ is intended to retain the basic structure of the separate Poisson characteristic functions that make up the linear combination. This implies that in the continuum limit the resultant expression for $E_{(K)}(h)$ is generally non-Gaussian. It is our proposal to look for the ground-state distribution of various scalar models within the family of characteristic functions offered above, and in particular for those models that are pseudofree models (thus having a limited number of parameters). As emphasized previously, such pseudofree models may serve as starting points for a perturbation analysis for more interesting interacting models.

On one hand, it may even be possible to find the ground-state distribution for some interacting models within the family obtained by linear superposition. On the other hand, although the class of ground-state distributions $\Psi(\phi)^2$ that may be described by the linear superposition of infinitely divisible distributions may be relatively large, the reader may well ask by what reason could we expect to find an acceptable description of (say) a $\phi^4_n$, $n \geq 5$, model within that particular class of characteristic functions.

To address this question consider the following one-dimensional quantum mechanical problem: A system with classical Hamiltonian $H(p, q)$ is quantized by adopting the quantum Hamiltonian

$$\mathcal{H} = H(P, Q) + \hbar Y(P, Q),$$

where $Y$ is usually present to account for possible factor ordering ambiguity. However, $Y$ could in fact be a quite general operator (so long as $\mathcal{H}$ is self adjoint) because in the classical limit in which $\hbar \to 0$ the original classical Hamiltonian $H(p, q)$ is recovered. This kind of ambiguity is always present in quantum theory, and the traditional way to deal with it is to “appeal to experiment”. (Remark: For example, such additional terms are surely present in any analysis of nonrenormalizable models based on the counterterms suggested by perturbation theory.)
It is possible that approaching a wide class of scalar models through the family of characteristic functions represented by $C_{(K)}(h)$ may correspond to a realization of certain models with the presence of selected counterterms (the analogue of $Y(P,Q)$ above). Since a principal goal of the present study is to achieve nontriviality, any satisfactory set of counterterms will suffice initially; limitations that may arise from an appeal to experiment can be considered later.

In the following section we will discuss correlation functions for the case of a ground-state distribution which is a generalized Poisson distribution. That same discussion can easily be extended to also include those distributions that are given by the linear superposition of Poisson distributions.

6 Correlation Functions

Even without fully defining our choice for a full-time, lattice-space action function including the required auxiliary potential, we can nevertheless draw some important general conclusions. In particular, let us show that the full-time correlation functions can be controlled by their sharp-time behavior along with a suitable choice of test sequences.

6.1 Correlation function bounds for general distributions

Let the notation

$$
\phi_u \equiv \Sigma_k u_k \phi_k a^n
$$

denote the full-time summation over all lattice fields where \( \{u_k\} \) denotes a suitable test sequence. We also separate out the temporal part of this sum in the manner

$$
\phi_u \equiv \Sigma_{k_0} a \phi_{u'} \equiv \Sigma_{k_0} a \Sigma'_{k} u_k \phi_k a^n .
$$

Observe that the notation $\phi_{u'}$ (with the prime) implies a summation over only the spacial lattice points for a fixed (and implicit) value of the temporal lattice value, $k_0$. 

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Let the notation \( \langle \cdot \rangle \) denote full-time averages with respect to the field distribution determined by the lattice action, and then let us consider full-time correlation functions such as
\[
\langle \phi^{(1)} \phi^{(2)} \cdots \phi^{(2q)} \rangle = \sum_{k_0^{(1)}, k_0^{(2)}, \ldots, k_0^{(2q)}} a^{2q} \langle \phi^{(1)} \phi^{(2)} \cdots \phi^{(2q)} \rangle, \quad q \geq 1,
\]
where the expectation on the right-hand side is over products of fixed-time summed fields, \( \phi^{(i)} \), for possibly different times, which are then summed over their separate times. All odd correlation functions are assumed to vanish, and furthermore, \( \langle 1 \rangle = 1 \) in this normalized spacetime lattice field distribution.

It is also clear that
\[
|\langle \phi^{(1)} \phi^{(2)} \cdots \phi^{(2q)} \rangle| \leq \sum_{k_0^{(1)}, k_0^{(2)}, \ldots, k_0^{(2q)}} a^{2q} \left| \langle \phi^{(1)} \phi^{(2)} \cdots \phi^{(2q)} \rangle \right|.
\]

At this point we turn our attention toward the spatial sums alone.

We appeal to straightforward inequalities of the general form
\[
\langle AB \rangle^2 \leq \langle A^2 \rangle \langle B^2 \rangle.
\]
In particular, it follows that
\[
\langle \phi^{(1)} \phi^{(2)} \phi^{(3)} \phi^{(4)} \rangle^2 \leq \langle \phi^{(1)} \rangle^2 \langle \phi^{(2)} \rangle^2 \langle \phi^{(3)} \rangle^2 \langle \phi^{(4)} \rangle^2,
\]
and, in turn, that
\[
\langle \phi^{(1)} \phi^{(2)} \phi^{(3)} \phi^{(4)} \rangle^4 \leq \langle \phi^{(1)} \rangle^4 \langle \phi^{(2)} \rangle^4 \langle \phi^{(3)} \rangle^4 \langle \phi^{(4)} \rangle^4 \leq \langle \phi^{(1)} \rangle^4 \langle \phi^{(2)} \rangle^4 \langle \phi^{(3)} \rangle^4 \langle \phi^{(4)} \rangle^4.
\]
By a similar argument, it follows that
\[
|\langle \phi^{(1)} \phi^{(2)} \cdots \phi^{(2q)} \rangle| \leq \prod_{j=1}^{2q} \left[ \langle \phi^{(2)} \rangle^2 \right]^{1/2q},
\]
which has bounded any particular mixture of spatial correlation functions at possibly different times, by a suitable product of higher-power expectations each of which involves field values ranging over a spatial level, all at a single fixed lattice time. By time translation invariance of the various single time correlation functions we can assert that
\[
\langle \phi^{(2r)} \rangle.
\]
which is defined at time \( k_0^{(j)} \), is actually independent of the time and, therefore, the result could be calculated at any fixed time. In particular, we can express such correlation functions as

\[
\langle \phi_u^2 \rangle = \int \phi_u^2 \Psi(\phi)^2 \Pi_k d\phi_k .
\]

Adding this relation to those established above completes a bound on the multi-time correlation function in terms of moments of the ground-state distribution.

### 6.2 Correlation function bounds for Poisson distributions

Observe that the bounds we have discussed up to this point apply to arbitrary ground-state distributions \( \Psi(\phi)^2 \). Let us now specialize to generalized Poisson distributions with which we shall be able to make additional and more specific remarks.

For Poisson distributions, we note that the moments of the ground-state distribution for the correct (but unknown) ground-state may be directly related to the truncated moments for the ground-state distribution, which in turn are directly determined by the “moments” of the weight function \( \rho(\phi, a) \). Let us recall the notation introduced earlier, namely

\[
(\phi_u^2)^p \equiv \int \phi_u^2 \rho(\phi, a) \Pi_k d\phi_k , \quad p \geq 1 ,
\]

which are not moments but nroments (since \( \int \rho(\phi, a) \Pi_k d\phi_k = \infty \)). These nooment expressions coincide exactly with the truncated moments of the ground state distribution. Recall that ordinary and truncated moments are related by the generating function relation

\[
\langle e^{\alpha \phi_u^*} \rangle = \exp[\langle e^{\alpha \phi_u^*} - 1 \rangle^T] ,
\]

which holds for all \( \alpha \), where \( T \) denotes truncated. Therefore, in our case,

\[
\langle \phi_u^2 \rangle^T \equiv (\phi_u^2)^p , \quad p \geq 1 .
\]

For example, we list a few relations that follow from this connection:

\[
\langle \phi_u^2 \rangle = (\phi_u^2)^p ,
\]

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\[ \langle \phi_{u'}^4 \rangle = (\phi_{u'}^4) + 3(\phi_{u'}^2)^2, \]
\[ \langle \phi_{u'}^6 \rangle = (\phi_{u'}^6) + 15(\phi_{u'}^2)(\phi_{u'}^4) + 15(\phi_{u'}^2)^3, \]
\[ \langle \phi_{u'}^8 \rangle = (\phi_{u'}^8) + 28(\phi_{u'}^2)(\phi_{u'}^6) + 35(\phi_{u'}^4)^2 + 210(\phi_{u'}^2)^2(\phi_{u'}^4) + 105(\phi_{u'}^2)^4, \]

etc.

The conclusion of this exercise is that all of the multi-time correlation functions are bounded by terms that are composed from the moments determined from the weight function \( \rho(\phi, a) \). If we can choose \( \rho(\phi, a) \) so that it has a suitable continuum limit, and in such a way that the moments remain finite in that limit, then we will have obtained a bound on every full-time correlation function in the continuum limit. Convergence (of subsequences, if necessary) of these correlation functions follows. Moreover, the ground-state distribution stays firmly within the family of Poisson distributions and, by working with the various moments \( (\phi_{u'}^{2q}) \), we have avoided the fate of a Gaussian, i.e., trivial, limiting distribution.

7 From Noments to Genuine Averages

7.1 A two-point normalization

The discussion regarding correlation functions has shown that many functions of interest can be evaluated exactly, or are suitably bounded, by terms involving various noments such as

\[ (\phi_{u'}^{2q}) = R(a) \int (\sum_k u_k \phi_k a^s)^{2q} \frac{e^{-\sum_k A_k l \phi_k a^{2s}}}{\Pi_k [\sum_l J_{k-l} \phi_l^2]^{1/2}} \Pi_k d\phi_k, \]

where \( q \geq 1 \) and \( u_k \) denotes a test sequence here used in space alone. Expansion of the power \( 2q \) leads to expressions of the sort

\[ (\phi_{i_1} \phi_{i_2} \ldots \phi_{i_{2q}}) = R(a) \int \phi_{i_1} \phi_{i_2} \ldots \phi_{i_{2q}} \frac{e^{-\sum_k A_k l \phi_k a^{2s}}}{\Pi_k [\sum_l J_{k-l} \phi_l^2]^{1/2}} \Pi_k d\phi_k. \]

It is these moments that we wish to discuss in this section with the possible aim of approximately evaluating them using Monte Carlo techniques.
In order to invoke Monte Carlo calculational methods, however, it is clear that we need to introduce a normalized probability measure, and for this purpose it will suffice to fix the factor $R(a)$, at least implicitly. We are free to fix this normalization based on the freedom involved in the usual field-strength renormalization. And we do so by selecting a carefully chosen and suitably weighted two-point function and then declare that the moment of this expression is unity.

To this end, let us choose $Y_{k,l}$ – where, in this case, $k$ and $l$ involve only the spatial components – as the elements of a symmetric matrix $Y \equiv \{Y_{k,l}\}$ which is positive definite. Furthermore the elements $Y_{k,l}$ are chosen so that their “high frequency” components are suitably reduced. We will offer two examples of what we deem to be suitable matrices, and from those it should become clear just what we have in mind regarding the “high frequency” components. The first example we choose for $Y_{k,l}$ is suitable for a lattice formulation of the problem, such as may arise in Monte Carlo calculations, where the lattice size $L$ and the lattice spacing $a$ may vary, but remain finite, and no attempt at approaching the continuum limit is contemplated; this example will be used in Sec. 7.1. The second example we choose for $Y_{k,l}$ is one designed to be use for an approach to the continuum limit; the second example will be used in Sec 7.2 in which the continuum limit is discussed.

Observe that the unit matrix $\delta_{k,l}$ is positive definite, but it does not discriminate between “low” and “high” frequencies. To describe the unit matrix in other terms, we first assume that $L$, the number of lattice points along each axis, is odd, or in other words, $L = 2P + 1$, where $P$ is a positive integer. We choose two integers $r$ and $s$, where $-P \leq r \leq P$, and similarly for $s$, and observe that

$$\frac{1}{L} \sum_{m=-P}^{P} e^{2\pi i (r-s)m/L} = \delta_{r,s}.$$  

Now, as a preliminary to making our choice for $Y_{k,l}$, we dampen the weight of the high frequencies in the previous sum by defining

$$Z_{r,s} \equiv \frac{1}{L} \sum_{m=-P}^{P} e^{-|m|T} e^{2\pi i (r-s)m/L},$$

where $T > 0$ is a damping parameter to be chosen. Clearly, the resultant function is still positive definite as desired. Indeed, the sum can be evaluated
in closed form as
\[
\frac{1 - e^{-2T} - 2e^{-(P+1)T} \cos[2\pi \Delta (P + 1)/L] + 2e^{-(P+2)T} \cos[2\pi \Delta P/L]}{L\{1 - 2e^{-T} \cos[2\pi \Delta/L] + e^{-2T}\}},
\]
where \(\Delta \equiv r - s\). Finally, to define our first choice for \(Y_{k,l}\), we let
\[
Y_{k,l} \equiv \prod_{j=1}^{s} Z_{k_j,l_j},
\]
where, as noted above, \(k\) and \(l\) involve only the spatial components. If, on the other hand, \(L\) is even, i.e., \(L = 2P\), for integral \(P > 0\), then we choose
\[
Z_{r,s} \equiv \frac{1}{L} \sum_{m=-P+1}^{P} e^{-|m|T} e^{2\pi i (r-s)m/L},
\]
which has the closed form
\[
Z_{r,s} = [1 - e^{-PT} (-1)^\Delta] \left\{ \frac{2[1 - e^{-T} \cos(2\pi \Delta/L)]}{[1 - 2e^{-T} \cos(2\pi \Delta/L) + e^{-2T}]} - 1 \right\}.
\]
The quantity \(Y_{k,l}\) is constructed from \(Z_{r,s}\) in the same way as for the odd \(L\) case.

Although not needed immediately, we also choose our second form for \(Y_{k,l}\) suitable for discussing the continuum limit. For the following, let \(z \in \mathbb{R}\), and let the orthonormal set of Hermite functions be called \(h_n(z)\), \(0 \leq n < \infty\), with the usual property that
\[
\sum_{n=0}^{\infty} h_n(z'') h_n(z') = \delta(z'' - z').
\]
We dampen the high frequencies once again and define
\[
Z(z'', z') \equiv \sum_{n=0}^{\infty} e^{-nT} h_n(z'') h_n(z'),
\]
where again \(T > 0\) is a parameter to be chosen. This sum can be evaluated in closed form as
\[
Z(z'', z') = \frac{1}{\sqrt{2\pi T}} \exp[-(1/2)(z''^2 + z'^2) \coth(T) + z'' z' \csch(T)].
\]
Clearly, the kernel $Z(z'', z')$ is positive definite. Finally, we define the function $Y(x, y)$ – where both $x = \{x_1, x_2, \ldots, x_s\}$ and $y = \{y_1, y_2, \ldots, y_s\}$ are in $\mathbb{R}^s$ – by the expression

$$Y(x, y) \equiv \prod_{j=1}^s Z(x_j, y_j).$$

Note that since each Hermite function qualifies as a test function, we may refer to $Z(z'', z')$ and $Y(x, y)$ as positive definite “test kernels”. The given choice $Y(x, y)$ is suitable for the continuum limit for the whole space $\mathbb{R}^s$, but if one chooses to use it for a large lattice, before taking the continuum limit, it would be acceptable to use

$$Y_{k,l} \equiv Y(ka, la),$$

namely, the values that $Y(x, y)$ assumes on the lattice points themselves.

Having chosen a suitable, positive definite matrix $Y_{k,l}$, as discussed above, we are in a position to establish our normalization criterion. In particular, for a given, fixed, choice of the matrix $Y$, we (arbitrarily) declare that the moments

$$R(a) \int \Sigma_{k,l} \phi_k Y_{k,l} \phi_l a^{2s} e^{-\Sigma_{k,l} \phi_k A_{k,l} \phi_l a^{2s}} \frac{1}{\Pi_k' [\Sigma_{l,k} J_{l-k} \phi_l^2]^{1/2}} \Pi_k' d\phi_k = 1.$$

This normalization condition fixes the constant $R(a)$. In effect, we are saying that a certain two-point function with a suitably chosen positive definite weighting has a moment of unity. We can put this normalization to good use for us as follows.

For all $q \geq 1$, we rewrite the moment of general interest as

$$\langle \phi_{l_1} \phi_{l_2} \ldots \phi_{l_{2q}} \rangle = R(a) \int \phi_{l_1} \phi_{l_2} \ldots \phi_{l_{2q}} e^{-\Sigma_{k,l} \phi_k A_{k,l} \phi_l a^{2s}} \frac{1}{\Pi_k' [\Sigma_{l,k} J_{l-k} \phi_l^2]^{1/2}} \Pi_k' d\phi_k$$

$$= \int \frac{\Sigma_{k,l} \phi_k Y_{k,l} \phi_l a^{2s} e^{-\Sigma_{l,k} \phi_l A_{l,k} \phi_k a^{2s}}}{\Pi_k' [\Sigma_{l,k} J_{l-k} \phi_l^2]^{1/2}} \Pi_k' d\phi_k$$

$$\equiv \int \frac{\phi_{l_1} \phi_{l_2} \ldots \phi_{l_{2q}}}{\Sigma_{k,l} \phi_k Y_{k,l} \phi_l a^{2s}} d\sigma(\phi).$$
This last equation establishes the moment of interest as a genuine average involving a normalized probability measure

\[
\frac{\int \frac{\left[ \sum_{k,l}^\prime \phi_k Y_{k,l} \phi_l a^{2s} \right] e^{-\sum_{k,l}^\prime \phi_k A_{k,l} \phi_l a^{2s}} \Pi_k^l d\phi_k}{\Pi_k^l \left[ \sum_{l,t}^\prime J_{k,l} \phi_l^2 \right]^{1/2}}}{\Pi_k^l d\phi_k}
\]

It is the final expression for \( (\phi_{l_1} \phi_{l_2} \ldots \phi_{l_2q}) \) we would like to evaluate approximately by means of conventional Monte Carlo methods.

We have arrived at the expression of moments as genuine averages on the basis of a convenient choice of normalization given above. Note that if instead we had chosen to normalize matters so that

\[
R(a) \int \frac{\left[ \sum_{k,l}^\prime \phi_k Y_{k,l} \phi_l a^{2s} \right] e^{-\sum_{k,l}^\prime \phi_k A_{k,l} \phi_l a^{2s}} \Pi_k^l d\phi_k}{\Pi_k^l \left[ \sum_{l,t}^\prime J_{k,l} \phi_l^2 \right]^{1/2}} = b ,
\]

rather than \( b = 1 \) as above, then the final result would have been

\[
(\phi_{l_1} \phi_{l_2} \ldots \phi_{l_2q}) = b \int \frac{\phi_{l_1} \phi_{l_2} \ldots \phi_{l_2q}}{\sum_{k,l}^\prime \phi_k Y_{k,l} \phi_l a^{2s}} d\sigma(\phi) ,
\]

where the normalized probability measure \( d\sigma(\phi) \) is unchanged. Hereafter, we assume that \( b = 1 \).

It should be noted that there is a special advantage derived from the fact that the moments can be expressed as suitable averages over a particular probability distribution. This advantage follows from the fact that inequalities such as

\[
\int \left[ \frac{(\phi_{l_1} \phi_{l_2})}{\sum_{k,l}^\prime \phi_k Y_{k,l} \phi_l a^{2s}} \right]^2 d\sigma(\phi) \geq \left[ \int \frac{(\phi_{l_1} \phi_{l_2})}{\sum_{k,l}^\prime \phi_k Y_{k,l} \phi_l a^{2s}} d\sigma(\phi) \right]^2
\]

hold.

### 7.2 Continuum limit

Finally, we briefly take up the question regarding the continuum limit. There are several important and distinct aspects of the continuum limit, namely,
not only $a \to 0$ but $N' \to \infty$, and in fact, it is also necessary that the spacial lattice volume $N'a^8 \to \infty$. Let us first discuss the situation for large $N'$.

When dealing with a large number of integration variables, i.e., $N' \gg 1$, it is important to note that there are important – even profound – differences between Gaussian-like integrals and Poisson-like integrals. In presenting the following discussion we closely follow a section of [3] dealing with an idealized set of examples. For all $p \geq 1$, consider the two sets of integrals

\[ I_G(2p) = \int (\Sigma_k \phi_k^2)^p e^{-A \Sigma_k \phi_k^2} \Pi_k' d\phi_k, \]

as representative of the Gaussian-like expressions, and

\[ I_P(2p) = \int (\Sigma_k \phi_k^2)^p e^{-A \Sigma_k \phi_k^2} [\Sigma_k \phi_k^2]^{-N'/2} \Pi_k' d\phi_k, \]

as representative of the Poisson-like expressions. These integrals are “caricatures” of the ones we study, but the results are nevertheless informative.

To help in our study, let us introduce hyper-spherical coordinates [3] (c.f., also [9]) defined by

\[
\begin{align*}
\phi_k &\equiv \kappa \eta_k, & 0 \leq \kappa < \infty, & -1 \leq \eta_k \leq 1, \\
\Sigma_k \eta_k^2 &\equiv 1, & \Sigma_k \phi_k^2 &\equiv \kappa^2.
\end{align*}
\]

Here $\kappa$ denotes an overall radius variable while $\{\eta_k\}$ denotes an $N'$-dimensional direction field. In terms of these variables, the Gaussian-like integrals take on the form

\[ I_G(2p) = 2 \int \kappa^{2p} e^{-A \kappa^2} \kappa^{N'-1} d\kappa \delta(1 - \Sigma_k \eta_k^2) \Pi_k' d\eta_k. \]

Evaluation of this expression for large $N'$ is dominated by the factor $\kappa^{N'-1}$, and a steepest descent method can be used to evaluate the integral over $\kappa$ to a suitable accuracy. To leading order, it follows that the stationary point is given by $\kappa = (N'/2A)^{1/2}$. As a consequence, for each value of $A$, the integrand is supported on a disjoint set of $\kappa$ as $N' \to \infty$. This well-known fact leads to divergences in perturbation calculations. For example, let us calculate

\[ I^*_G(2) = \int (\Sigma_k \phi_k^2)^2 e^{-A \Sigma_k \phi_k^2} \Pi_k' d\phi_k \]
for a different value of $A$ by the perturbation series,

$$I_G^*(2) = I_G(2) - \Delta A I_G(4) + \frac{1}{2}(\Delta A)^2 I_G(6) - \cdots,$$

where $\Delta A \equiv A^* - A$. Since $I_G(2p)/I_G(2) \propto N'^{p-1}$, this series exhibits divergences as $N' \to \infty$.

Let us now consider the Poisson-like integrals expressed in the same hyper-spherical coordinates. It follows that

$$I_P(2p) = 2 \int \kappa^{2p} e^{-\kappa^2} \left[ \kappa^2 \right]^{-N'/2} \kappa^{N'-1} d\kappa \delta(1 - \Sigma' \eta_k^2) \Pi' d\eta_k .$$

Here we see no large $\kappa$-power in the integrand, and it follows, e.g., that

$$\frac{I_P(4)}{I_P(2)} = \frac{\int \kappa^3 e^{-\kappa^2} d\kappa}{\int \kappa e^{-\kappa^2} d\kappa} = \frac{1}{A},$$

which is finite and independent of $N'$. Hence a perturbation calculation of the change of $A$ to $A^*$ for $I_P(2)$, for example, involves no divergences as $N' \to \infty$! Further examples of the difference of Gaussian- and Poisson-like integrals is offered in [3], but the main point has already been made showing the profound difference between these two types of integral for large $N'$.

How do these examples impact on our main discussion? Consider the expression for a simple moment given by

$$(\phi_t^2) = R(a) \int \phi_t^2 \frac{e^{-\Sigma' \eta_k A_k \phi_t a^2}}{\Pi' \left[ \Sigma' J_{k-l} \phi_t^2 \right]^{1/2}} \Pi'_k d\phi_k .$$

Expressed in hyper-spherical coordinates, this integral becomes

$$(\phi_t^2) = 2 R(a) \int \kappa^2 \eta_t^2 \frac{e^{-\kappa^2 \Sigma' \eta_k A_k \eta_t^2}}{\Pi'_k \left[ \Sigma' J_{k-l} \eta_t^2 \right]^{1/2}} \kappa^{-1} d\kappa \delta(1 - \Sigma' \eta_k^2) \Pi'_k d\eta_k ,$$

and we see an analogous absence of the parameter $N'$ regarding the integral over $\kappa$. This favorable situation has arisen from the presence of a denominator factor which, although different in detail, has an overall power of $\kappa$ identical to that of the elementary examples for $I_P(2)$. Ideally, one might like to choose a denominator factor that is as local as possible on the lattice, and that would mean choosing a set of $\{\beta_k\}$ parameters of the form $\beta_k = \beta \delta_{k,0}$. 

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This choice has the desired locality property, but it does not lead to an integrable denominator, as noted previously and which is further discussed in the Appendix. The choice of $\beta_k = J_k$—that is, our featured $k^{\text{plus}}$ set of values— is a suitable compromise that has locality in the continuum limit but retains enough lattice spread to always ensure an integrable denominator. In calculations of our principal interest, and with our featured choice of the denominator factor, we expect to benefit from the same lack of divergences due to the missing large $N'$ dependence of the hyper-spherical radius $\kappa$ as was exhibited in the elementary examples.

Next we consider the problems associated with the limit $a \to 0$ along with $N' \to \infty$, ideally, so that $N'a^s \to \infty$, but at the very least, so that $N'a^s \to V'$, where $0 < V' < \infty$, and preferably $V'$ is “large” in some sense. To appreciate the kind of problems faced it is useful to examine first the usual situation for a “free” theory and its continuum limit.

Consider the lattice formulation for the ground-state distribution of the free (= Gaussian) theory given by

$$C_F(h) = M \int e^{\Sigma'_k h_k a^s} e^{-\Sigma'_k,\ell h_k B_{k,\ell} \phi \phi^{*}} \Pi'_k d\phi_k = e^{-(1/4)\Sigma'_k,\ell h_k B_{k,\ell}^{-1} h_{\ell} a^s},$$

where $M$ denotes a normalization so that $C_F(0) = 1$. It is clear in this case that the two-point function is given by

$$\langle \phi_r \phi_s \rangle = M \int \phi_r \phi_s e^{-\Sigma'_k,\ell h_k B_{k,\ell} \phi \phi^{*}} \Pi'_k d\phi_k = \frac{1}{2} B^{-1}_{r,s} a^{-s},$$

i.e., essentially the $r, s$ matrix element of $B^{-1}$, the inverse matrix to $B$. A suitable continuum limit of this expression starts by considering the smeared expression

$$\langle (\Sigma'_r u_r \phi_r a^s)^2 \rangle = \frac{1}{2} \Sigma'_{r,s} u_r B^{-1}_{r,s} u^s a^s,$$

for sequences $\{u_r\}$, for example, such that $\Sigma'_{r} u^2_r a^s = 1$. In this case, the continuum limit should assume the form

$$\langle (\int u(x) \phi(x) d^d x)^2 \rangle = \frac{1}{2} \int u(x) u(y) F(x, y) d^d x d^d y,$$

for some generalized function $F(x, y)$ and for all functions $u(x) \in C^\infty_0$, for example, for which $\int u(x)^2 d^d x = 1$. The key concept is the limiting behavior
of the lattice two-point function to a suitable generalized function in the continuum limit, a well-known property for a free theory. Observe that the properties of \( F(x, y) \) are directly determined by the properties of the matrix \( B^{-1} \), the inverse of the matrix \( B \).

Now let us consider a similar limit for the ground-state distribution determined by a generalized Poisson distribution. In particular, we first focus on the two-point function

\[
P_{r,s} \equiv \langle \phi_r \phi_s \rangle = R(a) \int \phi_r \phi_s e^{-\Sigma'_{k,l} \phi_k A_{k,l} \phi_l a^2} \prod'_k d\phi_k ,
\]

where, as discussed above, normalization is set by

\[
R(a)^{-1} = \int \left[ \Sigma'_{k,l} \phi_k Y_{k,l} \phi_l a^2 \right] e^{-\Sigma'_{k,l} \phi_k A_{k,l} \phi_l a^2} \prod'_k d\phi_k .
\]

As in the Gaussian example, we now consider the expression

\[
\langle (\Sigma'_{r,s} u_r \phi_r a^s)^2 \rangle = \Sigma'_{r,s} u_r P_{r,s} u_s a^2 ,
\]

and, for a satisfactory continuum limit, we ask that as \( a \to 0 \) and \( N'a^s \to \infty \), or at least \( N'a^s \to V' \gg 1 \), we find that

\[
\langle (\int u(x) \phi(x) d^s x)^2 \rangle = \int u(x) u(y) G(x,y) d^s x d^s y ,
\]

for a suitable generalized function \( G(x,y) \), say, for the same set of smooth functions \( u \) considered above. In particular, according to our normalization procedure for determining \( R(a) \), we have required that the continuum choice of the positive definite test kernel, \( Y(x,y) \), such as the example introduced in Sec. 7.1, satisfy

\[
\int Y(x,y) G(x,y) d^s x d^s y = 1 .
\]

Note that, quite unlike the Gaussian case, the matrix \( A \) is only indirectly involved in the determination of the generalized function \( G(x,y) \). Indeed, to have a suitable form for \( G(x,y) \), there is no guarantee that the matrix \( A \) has any clear continuum limit by itself, i.e., there is no requirement that

\[
\Sigma'_{k,l} \phi_k A_{k,l} \phi_l a^2 \to \int \phi(x) A(x,y) \phi(y) d^s x d^s y
\]
for a suitable generalized function \( A(x, y) \) in the continuum limit. All that is required is that \( G(x, y) \) be suitable – as well as suitable distributional behavior for all the higher-order correlation functions as well!

It would be helpful to build up some computational experience for correlation functions for the generalized Poisson distributions in order to eventually help select matrices such as \( A \) by connecting them to the generalized (correlation) functions (such as \( G \)) to which they give rise. Perhaps Monte Carlo calculations could make a contribution to that effort.

In summary, in our all-too-brief discussion of the continuum limit, we have observed: (i) the atypical behavior of the limit in which \( N' \to \infty \), namely, in the case of generalized Poisson distributions, how, in hyper-spherical coordinates, the appearance of \( N' \) as a large power of the overall field radius is absent, thereby rendering perturbation series termwise finite in contrast to a comparable calculation for Gaussian-like integrals; and (ii) the different role of internal parameters (e.g., matrices \( A \) and \( B \)) in the ground-state distribution for Poisson- and Gaussian-like distributions, and how they effect the form taken by the continuum limit of correlation functions.

8 Conclusions

In discussing lattice quantum field theory models, it is conventional – and, of course, very natural – to start with a choice for the lattice action. A lattice action for a model such as \( \varphi^4_n; n \geq 5 \), however, necessitates that some difficult choices must be made for the appropriate counterterms. Appropriate in this sense means counterterms that have the virtue that the resultant model and its continuum limit will faithfully describe the original model chosen, as well as contain the original motivating classical model in the limit that \( \hbar \to 0 \). This is a lot to ask, and it is not surprising that the auxiliary potential (i.e., counterterm) generally chosen for this purpose does not perform as desired.

In this paper we have taken a different approach. First of all, recognizing that counterterms suggested by regularized, renormalized perturbation theory are inappropriate for nonrenormalizable models (see Sec. 2), we have felt compelled to direct our initial focus away from the lattice action and place it instead on the sharp-time, ground-state distribution function \( \Psi(\phi)^2 \). To avoid triviality, this distribution must be non-Gaussian, and to avoid the basin of attraction that leads to a Gaussian distribution (and its associated
triviality), we have chosen to emphasize the possible relevance of the only other class of suitable infinitely divisible distributions, namely, the generalized Poisson distributions. Such distributions avoid manifest triviality by being non-Gaussian, and they are robust to survival – as are the Gaussian distributions – under general limiting conditions.

We found that ground-state distributions in general, and Poisson distributions in particular, were faithful replacements for the lattice action which implicitly – even though somewhat indirectly – was determined by the ground-state distribution. Moreover, and this is the important point, the class of lattice actions associated with generalized Poisson distributions are automatically assured to lead to manifest nontriviality by ensuring a non-Gaussian continuum limit.

Although we could be fairly specific about which Poisson distribution we favored, it was nevertheless extremely difficulty to see just which lattice action was implicitly connected with that distribution. Thus we felt necessary to abandon any focus on individual models, and rather focus on classes of models hoping that such a wider net might capture one or another model of possible interest.

In the putative continuum analysis of Sec. 3.4, it was emphasized that suitable matrix elements of the canonical momenta, as well as those of the Hamiltonian, were uniquely specified by our choice of the characteristic function associated with the ground-state distribution. It is interesting to further note that if it were possible to actually determine the associated canonical coherent states defined with the ground state itself serving as the fiducial vector, then additional matrix elements of interest could be computed. In particular, incorporating and extending the notation for $|h⟩$ of Sec. 3.4, we let

$$|h, g⟩ \equiv e^{i\int [h(x)\hat{\phi}(x) - g(x)\hat{\pi}(x)] \, dx}\, |0⟩$$

for which it follows [10] that

$$\langle h, g|\hat{\phi}(x)|h, g⟩ = g(x) , \quad \langle h, g|\hat{\pi}(x)|h, g⟩ = h(x) ,$$

and more significantly (assuming a $\phi^4_n$ model under consideration) that

$$\lim_{\hbar \to 0} \langle h, g|\mathcal{H}|h, g⟩ = \int \{\frac{1}{2}h(x)^2 + \frac{1}{2}[\nabla g(x)]^2 + \frac{1}{2}m^2 g(x)^2 + \lambda g(x)^4\} \, dx ,$$

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implying the close connection of the diagonal coherent state matrix elements of the quantum Hamiltonian with the classical Hamiltonian itself [10]. All in all, these conditions place strong limits on the ground-state wave function, even if they are highly implicit.

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Appendix: Fundamental Restrictions on the Choice of the Weight Function \( \rho(\phi, a) \)

In this Appendix we address the question of acceptable values for \( \gamma \) and acceptable choices for the set \( \{\beta_k\} \) that are both part of the canonical form for \( \rho(\phi, a) \) we have adopted in Sec. 4. Therefore, we again consider the integral given by

\[
R(a) \int \frac{\phi_t^{2p} e^{-U(\phi, a)}}{\Pi_k' \left[ \sum_l' \beta_{k-l} \phi_l^2 \right]^{1/2}} \Pi_k' d\phi_k
\]

which should diverge when \( p = 0 \) and converge whenever \( p \geq 1 \). We assume that \( U(\phi, a) \) controls large \( \{\phi_k\} \) behavior for all \( p \), and so the question of divergence or convergence is related to the small \( \{\phi_k\} \) behavior where \( U(\phi, a) \approx 0 \), and where \( U(\phi, a) \) is not a contributing factor. Indeed, for the general question of divergence or convergence the specific form of \( U(\phi, a) \) is not of particular relevance and it may effectively be replaced by

\[
U(\phi, a) = \sum_k' \phi_k^2 a^s,
\]

leading to the expression of interest

\[
I_p \equiv R(a) \int \frac{\phi_t^{2p} e^{-\Sigma_k' \phi_k^2 a^s}}{\Pi_k' \left[ \sum_l' \beta_{k-l} \phi_l^2 \right]^{1/2}} \Pi_k' d\phi_k , \quad p = 0, 1, 2, \ldots
\]
It is convenient at this point to again introduce hyper-spherical coordinates for which 
\[ \phi_k \equiv \kappa \eta_k, \quad \kappa \geq 0, \quad -1 \leq \eta_k \leq 1, \]\nwhere \( \Sigma'_k \eta_k^2 \equiv 1 \), and \( \Sigma'_k \phi_k^2 = \kappa^2 \). In terms of these coordinates, the integral of interest reads

\[ I_p = 2R(a) \int \frac{k^{2p} \eta_k^{2p} e^{-k^2 a^s \kappa^N - 1} d\kappa \delta (1 - \Sigma'_k \eta_k^2) \Pi'_k d\eta_k}{\kappa^{2\gamma N'} \Pi'_k [\Sigma'_l \beta_{k-l} \eta_l^2]^{\gamma}}. \]

For \( p = 0 \) we attribute divergence of this integral to a divergence at \( \kappa = 0 \). Consequently, we need \( 2\gamma N' \geq N' \), i.e., \( \gamma \geq 1/2 \). For \( p \geq 1 \), convergence at \( \kappa = 0 \) requires that \( 2\gamma N' < N' + 2 \), i.e., \( \gamma < 1/2 + 1/N' \). Thus we are led to the bounds

\[ \frac{1}{2} \leq \gamma < \frac{1}{2} + \frac{1}{N'}. \]

Since the limit \( N' \to \infty \) is eventually to be taken, we already satisfy this situation by adopting \( \gamma = 1/2 \), as noted earlier. However, we are not quite finished because convergence for \( p \geq 1 \) also requires convergence of the integrals over the \( \{ \eta_k \} \), and as we now shall see, this convergence requires some restriction on the set of coefficients \( \{ \beta_k \} \).

The first remark regarding such integrals is to observe that with \( \gamma = 1/2 \) it is not possible that \( \beta_k \propto \delta_{k0} \), for in that case integrals over each \( \eta_k \), save when \( k = t \), lead to a divergence. Consequently, \( \beta_k \) must be nonvanishing for additional sites. To study how many additional sites are sufficient, it is convenient to express our basic integral in yet another form, namely

\[ I_p = R(a) \int \frac{\phi_t^{2p} e^{-\Sigma'_k \phi_k^2 a^s} \Pi'_k d\phi_k}{\Pi'_k [\Sigma'_l \beta_{k-l} \phi_l^2]^{1/2}} \Pi'_k d\phi_k \]
\[ = R(a) a^{s N'/2} \pi^{-N'/2} \int \phi_t^{2p} e^{-\Sigma'_k \phi_k^2 a^s} \Pi'_k d\phi_k \]
\[ \times \int [\Pi'_k \Lambda_k^{(-1/2)}] e^{-\Sigma'_k \Lambda_k [\Sigma'_l \beta_{k-l} \phi_l^2] a^s} \Pi'_k d\Lambda_k \]
\[ = R(a) a^{s N'/2} \pi^{-N'/2} \int \phi_t^{2p} [\Pi'_k \Lambda_k^{(-1/2)}] e^{-\Sigma'_k (1 + \Sigma'_l \beta_{k-l} \Lambda_l) \phi_k^2 a^s} \Pi'_k d\Lambda_k \Pi'_k d\phi_k, \]

where the integral for each \( \Lambda_k \) variable runs from 0 to \( \infty \). Exchanging the order of integration leads to

\[ I_p \equiv K \int \frac{[\Pi'_k \Lambda_k^{(-1/2)}] \Pi'_k d\Lambda_k}{[1 + \Sigma'_l \beta_{k-l} \Lambda_l]^p [\Pi'_k (1 + \Sigma'_l \beta_{k-l} \Lambda_l)^{1/2}]} \],

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where

\[ K \equiv 2^{-p} (2p - 1)!! R(a) a^{-sp} . \]

Observe that when the basic integral \( I_p \) is expressed in the \( \{ \lambda_k \} \) variables, divergence or convergence concerns the behavior of the integrand for large \( \{ \lambda_k \} \).

Initially, let us again show, when \( p = 0 \), that \( I_0 = \infty \) – this time in terms of the \( \{ \lambda_k \} \) integration variables. This result follows from the lower bound given by

\[
I_0 \geq K \int \frac{[\prod_k \lambda_k^{(-1/2)}] \prod_k d\lambda_k}{[\prod_k (1 + \Sigma' \beta \lambda_1)]^{1/2}} ,
\]

which holds because \( \beta_k \leq 1 \). If we introduce coordinates of the form \( \lambda_k \equiv \zeta \eta_k^2 \), \( \zeta \geq 0, -1 \leq \eta_k \leq 1 \), and \( \Sigma' \eta_k^2 = 1 \), then it follows that

\[
I_0 \geq 2^{N'} K \int \frac{\zeta^{(N'/2-1)} d\zeta}{[1 + \zeta]^{N'/2}} \int \delta(1 - \Sigma' \eta_k^2) \prod_k d\eta_k .
\]

The first integral diverges, while the remaining factors are positive. Thus we have again shown that \( I_0 = \infty \), as expected.

For \( p \geq 1 \), we require that \( I_p < \infty \). First we observe by construction that \( I_q \leq I_p \) whenever \( q \geq p \). Therefore, to show that \( I_p < \infty \) for \( p \geq 1 \), it suffices to show that \( I_1 < \infty \). Thus we focus attention on

\[
I_1 \equiv K \int \frac{[\prod_k \lambda_k^{(-1/2)}] \prod_k d\lambda_k}{[1 + \Sigma' \beta \lambda_1] [\prod_k (1 + \Sigma' \beta - i \lambda_1)]^{1/2}}
\]

If this integral were to diverge it must diverge for large \( \lambda \). We can reach large \( \lambda \) in many different directions. In particular, we can imagine that \( P, 1 \leq P \leq N' \), of the \( \lambda \) variables are all approaching infinity in a certain direction. As an example, consider \( P = 3 \) and \( \lambda_1 = .5 \zeta, \lambda_2 = .2 \zeta, \lambda_3 = .1 \zeta \), as \( \zeta \to \infty \), while all other \( \lambda \) variables have finite values. The “direction” of approach to infinity in this case refers to the relative size of the growing \( \lambda \) terms, i.e., the direction as determined by \( \eta_1^2 = .5, \eta_2^2 = .2, \) and \( \eta_3^2 = .1 \).
Staying with this example for a moment, the three large $\lambda$ values will arise in several terms in the denominator because of the possible many-fingered nature of the $\beta$ terms. We divide the set of distinct space-like $k$ values into the set $P$ which contains one or more of the $P$ variables approaching infinity and the set $Q$ containing the remaining $N' - P$ terms for which the variables $\lambda$ do not approach infinity. In dividing the appropriate sets below, we have added an asterisk to the set of $\lambda_k$ values that are approaching infinity to serve as a reminder of that fact. Thus we are led to the expression

$$I_1 = K \int \int \frac{[\Pi'_{k \in P} \lambda_k^{*-1/2}]}{[1 + \Sigma'_{l \in P} \beta_{l-1} \lambda_l^* + \Sigma'_{l \in Q} \beta_{l-1} \lambda_l]} \times \frac{[\Pi'_{k \in Q} \lambda_k^{*-1/2}] \Pi'_{k \in P} d\lambda_k^* \Pi'_{k \in Q} d\lambda_k}{[\Pi'_k(1 + \Sigma'_{l \in P} \beta_{k-1} \lambda_l^* + \Sigma'_{l \in Q} \beta_{k-1} \lambda_l)^{1/2}]}.$$  

In the numerator of this expression there are $P$ variables approaching infinity at the same time, where $1 \leq P \leq N'$, and we need to consider all possibilities. For the denominator, there are several cases to consider. We assume that the set $\{\beta_k\}$ has $1 + S_0$ nonvanishing terms and that they are connected to the “home coordinate”, $k = 0$. In particular, let us focus on the choice $k^{\text{plus}}$ (consisting of the point $k$ and its $2s$ nearest neighbors in spatial directions) for which $S_0 = 2s$. As a consequence, there are large $\lambda$ variables in $P + S (+1)$ factors in the denominator; here the term $S$, which may vary from case to case, arises from the many-fingered nature of $\{\beta_k\}$, and the term $(+1)$ adds 1 only if a large $\lambda$ factor appears in the denominator factor associated with $p = 1$. Convergence of $I_1$ is ensured if for all $P$, $1 \leq P \leq N'$, the choice of $\{\beta_k\}$ ensures that $S (+1) \geq 1$ in all possible combinations of the large $\lambda$ values. A few specific examples may help clarify how the choice of $k^{\text{plus}}$ fulfills all the necessary requirements.

For example, if $P = 1$ then $S (+1) = S_0 (+1) \geq 1$. If $P = N' - 1$ includes all the lattice points except one, then $S (+1) = 1$; if $P = N'$, and thus includes all lattice points, then $S (+1) = 1$. If $P = L$, all of which lie in a single coordinate direction (with period $L$), then $S (+1) = 2(s-1)L (+1) \geq 1$. Incidentally, this example shows that fingering in just one spatial direction is not sufficient – and, by analogy, fingering in $(s - 1)$ spatial directions is also insufficient. It is necessary to finger in all $s$ different spatial directions.

A picture may also help; see Fig. 1. Although not directly relevant to nonrenormalizable fields, we give a two-dimensional example ($s = 2$) for
which the points with large $\lambda$ values are marked with solid circles $\bullet$, the additional points in which those large values appear in denominator factors are marked with an open circle $\circ$, and the point $k = t$ is marked with an $\times$. In the example of this picture, $P = 2$, $S_0 = 4$, $S = 6$, and since the $p = 1$ term for this example does not contribute, the $(+1)$ term is absent.

Fig. 1: A two-dimensional example with two large $\lambda$ values (dark circles) which also have a presence in six additional locations (open circles). The place marked by $\times$ is where the additional moment is located.

References


