On the two-dimensional magnetic reconnection with nonuniform resistivity

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In this paper two theoretical approaches for the calculation of the rate of quasi-stationary, two-dimensional magnetic reconnection with nonuniform anomalous resistivity are considered in the framework of incompressible magnetohydrodynamics (MHD). In the first, “global” equations approach the MHD equations are approximately solved for a whole reconnection layer, including the upstream and downstream regions and the layer center. In the second, “local” equations approach the equations are solved across the reconnection layer, including only the upstream region and the layer center. Both approaches give the same approximate answer for the reconnection rate. Our theoretical model is in agreement with the results of recent simulations of reconnection with spatially nonuniform resistivity by Baty, Priest and Forbes (2006), contrary to their conclusions.

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I. INTRODUCTION

Magnetic reconnection is one of the most important processes of plasma physics, and is believed to play the central role in observed phenomena in laboratory and cosmic plasmas. At the same time there has been a long-standing debate about the correct theoretical model of magnetic reconnection. Most previous theoretical and numerical work focused on reconnection processes in two-dimensions, in which all physical scalars and vectors are independent of the third coordinate (z). There exist two original and well known models of magnetic reconnection, which predict a slow magnetic reconnection rate in hot low-density plasmas. Second, the Petschek model, in which a fast reconnection rate is achieved by introducing switch-off magnetohydrodynamic (MHD) shocks attached to the ends of a relatively short reconnection layer in the downstream regions. Many numerical simulations have been carried out to discriminate between these two models. More recent high-resolution simulations generally favor the Sweet-Parker model of slow reconnection in the case of constant resistivity, and do not confirm the Petschek theoretical picture for the geometry of the reconnection layer with shocks. However, reconnection becomes much faster and Petschek-like if resistivity is not constant and is enhanced locally in the reconnection layer (numerical studies of this case were pioneered by Ugai and Tsuda, by Hayashi and Sato, and by Scholer).

These results have been called into question in a recent paper, Baty, Priest and Forbes (2006), which claims that Petschek reconnection can occur when the resistivity is not absolutely constant but varies by an arbitrarily small amount. This paper has motivated us to show on the basis of an earlier paper of ours, Malyshkin et al. [14], that the Petschek shocks are produced not by the variation of the resistivity but by the rate of variation which for their case is very large and unlikely to happen naturally.

In 2001 Kulsrud provided some theoretical insight into magnetic reconnection process which qualitatively explained the results of recent simulations. He considered quasi-stationary, two-dimensional magnetic reconnection in the classical Sweet-Parker-Petschek reconnection layer with zero guide field ($B_z = 0$), zero plasma viscosity and an anomalous resistivity that is a piecewise constant function of the electric current. Kulsrud was first to suggest that one has to calculate the half-length of the reconnection layer $L'$ from the MHD equations and the jump conditions on the Petschek shocks, instead of treating $L'$ as a free parameter (as Petschek erroneously did when chose $L'$ to be equal to its minimal possible value under the condition of no significant disruption to the plasma flow). As a result, in the case of constant resistivity Kulsrud correctly estimated the layer half-length $L'$ to be approximately equal to the global magnetic field scale, $L' \approx L$. In this case the Petschek reconnection rate reduces to the slow Sweet-Parker reconnection rate, in agreement with numerical simulations. The second result obtained by Kulsrud is that in the case when resistivity is non-constant but anomalous and enhanced (e.g. by plasma instabilities), the reconnection rate can become considerably faster than the Sweet-Parker rate.

In our recent paper (Malyshkin, Linde and Kulsrud [14]), to which we will hereafter refer as MLK2005 paper, we put Kulsrud’s derivations on a rigorous analytical basis and extended his model by using a new theo-
retical approach to calculate the reconnection rate. This approach is based on “local” analytical derivations across a thin reconnection layer, and it is applicable to the case when resistivity is anomalous, nonuniform and an arbitrary function of the electric current and of the spatial coordinates. We included the case of non-zero guide field \( B_x \neq 0 \) and non-zero plasma viscosity in our model. We found an approximate formula for the reconnection rate which confirmed Kulsrud’s theoretical results.

The present paper has two goals. First, to calculate the reconnection rate, and second, to compare it with the recent simulations of Baty et al. [15].

In the next section we simultaneously follow two theoretical approaches to the calculation of the reconnection rate. In the first approach, which we call the “global” equations approach, we derive and solve approximate MHD equations for a whole reconnection layer, including the upstream and downstream regions and the layer center. These theoretical derivations are similar to those done before, except for an important difference. Namely, we take into consideration an additional important equation, the spatial homogeneity of the \( z \)-component of electric field \( E_z \) along the reconnection layer. This equation, together with the jump condition on Petschek shocks, allows us to find the reconnection layer length \( L' \), which must be determined consistently [12]. The second theoretical approach, which we call the “local” equations approach, basically coincides with the calculations done in the MLK2005 paper. In this approach we derive and solve approximate MHD equations across a thin reconnection layer, including only the upstream region and the layer center. Both approaches give the same approximate formula for the reconnection rate, which is valid in the general case of an arbitrary nonuniform anomalous resistivity [see Eq. (11)]. Using these two approaches simultaneously allows us to better understand the physics of a reconnection process.

In Sec. III we pursue the second goal of this paper. We consider the case treated by Baty et al. [12], when resistivity is a prescribed function of the two spatial coordinates \( x \) and \( y \) (remember that we consider two-dimensional reconnection, so that no physical quantities depend on \( z \)). We argue that our theoretical model is in agreement with the results of these recent simulations of reconnection by Baty, Priest and Forbes [15]. This agreement contradicts their theoretical conclusion that Petschek shocks exist with constant resistivity, and we explain why. Finally, in Sec. IV we discuss our results.

II. RECONNECTION WITH NONUNIFORM ANOMALOUS RESISTIVITY

In this section we consider magnetic reconnection with nonuniform anomalous resistivity in the classical two-dimensional Sweet-Parker-Petschek reconnection layer, shown in Fig. 1. The layer is in the \( x \)- and \( y \)-axes being perpendicular to and along the layer respectively. The length of the layer is \( 2L' \), which is approximately equal to or smaller than the global magnetic field scale \( L \) that will be introduced below. The thickness of the layer, \( 2\delta_o \), is much smaller than its length, i.e. \( 2\delta_o \ll 2L' \). The classical Sweet-Parker-Petschek reconnection layer is assumed to have a point symmetry with respect to its geometric center point \( O \) in Fig. 1 and reflection symmetries with respect to the axes \( x \) and \( y \).

Thus, for example, the \( x \)- and \( y \)-components of the plasma velocities \( \mathbf{V} \) and of the magnetic field \( \mathbf{B} \) have the following simple symmetries: \( V_x(\pm x, \mp y) = \pm V_x(x, y) \), \( V_y(\pm x, \mp y) = \mp V_y(x, y) \), \( B_x(\pm x, \mp y) = \mp B_x(x, y) \) and \( B_y(\pm x, \mp y) = \pm B_y(x, y) \). There could be a pair of Petschek shocks attached to each of the two reconnection layers in the downstream regions. Because of the MHD jump conditions on the Petschek shocks, there must be a nonzero perpendicular magnetic field \( B'_c \) in the downstream region at point \( O' \) in Fig. 1 [12].

If the plasma viscosity is small, then the plasma outflow velocity \( V_{out}' \) in the downstream region at point \( O' \) is approximately equal to the Alfvén velocity \( V_A \) calculated in the upstream region at point \( M \) (refer to Fig. 1). The plasma inflow velocity \( V_R \) in the upstream region at point \( M \), outside the reconnection layer, is much smaller than the outflow velocity, \( V_R \ll V_A \). Finally, the magnetic field \( B_m \) at point \( M \), outside the layer, is mostly in the direction of the layer (i.e. in the \( y \)-direction).

Now let us list four assumptions that we make about reconnection process. First, we assume that the characteristic Lundquist number is large, which (by our definition) means that resistivity is negligible outside the reconnection layer. Second, we assume the plasma flow is incompressible. Third, for simplicity we neglect plasma viscosity. (The case of nonzero viscosity is treated in the MLK2005 paper [14].) Fourth, we assume the reconnection process is quasi-stationary. This is true if the reconnection is slow, \( V_R/V_A \ll 1 \), and that there are no plasma instabilities in the reconnection layer. For an incompressible viscousless plasma the assumption of slow reconnection is equivalent to the assumption that the reconnection layer is thin, \( \delta_o/L' \approx V_R/V_A \ll 1 \). The above assumptions are standard in the Sweet-Parker and Petschek reconnection models. Please note that we make...
no assumptions about the values of the guide field \( B_z \). In other words, our derivations apply to what is called “two-and-a-half dimensional” reconnection.

For brevity, we use physical units in which the speed of light and four times \( \pi \) are replaced by unity, \( c = 1 \) and \( 4\pi = 1 \). In these units the MHD equations that we need to find the reconnection rate are as follows. Faraday’s law \( \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \) for the \( x \)- and \( y \)-components of magnetic field in two dimensions is

\[
\partial E_z/\partial y = -\partial B_z/\partial t \approx 0, \quad \partial E_z/\partial x = \partial B_y/\partial t \approx 0,
\]

where \( \partial \mathbf{B}/\partial t \approx 0 \) because of the quasi-stationarity of reconnection. From Eqs. (1) we see that the electric field \( z \)-component is constant in space, i.e. \( E_z = E_z(t) \) is a function of time only. Next, neglecting the displacement current in the framework of MHD, Ampere’s law for the \( z \)-component of the current is

\[
j_z = (\nabla \times \mathbf{B})_z = \partial B_y/\partial x - \partial B_x/\partial y.
\]

Ohm’s law for the spatially uniform \( z \)-component of the electric field is

\[
E_z(t) = -V_y B_y + V_y B_x + \eta j_z = \text{constant in space}, \quad (3)
\]

where resistivity \( \eta = \eta(j_z,x,y) \) is an arbitrary function of the electric current \( z \)-component and of the two-dimensional coordinates \( x \) and \( y \). Next, the equation for plasma acceleration along the layer in the \( y \)-direction is

\[
r(\mathbf{V} \cdot \nabla) V_y = -(\partial/\partial y)[P + B_z^2/2 + B_y^2/2] + (\mathbf{B} \nabla) B_y, \quad (4)
\]

where \( r \) is the constant plasma density, \( P \) is the sum of the plasma pressure and the guide field pressure \( B_z^2/2 \). In addition, we have

\[
\partial_x V_z + \partial_y V_y = 0, \quad \partial_x B_z + \partial_y B_y = 0 \quad (5)
\]

because the field and the velocity are divergence-free.

Now, from MHD equations (2)–(5) we obtain the following table of “global” and “local” equations for the reconnection layer:

<table>
<thead>
<tr>
<th>Global Equations</th>
<th>Local Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st, Ampere’s law</td>
<td>( j_o \equiv (j_z)_o \approx (\partial_x B_y)_o \approx B_m/\delta_o )</td>
</tr>
<tr>
<td>2nd, Incompressibility</td>
<td>( V_R L' \approx V'_o \delta_o )</td>
</tr>
<tr>
<td>3rd, Plasma acceleration</td>
<td>( (\partial_y V_y)_o \approx -(\partial_x V_z)_o \approx V_R/\delta_o )</td>
</tr>
<tr>
<td>4th, Jump condition on shocks</td>
<td>( B'_o/\sqrt{\rho} = V'_o \approx V_R )</td>
</tr>
<tr>
<td>5th, ( E_z = \text{const across the layer} )</td>
<td>( \eta_o j_o = V_R B_m )</td>
</tr>
<tr>
<td>6th, ( E_z = \text{const along the layer} )</td>
<td>( \eta_o j_o = \eta' j'_o + V'_o B'_o/\delta_o = 0 = \partial_y^2 (\eta j_z)_o + 2(\partial_y V_y)_o (\partial_y B_z)_o )</td>
</tr>
<tr>
<td>7th, Unknown quantities</td>
<td>( j_o, \delta_o, V_R, L', V'_o, B'_o )</td>
</tr>
</tbody>
</table>

Here quantities with subscript \( o \) are taken at the reconnection layer central point \( O \), while quantities with the prime sign are taken at point \( O' \) in the downflow region (refer to Fig. 1). The column “Global Equations” includes equations that are written at points \( O, M \) and \( O' \), these equations represent the “global” equations approach to the calculation of the reconnection rate. The column “Local Equations” includes equations that are written only at points \( O \) and \( M \) and represent the “local” equations approach. Note that equations on lines 1 and 5 enter both columns in the same form. The first line of the table includes the Ampere’s law equation (2), with the \( \partial_y B_z \) term neglected because it is small, and the \( \partial_x B_y \) term estimated at point \( O \). The second line of the table includes the plasma incompressibility condition (5) in its “global” and “local” forms. The “global” form is the mass conservation equation, while in the “local” form the \( \partial_y V_z \) term is estimated at point \( O \). The third line contains equations for plasma acceleration given by Eq. (4). The “global” equation \( V'_o \approx V_A \) reflects the well known result that in the absence of viscosity the plasma outflow velocity in the downstream region is approximately equal to the Alfvén velocity calculated in the upstream region (1,2). The “local” equation for plasma acceleration results from differentiation of Eq. (4) with respect to \( y \) and taking the symmetries of the reconnection layer into account. The \( j_o (\partial_y B_z)_o \) term in this equation is the magnetic tension force, while the pressure term is equal to

\[
(\partial_y^2 P)_o = -\partial_y^2 (B_z^2/2)_m = B_m (\partial_y^2 B_y)_m < 0. \quad (7)
\]

Thus, the drop of pressure \( P \) (which includes the plasma and guide field pressure) along the layer is equal to the pressure drop of the parallel magnetic field outside the layer. This result follows from the force balance condition for the plasma across the reconnection layer (in analogy with the Sweet-Parker derivations), and its rigorous proof can be found in the appendix A of the MLK2005 paper (14). The last equality in Eq. (7) comes from the layer reflection symmetry with respect to the \( x \) axis.

Next, the fourth line in Table 6 includes the standard jump condition on the switch-off MHD Petschek shocks.
attached to the ends of the reconnection layer\footnote{12,13}. This condition is a “global” equation, and the plasma
incoming velocity \(V'_R\) at point \(M'\) is estimated as being
approximately equal to the plasma incoming velocity \(V_R\)
at point \(M\) (refer to Fig. 1). There is no corresponding
“local” equation because we do not consider the down-
stream region and Petschek shocks in the “local” equations
approach! Equations in the fifth and sixth lines of Table 6
directly result from Ohm’s law equation \(3\). Namely, we use the spatial homogeneity of \(E_z(t)\), which is a con-
sequence of the quasi-stationarity of reconnection.
To obtain the single equation in the fifth line and the
“global” equation in the sixth line, we equate the Ohm’s
law expression for \(E_z\) at points \(O, M\) and \(O'\). To obtain
the “local” equation in the sixth line, we take the second
order partial derivative \(\partial^2_{y}\) of Eq. \(3\) at point \(O\). Finally,
the seventh line in Table 6 lists all unknown physical
quantities to be estimated in the “global” and “local”
equations approach. Note that quantities \(j_o, \delta_o, V_R, \)
\((\partial_y V'_y)\) and \((\partial_y B_z)\) are “local” (i.e. defined at the layer
central point \(O\) and at point \(M\) in the upstream region),
while quantities \(L'\), \(V'_o\) and \(B'_z\) are “global” (i.e. defined
at point \(O'\) in the downstream region).
There are a few additional equations that we need.
First, we use the second-order Taylor expansion of \(\eta_j\)
along the y-axis to estimate the \(\eta'_j\) term in the 6th line in
Table 6
\[
\eta'_j \approx \eta_j x + (L^2/2) \{ j_o (\partial^2_{y}) \eta \} +
+ \{ \eta_o + j_o (\partial \eta / \partial j) \} (\partial^2_{y} j_o). \tag{8}
\]
Note that the first-order Taylor expansion terms are zero
because of the symmetry. Second, we define the field
\(\text{global scale } L\) and the resistivity scale \(\eta_o\) as \footnote{20}
\[
L^2 \equiv -2B_m / (\partial^2_{y} B_m), \quad \eta^2 \equiv -2\eta / (\partial^2_{y} \eta). \tag{9}
\]
Third, the y-scale of the current \(j_z\) to a factor of order
unity, turns out to be about the same as the y-scale of
the outside magnetic field,
\[
j^{-1} (\partial^2_{y} j_z) \approx B_m^{-1} (\partial^2_{y} B_y) = -2L^2. \tag{10}
\]
This result can be understood by taking the second order
partial derivative \(\partial^2_{y}\) of the Ampere’s law equation \(j_o =
B_m / \delta_o\) given in the first line in Table 6 while keeping
\(\delta_o\) constant because the partial derivative in \(y\) is to be
taken at a constant value of \(x = \delta_o\). The detailed proof
of Eq. \(10\) is given in the appendix B of the MLK2005
paper.
Now we have all equations necessary to find the re-
connection rate and all other unknown physical par-
eters. In the “Global Equations” column in Table 6
we have six equations and six unknowns, and in the
“Local Equations” column we have five equations and
five unknowns. Using the “local” equations and equa-
tions \footnote{17,} \footnote{18} and \footnote{11}, we obtain the following approx-
imate algebraic equation for the 2-current \(j_o\) at the re-
connection layer central point \(O\):
\[
3 + j_o / \eta_o \cdot \left( \partial \eta / \partial j \right)_o + L^2 / \eta_o \approx \eta_o^2 j_o L^2 / \eta_o, \tag{11}
\]
where resistivity \(\eta_o \equiv \eta_j x = 0, \eta_o = 0\) is a func-
tion of \(j_o\). The “global” equations give a similar result
with 3 replaced by 1 in Eq. \(11\), which is a less accurate
result due to additional approximations made in Eq. \(8\)
and in the formula \(V'_R \approx V_R\) in the fourth line in Table 6.
Given the resistivity function \(\eta = \eta_j x, y\), as well
as the magnetic field \(B_m\) and its global scale \(L =
-2B_m / (\partial^2_{y} B_y)\) both calculated at point \(M\) in the
upstream region, we can solve the algebraic Eq. \(11\) for
the current \(j_o\). Once \(j_o\) is calculated, we can easily find
the reconnection rate, which is the rate of destruction of
magnetic flux at point \(O\) and is equal to the electric field
z-component \(E_z(t) = \eta_j x\). We can also find all the other
physical parameters, by using the equations in Table 6
\[
\delta_o \approx B_n / j_o, \tag{12}
\]
\[
V_R \approx \eta_j j_o / B_m \ll V_A, \tag{13}
\]
\[
\left( \partial_y V'_y \right) \approx \eta_o^2 B_m / \eta_o \approx V_R / \delta_o, \tag{14}
\]
\[
\left( \partial_y B_z \right) \approx j_o \left( V'_R / V_A^2 - 2B_m / j_o L^2 \right), \tag{15}
\]
\[
B'_z \approx V_R / \sqrt{B_m} \approx B / B_m / V_A, \tag{16}
\]
\[
V'_o \approx V_A, \tag{17}
\]
\[
L' \approx \delta_o V_A / \left( \partial_y V'_y \right) \approx \left( \partial_y V'_y \right) / \left( \partial_x B_z \right). \tag{18}
\]
The last approximate equality in Eq. \(18\) is valid be-
cause the second term on the right-hand-side of Eq. \(15\)
can be neglected. This is because \(3B_m / j_o L^2 \leq V_R / V^2\),
which follows from Eqs. \(13\) and \(11\) (also refer to foot-
note \(28\)).
Equations \footnote{11-18} are very general results for quasi-
stationary magnetic reconnection with no assumptions
about the functional form of resistivity and the guide
field value. Now let us look at the three terms on the
left-hand side of Eq. \(11\). When resistivity is constant,
the only term left is the first term. The second term is
clearly related to the dependence of anomalous resistivity
on the current. The third term becomes important when
resistivity is \textit{ad hoc} localized in space. As a result, in
the end of this section we consider three special cases of
magnetic reconnection, in which Eqs. \(11\)-\(18\) reduce to
simpler formulas. These three cases correspond to domin-
ation of the first, second and third terms respectively
on the left-hand-side of Eq. \(11\), and they are as follows.
The first case is the quasi-uniform resistivity case,
when 
\[
1 \gg \max [(j_o / \eta_o) \left( \partial \eta / \partial j \right)_o, L^2 / \eta_o,],\]  
then
\[
j_o \approx (B_m / L)^{1/2}, \quad \text{where } S = V_A L / \eta_o, \tag{19}
\]
\[
V_R / V_A \approx S^{-1/2}, \quad \delta_o \approx L^{-1/2}, \quad L' \approx L.
\]
Here we introduce the Lundquist number \(S_o \equiv V_A L / \eta_o\)
and assume for our estimates that \(3^{1/2} \approx 1\). Equa-
tions \footnote{19} are the familiar Sweet-Parker results \footnote{1,2}.
Thus, if resistivity is quasi-uniform or uniform, then reconnection is Sweet-Parker.

The second case is the Petschek-Kulsrud reconnection, if \((j_o/\eta_o)(\partial\eta/\partial j_o) \gg \max[1, L^2/l_o^2]\), then

\[
V_R/V_A \approx \delta_o/L' \approx \left[(B_m/V_AL^2)(\partial\eta/\partial j_o)\right]^{1/3}, \tag{20}
\]

\[
L' \approx L [(j_o/\eta_o)(\partial\eta/\partial j_o)]^{-1/2} \ll L.
\]

This is the case of fast reconnection with Petschek geometry and shocks. Equations (20) were first derived by Kulsrud [12]. This is the reason we call this case Petschek-Kulsrud reconnection.

The third case is the case of reconnection with spatially localized resistivity, if \(L^2/l_o^2 \gg \max[1, (j_o/\eta_o)(\partial\eta/\partial j_o)]\), then

\[
\begin{align*}
     j_o & \approx (B_m/l_o)\delta_o^{1/2}, & & \text{where } S_o \equiv V_A l_o/\eta_o, \tag{21} \\
     V_R/V_A & \approx S_o^{-1/2}, & & \delta_o \approx l_o S_o^{-1/2}, & & L' \approx l_o \ll L.
\end{align*}
\]

Here we introduce the effective Lundquist number \(S_o \equiv V_A l_o/\eta_o\) that is based on the resistivity scale \(l_o\) given by Eq. (19). Equations (21) are the same as the Sweet-Parker equations (19) with the global field scale \(L\) replaced by the resistivity scale \(l_o\). When resistivity is strongly localized, \(l_o \ll L\), the reconnection becomes fast yielding Petschek geometry and shocks.

We postpone the detailed analysis and discussion of our theoretical results until the last section of the paper.

III. RECONNECTION WITH SPATIALLY LOCALIZED RESISTIVITY

In this section we compare our theoretical results to the results of recent simulations of reconnection with spatially nonuniform resistivity by Baty, Priest and Forbes (2006) [13]. They took the resistivity as

\[
\eta(x, y) = \eta_1 + (\eta_o - \eta_1) \exp \left[-(x/l_y)^2 - (y/l_y)^2\right], \tag{22}
\]

and considered different values of parameters \(\eta_o, \eta_1, l_x\) and \(l_y\). They found that the reconnection rate does not depend on the value of \(l_x\). This is in agreement with our Eq. (14), which includes only \(\partial^2_\eta (x, y)\) derivative via resistivity scale \(l_o\) given by Eq. (19). As for the dependence on the other parameters, in all their simulation runs Baty et al. found the Petschek geometrical configuration with shocks and reconnection rate faster than the Sweet-Parker rate. They paid special attention to the case when \(\eta_o - \eta_1\) in Eq. (22) is small and the resistivity is weakly nonuniform. They called this case a “quasi-uniform resistivity” case, and observed the Petschek solution in this case as well. At the same time, we theoretically derived the Sweet-Parker solution for the quasi-uniform resistivity case, given by Eqs. (18). Thus, there is a disagreement between our theoretical results and the claims by Baty et al., which needs to be addressed.

The reason for this disagreement is that the resistivity used by Baty et al. was not actually quasi-uniform in all their simulation runs. In fact, when resistivity depends only on coordinates, the condition of a quasi-uniform resistivity is \(1 \gg L^2/l_o^2\), refer to Eqs. (19). In other words, the resistivity localization scale \(l_o\) must be much larger than the field global scale \(L\). In their most uniform resistivity simulation run Baty et al. used \(\eta_o = 10^{-4}, \eta_1 = 9.3 \times 10^{-5}, l_y = 0.1, l_o = 0.378, L = 1\). According to Eq. (22), in this case \(l_o \approx l_y \sqrt{\eta_0/\eta_1} = 0.378\), which is considerably smaller than \(L = 1\). Thus, while the resistivity function \(\eta(x, y) = 9.3 \times 10^{-5} + 7 \times 10^{-6} \exp\left[-(y/0.1)^2 - (x/l_y)^2\right]\) would appear close to uniform when its graph is plotted, its second derivative \(\partial^2_\eta (x, y)\) is large due to small value of \(l_y = 0.1\). As a result, this resistivity should be viewed as far from uniform and as rather well localized, \(L^2/l_o^2 \approx 7\). In their other simulation runs Baty et al. used even stronger localized resistivity with larger values of \(\eta_o - \eta_1\) in Eq. (22). In fact, they were not able to run any simulations with strictly uniform resistivity \(\eta = \text{const}\) because of an instability of the reconnection layer. The reason for this instability possibly lies in the boundary conditions that Baty et al. used, which might conflict with our Eq. (14). The latter must be satisfied for quasi-stationary reconnection.

We would like to quantitatively compare our theoretical results to the results obtained by Baty et al. in their most uniform resistivity simulation run. They had \(l_o = 0.378, l_y = 0.1, \eta_o = 10^{-4}, V_A \approx 1, B_m \approx 0.9\) and they found \(j_o \approx 120\) for the current at the reconnection layer central point O [13, 14]. Our approximate Eqs. (21), derived for the localized resistivity case, give \(j_o \approx (B_m/l_o)(V_A l_o/\eta_o)^{1/2} = 146\), which is close to \(j_o \approx 120\) observed in the simulation. The small disagreement could be due to plasma compressibility, or due to the finite Lundquist number used in the simulation and due to the approximate nature of our theoretical model.

We conclude that our theoretical model for magnetic reconnection is in an agreement with the simulations by Baty et al. [13]. Our model is in reasonable agreement with several other previous numerical simulations of reconnection with spatially nonuniform resistivity [7, 8, 11, 17, 18].

IV. DISCUSSION

Let us now discuss our major results.

First, equation (14) that determines the reconnection current \(j_o\) and the quasi-stationary reconnection rate \(j_o/\eta_o\) is derived by using the “local” equations theoretical approach (see the last column of Table 6). This approach involves only “local” quantities and equations that are defined on the interval \(OM\) across the reconnection layer, refer to Fig. 1. Thus, the quasi-stationary reconnection rate in a thin two-dimensional layer is determined locally, in the layer central point \(O\) and in the upstream region outside the layer at point \(M\). In other words, the rate is fully determined by a particular functional form of anomalous resistivity \(\eta(j_o, x, y)\) and by the lo-
cal configuration of the magnetic field in the upstream region. The later determines field $B_m$ and its scale $L \equiv [-2B_m / (\partial z^2 B_m)]^{1/2}$ at point $M$. As a result, the global properties of the reconnection layer (e.g., its length $L'$, the plasma outflow velocity $V_m'$, and the presence or absence of Petschek shocks) do not directly matter for the quasi-stationary reconnection rate. This is a very important result because the reconnection layer global geometry can be very complicated (e.g. in turbulent plasmas). Of course, there exists an indirect dependence of the reconnection rate on the field global properties because the local field configuration in the upstream region is determined by the field global configuration. The above statements are also true for all other “local” parameters – for the layer thickness $\delta_o$ and for the reconnection velocity $V_R$, refer to Eqs. (12) and (13).

Second, the reconnection rate can also be estimated by using the “global” equations theoretical approach, in which the whole reconnection layer is considered, including the downstream region at point $O'$ (refer to Fig. 1). The equations used in this approach are presented in the second column of Table 6. We would like to point out that two of these equation play the role in correct determination of the reconnection layer length $L'$ and geometry. The first key equation is the jump condition on the Petschek shocks, $B'_e / \sqrt{\rho} \approx V'_h \approx V'_o$, whose importance was first pointed out by Kulsrud [12]. The second key equation is the constancy of the electric field $z$-component along the reconnection layer. The first key equation is the jump condition on the Petschek shocks, $B'_e / \sqrt{\rho} \approx V'_h \approx V'_o$, whose importance was first pointed out by Kulsrud [12]. The second key equation is the constancy of the electric field $z$-component along the reconnection layer. The first key equation is the jump condition on the Petschek shocks, $B'_e / \sqrt{\rho} \approx V'_h \approx V'_o$, whose importance was first pointed out by Kulsrud [12]. The second key equation is the constancy of the electric field $z$-component along the reconnection layer. The first key equation is the jump condition on the Petschek shocks, $B'_e / \sqrt{\rho} \approx V'_h \approx V'_o$, whose importance was first pointed out by Kulsrud [12].

Third, for the case of a strong dependence of resistivity on the current, i.e., when $( j_o / \eta_o ) ( \partial \eta / \partial j_o ) \gg \max[1, L^2 / l_n^2]$, our results, given by Eqs. (20), coincide with the results obtained by Kulsrud [12]. Thus, both the “local” and “global” theoretical approaches confirm Kulsrud’s results and ideas, contrary to the doubts raised in paper by Baty et al. [13].

Fourth, we found that in the case of uniform or quasi-uniform resistivity the magnetic reconnection rate is the slow Sweet-Parker rate and not the fast Petschek rate, see Eqs. (19). This theoretical result follows from rigorous analytical derivations and agrees with numerical simulations. At the same time it contradicts the original Petschek theoretical model. Let us consider both the “global” and “local” analytical approaches in the case of a strictly uniform resistivity $\eta = \text{const} = \eta_o$, and let us explain why the Petschek reconnection layer geometry is not realized in this case. We take the “global” equations approach first. In the case of constant resistivity Eqs. (5) and (11) result in equation $(\eta_o j_o - \eta_j' z') / \eta_o j_o \approx (L' / L)^2$ for the fractional drop of the $j_z$ term along the reconnection layer. On the other hand, the constancy of the electric field $z$-component along the reconnection layer implies that this fractional drop is $(\eta_o j_o - \eta_j' z') / \eta_o j_o \approx V'_o B'_j / \eta_o j_o \approx 1$, where we use the “global” equations on lines 3 to 6 in the second column of Table 6. These two equations agree only if $L' \approx L$, which means that the geometry of the reconnection layer is Sweet-Parker and not Petschek. Next, we take the “local” equations approach. In this “local” approach we prefer not to consider the reconnection layer geometry and any “global” parameters, such as the layer length $L'$, for the calculation of the reconnection rate. Instead we argue as follows. Refer to the “local” equations in the last column of Table 6. Equations on lines 1, 2 and 5 result in an estimate of the plasma outflow velocity derivative $(\partial \eta V_b) / \eta_o \approx \eta_o j_o^2 / B_m^2$, see Eq. (14). Plasma acceleration equation on line 3 results in the upper estimate for the $B_z$ field derivative, $(\partial \eta V_b) / \eta_o \leq \rho (\partial \eta V_b)^2 / j_o$. As a result, we can find the upper estimate for the reconnection current $j_o$ from the “local” equation on line 6, which is the condition of constancy of the electric field $z$-component along the reconnection layer. If resistivity is uniform, this estimate turns out to be the Sweet-Parker value, $j_o \approx (B_m / L) (V_A L / \eta_o)^{-1/2}$, as given by Eq. (19). In addition, if one estimates the reconnection layer length as $L' \approx B'_e / \eta_o j_o \approx V_A / (\partial \eta V_b)$, he/she would again recover the Sweet-Parker result $L' \approx L$ [14]. We conclude that in the case of constant resistivity both the “global” and “local” equations approaches consistently lead to the Sweet-Parker reconnection rate and the Sweet-Parker geometry of the reconnection layer ($L' \approx L$).

Finally, let us point out that whether reconnection is unforced (free) or forced does not matter for our results and conclusions. Indeed, on one hand, in the case of unforced reconnection one solves Eqs. (11) and (13) for the current $j_o$ and for the reconnection velocity $V_R$. The solution will depend on the magnetic field $B_m$ in the upstream region at point $M$. On the other hand, in the case of forced magnetic reconnection the reconnection velocity $V_R$ is prescribed and fixed. In this case the field $B_m$ in the upstream region should be treated as an unknown parameter, and Eqs. (11) and (13) are to be solved together in order to find the correct quasi-stationary values of $j_o$ and $B_m$. In other words, in the forced reconnection case an initially weak outside magnetic field $B_m$ gets piled up to higher values until the resulting current $j_o$ in the reconnection layer becomes large enough to be able to match the prescribed velocity $V_R$ of magnetic flux and energy supply in the upstream region.

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[18] Equations in the CGS units can be obtained by substitutions $B \to B/\sqrt{4\pi}$ for magnetic field, $E \to cE/\sqrt{4\pi}$ for electric field, $j \to (\sqrt{4\pi}/c)j$ for electric current, and $\eta \to \eta$ for resistivity.
[19] We assume that $\eta(j_z, x, y)$ has finite derivatives in $y$ up to the second order and in $x$ and $j_z$ up to the first order. We consider $\eta$ to be a function of $j_z$ instead of the total current $j = (j_z^2 + j_x^2 + j_y^2)^{1/2}$. This is because the electrical conductivity in the $z$-direction can be reduced by plasma instabilities due to large values of $j_z$.
[20] Here we consider the natural case when $(\partial\eta/\partial j_z)_o \geq 0$ and $(\partial^2\eta/\partial y^2)_o \leq 0$ because plasma conductivity decreases as the current increases and we are interested in anomalous reconnection that is faster than the Sweet-Parker reconnection.