United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

**THE ONSET OF CHAOTIC SYMBOLIC SYNCRONIZATION
BETWEEN POPULATION INVERSIONS
IN AN ARRAY OF WEAKLY-COUPLED BOSE-EINSTEIN CONDENSATES**

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MIRAMARE – TRIESTE

July 2004

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Abstract

We investigate the onset of chaotic dynamics of the one-dimensional discrete nonlinear Schrödinger equation (DNLSE) with periodic boundary conditions in the presence of a single on-site defect. This model describes a ring of weakly-coupled Bose-Einstein condensates. We focus on the transition to global stochasticity in three different scenarios as the defect is changed. We make use of a suitable Poincaré section and continuation methods. Numerical continuation enables us to find different families of stationary solutions, where certain bifurcations lead to global stochasticity. The global stochasticity is characterized by chaotic symbolic synchronization between the population inversions of certain pairs of condensates. We have seen that the Poincaré cycles are useful to gain insight in the dynamics of this problem. Indeed, the return maps of the Poincaré cycles have been used successfully to follow the motion along the stochastic layers of different resonances in the chaotic self-trapping regime. Moreover, the time series of the Poincaré cycles suggests that in the global stochasticity regime the dynamics is, to some extent, Markovian, in spite of the fact that the condensates are phase locked with almost the same phase. This phase locking induces a peculiar local interference of the matter waves of the condensates.

1 Introduction

A large number of recent investigations have been devoted to study the combined effects of disorder and nonlinearity in one-dimensional lattices. A typical theoretical model within this category is the discrete nonlinear Schrödinger equation (DNLSE) [1]. This equation describes a large class of discrete nonlinear systems such as optical fibers [2, 3], polarons [4], small molecules such as benzene [4, 5], and, more recently, dilute Bose-Einstein condensates trapped in a multi-well periodic potential [6]. One of the main effects in these lattices are the localized excitations in perfectly periodic but strongly nonlinear systems [1, 7, 8]. We refer to discrete breathers, also known as intrinsic localized modes [1]. These are spatially localized, time-periodic and stable stationary solutions of the DNLSE [1] and have been observed in many physical systems [8]. Experimentalists are currently interested considering breathers in Bose-Einstein condensates [8].

The experimental observation of Bose-Einstein condensation (BEC) in a dilute gas of trapped atoms in an optical multiwell lattice, created by a far-detuned, standing-wave laser beam [9], has generated much interest in the dynamical properties of this new state of matter. The condensates can coherently tunnel between interwell barriers. The heights of the barriers can be adjusted since they are proportional to the intensity of the laser beam. These experimental techniques have allowed direct observation of several phenomena originally believed to belong to solid-state physics such as Zener tunneling in the presence of constant acceleration [9] and atomic Josephson current [10]. The evolution of this BEC is governed by the Gross-Pitaevskii equation and can be mapped, in the tight binding approximation, to a DNLSE [6]. Here, we consider the nonlinear dynamics of the DNLSE in a ring geometry in the presence of a single defect. In a BEC the defects can be created with additional lasers or magnetic fields. A double-well trapping potential was obtained by superimposing a sharp barrier induced by optical and magnetic traps [11]. In this way, two BEC were produced, one on each side of the barrier [11]. A theoretical model describing two interacting BEC considered this experiment [12]. Indeed, the possibility to carry out experiments with a small number of condensates in a ring geometry will become a reality, as soon as greater control of the microtraps technology is achieved [13].

The DNLSE is a vast subject with many different and relevant issues such as the dynamics of discrete breathers in one or two-dimensional infinite lattices [1], propagation of excitations in the presence of disorder [14], mobility and interaction of breathers [7]. Motivated by the dynamics of BEC, we concentrate on the special issue concerning the onset of chaos in the DNLSE with a small number of oscillators. The DNLSE is a nonlinear Hamiltonian system with M degrees of freedom, where M refers to the number of condensates. The DNLSE, as is well known, has two constants of motion [1]. Therefore, when $M = 2$, the DNLSE is integrable. However, when $M \geq 3$, the DNLSE can exhibit an amazing degree of complexity. The chaotic dynamics for $M = 3$ and $M = 4$ was studied, to our knowledge, for the first time almost two decades ago [15] considering, in particular, power spectra and the spectra of Lyapunov exponents of the

system. The nonlinear dynamics for $M = 3$ has also been considered more recently [16, 17]. For $M = 3$, symmetry arguments can show that the dynamics is that of an area-preserving two-dimensional map. As a result, it provides a rich behaviour, which is of interest for both theory and experiment of BEC. When $M = 3$, the passage of a trajectory from one stochastic region in phase space to another is blocked by KAM surfaces [18]. When $M > 3$, however, Arnold diffusion can take place, that is, in the phase space of the system, stochastic layers of different resonances intersect. Therefore, the motion will spread out over the whole system of intersecting layers [18]. Arnold conjectured that this instability is generic to many-dimensional nonlinear Hamiltonian systems. Indeed, in the context of DNLSE, Arnold diffusion was considered for $M = 4$ [19].

In this work, we study new types of chaotic solutions of the DNLSE with periodic boundary conditions and a single on-site defect. In a recent article [20] we have given a global picture of the dynamics in this system. Instead, now we focus on the transition to global stochasticity. In the aforementioned articles [16, 17, 19], the onset of chaos was not induced by on-site defects. The single one-site defect, in the present article induces an interesting complex dynamics. The number of condensates, $M = 7$, allows for the manifestation of phenomena such as Arnold diffusion [18]. The present study on the dynamical properties of this model is based upon Poincaré section and continuation methods in a parameter region of the on-site defect. The latter allowed us to find several families of stationary solutions. Moreover, we have found relevant information of the dynamics using suitable return maps of the Poincaré cycles, which, to our knowledge, is considered for the first time in multi-dimensional conservative systems. This approach allows us to unfold robust properties of this system which have, we believe, physical relevance. For instance, the population inversions of certain pairs of condensates occurs almost simultaneously. That is, these population inversions synchronize in an information sense [20, 21]. We show three different and systematic routes which lead to this behaviour. Moreover, the statistics of these Poincaré cycles display, surprisingly, an almost Markovian behaviour in spite of the fact that the condensates are phase locked and, therefore, the system shows coherent motion. In fact, a good interference between the matter waves of the condensates is the hallmark of phase locking.

This article has eight sections. The DNLSE and the initial conditions are discussed in Sec. II. In Sec. III, we introduce two types of qualitatively different instabilities induced by the single on-site defect. This allows us to define a suitable Poincaré section. Chaotic symbolic synchronization between spatially symmetric pairs of condensates is discussed in Sec. IV. The onset of chaotic self-trapping as a precursor of chaotic symbolic synchronization is considered in Sec. V. Two different branches of stationary solutions are studied with numerical continuation techniques, where certain bifurcations trigger the onset of the aforementioned symbolic synchronization. This study is carried out in Sec. VI. In Sec. VII, we discuss the phase locking effect between condensates and the resulting local interference. Finally, in Sec. VIII, we give the conclusions.

2 The Model and the Initial Conditions

We consider a one-dimensional ring of coupled BEC. When the height of the interwell barriers is much higher than the chemical potential of the system, the latter realizes an array of weakly coupled condensates, whose equation of motion satisfies a DNLSE [6] :

$$i \frac{\partial \Psi_m}{\partial t} + \Delta_m \Psi_m + K(\Psi_{m-1} + \Psi_{m+1}) + \rho |\Psi_m|^2 \Psi_m = 0, \quad (1)$$

where Ψ_m stands for the condensate complex amplitude in the m th well, ρ is the nonlinear coefficient arising from the interatomic interaction, K is proportional to the microscopic tunneling rate between adjacent sites, Δ_m stands for the on-site defect and is proportional to any external field superimposed on the lattice and, finally, t is the time. By introducing the dimensionless amplitude $\psi_m = \sqrt{\rho/2K} \Psi_m \exp(-i(\Delta + 2K)t)$, Eq.(1) transforms into the discrete nonlinear Schrödinger equation (DNLSE) given by :

$$i \frac{\partial \psi_m}{\partial \tau} + \delta_m \psi_m + (\psi_{m-1} + \psi_{m+1} - 2\psi_m) + 2 |\psi_m|^2 \psi_m = 0, \quad (2)$$

where $\delta_m = \frac{(\Delta_m - \Delta)}{K}$ stands for the defects, $\tau = Kt$ and Δ is any arbitrary number. As a result, the only parameters of the DNLSE in Eq.(2) correspond to the defects δ_m . This system is assumed to have periodic boundary conditions. The positive sign before the nonlinear term indicates that we are considering an attractive interatomic interaction between the condensates such as in the case for Lithium atoms [22]. However, our discussion can also apply to an even number of condensates with repulsive interaction, as shown below.

There are two integrals of motion in Eq.2. The first is the Hamiltonian from which Eq.2 is derived [1]. This is given by

$$H = \sum_{m=1}^M (|\psi_m - \psi_{m+1}|^2 - |\psi_m|^4 - \delta_m |\psi_m|^2). \quad (3)$$

The second constant is the norm, which is given by

$$N = \sum_{m=1}^M |\psi_m|^2. \quad (4)$$

Here M stands for the number of condensates. Moreover, Eq.2 is invariant with respect to the transformation $\delta_m \rightarrow -\delta_m$, $2 |\psi_m|^2 \rightarrow -2 |\psi_m|^2$ and $\psi_m \rightarrow \psi_m^* \exp(i\pi m - 4i\tau)$. As a result, we could also consider an even number of condensates with a repulsive interatomic interaction such as in the case of Rubidium atoms [22], since periodic boundary conditions hold.

We can rewrite the DNLSE by transforming into population-phase variables (N_m, θ_m) , where $\psi_m = \sqrt{N_m} \exp(-i\theta_m)$, to stress the physical meaning of the equations of motion. The equations

for N_m and θ_m are the following :

$$\begin{aligned}\frac{dN_m}{d\tau} &= 2\sqrt{N_m N_{m-1}} \sin(\theta_{m-1} - \theta_m) + 2\sqrt{N_m N_{m+1}} \sin(\theta_{m+1} - \theta_m), \\ \frac{d\theta_m}{d\tau} &= 2 - \delta_m - \sqrt{\frac{N_{m-1}}{N_m}} \cos(\theta_{m-1} - \theta_m) - \sqrt{\frac{N_{m+1}}{N_m}} \cos(\theta_{m+1} - \theta_m) - 2N_m.\end{aligned}\quad (5)$$

The initial conditions and the solutions have to fulfill the physical condition $N_m \geq 0$. When the phase difference between two condensates, say m and n , is such that $|\theta_m - \theta_n| < 2\pi$ for any τ , we say that these condensates are phase locked. That is, the absolute value of this phase difference remains bounded [23]. We will see next that in the chaotic regimes that we study $|\theta_m - \theta_n| \ll 1$ and, therefore, a constructive interference picture of matter waves sets up.

We will use as initial conditions a set of stationary solutions which arise when the defect $\delta_m = 0$ [20, 24]. To find these DNLSE stationary solutions, we deal with the nonlinear map approach [25, 26]. This map is obtained by making $\frac{dN_n}{d\tau} = 0$ and $\theta_n = \theta_m$ for any $n \neq m$ in Eq.5. Moreover, we can define the frequency of the resulting periodic orbit by setting $\frac{d\theta_m}{d\tau} = \lambda$, where λ is a constant. Therefore, the stationary solutions have the form $\psi_m(\tau) = \sqrt{N_m} \exp(-i\lambda\tau)$. As a result, the following cubic map (CM) is obtained :

$$\begin{aligned}X_{n+1} &= Y_n, \\ Y_{n+1} &= (\Gamma_n - 2|Y_n|^2)Y_n - X_n,\end{aligned}\quad (6)$$

where $\Gamma_n = 2 - \lambda - \delta_n$ and $X_n = \sqrt{N_n} > 0$ to account for the physical meaning of N_m . In the CM we will set $\Gamma = 2 - \lambda$ for which $\delta_n = 0$. The Jacobian J of this map is area preserving, i.e. $J = 1$. The fixed points $(\pm\sqrt{\Gamma/2-1}, \pm\sqrt{\Gamma/2-1})$ of the CM are elliptic for $2 < \Gamma < 4$. When $\Gamma = 2.5$, surrounding the elliptic fixed point, we find a period-seven orbit. Instead, when $\Gamma = 2.6$, surrounding the elliptic fixed point, we find a period-six orbit. Both periodic orbits are surrounded by island chains [24]. We will consider those DNLSE stationary solutions which are determined by the (elliptic) stable periodic orbits of the CM. These periodic orbits with periodicity 6 and 7 generate stable stationary solutions in a ring with 6 and 7 condensates respectively. The amplitudes of these two stationary DNLSE solutions $\sqrt{N_m}$ are shown in Fig. 1. Using the invariant transformation mentioned above, in the case with six condensates, we can prove that the stationary solution of Fig. 1b is stable when the interatomic interactions are repulsive. For the sake of completeness, let us mention that the homoclinic orbits of the CM determine the aforementioned breathers of the DNLSE [7] or a suitable variational approach [27]. We stress that as for the stationary solutions, the number M of condensates in the ring coincides with the periodicity of the stable periodic orbit of the CM and the initial phases of the condensates are the same [24].

3 Defect-Induced Oscillations and the Poincaré section

From now on we will consider the DNLSE with a single defect $\delta_3 < 0$ and $\delta_n = 0$ for $n \neq 0$. Moreover, we will focus on the case of a ring with seven condensates. The dynamics for the case with six condensates is qualitatively similar to the latter. We make use of the exact stationary DNLSE solutions of the previous section as initial conditions for this perturbed DNLSE. This contrast with a previous article [20], where we added small random perturbations of the order of 10^{-3} to the initial conditions to test for the stability and robustness of the solutions. As the parameter $\delta_3 < 0$ increases in absolute value, we find a transition from a quasiperiodic solution to a chaotic solution where the oscillations of N_m are localized within a small neighborhood of the stationary solutions. These regimes are observed in Fig.2a and Fig.2b for which $\delta_3 = -0.005$ and $\delta_3 = -0.00625$ respectively. These localized chaotic oscillations occur in a narrow interval $-0.0069 < \delta_3 < -0.0055$ and are not typical when $\delta_3 < 0$. In contrast, the presence of a defect $\delta_3 > 0$, in the quasiperiodic or chaotic regimes, typically induces localized oscillations of N_m [20]. These localized solutions are known as self-trapping solutions [16, 17, 19]. As the parameter $\delta_3 < 0$ decreases further, we find a typical scenario where the oscillations of N_m are delocalized as observed in Fig.2c. However, the initial behaviour of N_m is characterized by time sequences which remind us of quasiperiodic and localized chaotic motions.

In order to measure the extent of localization in the DNLSE, the quantity

$$\eta = -\frac{1}{\ln(M)} \sum_{i=1}^M \frac{\langle N_i \rangle}{N} \ln \frac{\langle N_i \rangle}{N}$$

has been used when $M = 3$ [16]. $\langle \rangle$ stands for statistical average and M is the number of condensates. The quantity η is unity when the populations N_i are the same on average, and is zero when only one of populations N_i is different from zero. In the chaotic self-trapping regime, when $\delta_3 = -0.00625$, we find that $\eta \approx 0.9056$. As a reference we can consider the stationary solution when $\delta_3 = 0$. In this case, we obtain $\eta \approx 0.88$. However, when $\delta_3 = -0.007$, the continuous-time chaotic delocalized solutions give $\eta \approx 0.9905$. As a result, the populations of the condensates N_i have the same statistical average. When $M = 3$, chaos usually leads to $\eta \approx 1$ for relatively small constants of motion N [16]. In these different oscillatory regimes, the change of $\delta_3 < 0$ induces an increment of the distance from the stationary solution when $\delta_3 < 0$ to the fixed initial condition given by the stationary solution when $\delta_3 = 0$. The distance between these two points is labeled as σ . The dependence of σ on $\delta_3 < 0$ is given by $\sigma \approx -2\delta_3 + C$, where C is a constant. This is shown in Fig. 2d. We have found the central resonances for a given $\delta_3 < 0$ using a root finding scheme with initial guesses in the vicinity of the stationary solution at $\delta_3 = 0$.

The picture that emerges is, therefore, that of an initial condition, given in Fig.1a, moving away from the central resonance for a given $\delta_3 < 0$. As in any Hamiltonian system, this leads to a closer vicinity to the stochastic layers where chaotic behaviour occurs. Stochastic layers

in phase space exists near resonances. Moreover, since the number of degrees of freedom of the system is larger than two, these stochastic layers are expected to form an interconnected network, which is known as an Arnold web [18]. The thickness of these layers, typically, expands with increasing perturbation [18]. This is observed when we decrease the parameter $\delta_3 < 0$ in the interval $-0.0069 < \delta_3 < -0.0055$. In fact, the time series of N_m of Fig.2b increase their oscillation amplitudes with decreasing $\delta_3 < 0$. At a given point, however, when the thickness of these layers is wide enough, resonance overlap and motion across certain stochastic layers occurs leading to the onset of a stronger chaotic motion. That is precisely what we see in Fig.2c. We will return to this issue below considering a suitable Poincaré surface of section.

Let us define the new variable $\mu \equiv N_L - N_R$, where $N_L = N_7 + N_1 + N_2$ and $N_R = N_4 + N_5 + N_6$ are the left wing and right wing populations respectively. In the quasiperiodic, self-trapping chaotic, and delocalized chaotic regimes, the trajectory intersects an 11-dimensional surface $\mu = 0$ transversally as observed in Fig. 3a, Fig. 3b and Fig. 3c,d. $\mu = 0$ defines an 11-dimensional surface due to the existence of the constants of motion H and N . The Poincaré surface of section is defined when the trajectory intersects this surface from $\mu < 0$ to $\mu > 0$ as shown in Fig. 3. That is, when $\mu = 0$ and $\frac{d\mu}{d\tau} > 0$. In other words, we define the Poincaré section exactly when the left wing population N_L becomes larger than the right wing population N_R . Let us recall that the Poincaré surface of section is a reduced phase space in which the volume-preserving property holds for Hamiltonian systems[18]. That is, the associated map of the Hamiltonian flow is also area-preserving. Since the DNLSE is a Hamiltonian flow with two constants of motion, there are four Floquet multipliers equal to one for each periodic orbit [18]. As a result, when $M = 7$, the associated volume-preserving map is ten-dimensional. That is why when $M = 3$, the related map is two-dimensional and the reason behind the presence of four vanishing Liapunov exponents in the DNLSE. As is well know, the definition of a Poincaré section lies at the heart of the study of a given Hamiltonian system. This will enlighten our understanding of key dynamical features.

4 Chaotic Symbolic Synchronization Between Spatially Symmetric Condensates

Let us consider first the self-trapping chaotic regime for which $\delta_3 = -0.00625$. This type of behaviour arises immediately after the quasiperiodic motion dissapears as we change the parameter $\delta_3 < 0$. At the Poincaré section, we find that $|N_1 - N_5| \sim 10^{-3}$, $|N_2 - N_4| \sim 10^{-3}$ and $|N_6 - N_7| \sim 10^{-3}$, as shown in Fig. 4a and Fig. 4c. That is, taking into account the magnitudes of N_m from Fig. 4d, we can agree that $N_1 \approx N_5$, $N_2 \approx N_4$ and $N_6 \approx N_7$ at the Poincaré section. A similar argument holds for the delocalized chaotic case when $\delta_3 = -0.007$, as shown in Fig. 5a and Fig. 5c. However, we may say that $N_1 - N_5$, $N_2 - N_4$ and $N_6 - N_7$ have the same sign during most of the time, as is clearly appreciated in Fig. 5b and Fig. 5d.

That is, these population differences, to a good extent, undergo inversion at the same time and have the same sign of $\mu \equiv N_R - N_L$. The magnitudes of the population inversions $N_1 - N_5$, $N_2 - N_4$ and $N_6 - N_7$ at the Poincaré section and at their maxima differ by almost two orders of magnitude. Moreover, this behaviour seems to be robust as confirmed by extensive numerical simulations. That is precisely what can make this synchronous inversion relevant, we believe, from an experimental standpoint. A qualitatively similar effect occurs when $\delta_3 = -0.0065$ as shown in Fig. 4b but lacks the robustness and magnitude of the previous parameter value.

The behaviour when $\delta_3 = -0.007$ is an example of synchronization of symbolic information (SSI) [28, 29] and holds, at least, for time units as large as $\tau \sim 10^5$. According to this notion, two arbitrary oscillators are perfectly synchronized in an information sense if they produce the same information, *i.e.*, symbols generated by one system map one-to-one to symbols emitted by the other system. Strictly speaking, this form of synchronization requires that the common information be emitted at precisely the same time. Fig. 5b and Fig. 5d suggest that the population inversions exhibit almost equivalent information at the same average rate. This, however, does not contradict the fact that the usual notion of synchronization cannot be applied to Hamiltonian systems, such as ours, since volume has to be preserved in phase space. Indeed, the trajectories of coupled systems have to collapse to the synchronization manifold [23], which is only possible in dissipative systems. In our system, instead, we compare the symbolic dynamic of these three chaotic population inversions. We will show below that the motion for which $\delta = -0.0065$ and $\delta = -0.007$ is chaotic as suggested by the largest positive Liapunov exponent.

5 Chaotic Self-Trapping as a Precursor of SSI

Let us focus on the dynamics at the Poincaré section. The time series when $\delta_3 = -0.0065$ and $\delta_3 = -0.007$ are shown in Fig. 4d and Fig. 6 respectively. Fig. 4d suggests that in the self-trapping regime there is some underlying structure in the dynamics, which is more clearly appreciated for N_3 . Instead, Fig. 6 suggests that there is no such a structure when $\delta_3 = -0.007$ and, moreover, this figure shows that the trajectory has been in the self-trapping region roughly during the time intervals $0 < \tau_i < 60$ and $3200 < \tau_i < 3600$. In fact, in a high precision calculation consisting of 10^5 time units, only these jumps between these two chaotic regions have taken place. The variable τ has been rescaled by dividing by 4π .

We show evidence suggesting that the dynamics when $\delta_3 = -0.0065$ mostly consists of a sequence of motions along the stochastic layers of different resonances in a manifold where energy H and the number of particles N are both constant. To support this, we address the dynamics shown in Fig. 7. We have considered the Poincaré cycles $\tau_{i+1} - \tau_i$ as return maps. The indices i label consecutive Poincaré sections and τ_i is the Poincaré recurrence time [18]. That is, we consider maps such as $\tau_{i+1} - \tau_i$ versus $\tau_{i+2} - \tau_{i+1}$. We have seen with amazing clarity that motion occurs near resonances. For instance, in Fig. 7a, the motion wanders stochastically

around a resonance torus of period five. In fact, if we sample this map each five Poincaré sections, we will only see a single stochastic layer. This occurs in the interval $2700 < i < 3900$. Fig. 7b suggests that the associated resonance is of period seven. Fig. 7c indicates that the stochastic layer is in the neighborhood of a resonance torus of period twelve. This seems to be a secondary generation resonance where the primary has period three. Fig. 7d suggests that the resonance related with the stochastic layer has a large periodicity, in fact, it looks like a single isolated stochastic layer. The intervals of the indices i of Fig. 7b, Fig. 7c and Fig. 7d, are $8580 < i < 8670$, $6600 < i < 7100$ and $5100 < i < 5380$ respectively. In Fig. 8a we have shown the time series of $\tau_{i+1} - \tau_i$ versus index i for $12000 < i < 26000$. During this long interval the motion sticks to the neighborhood of a period-five resonance torus as indicated by Fig. 8b. Indeed, by sampling every five Poincaré sections, we will only see a single stochastic layer as in Fig. 7a. This period-five resonance is, however, different from that of Fig. 7a. Therefore, we have seen that the stochastic layers of different resonances are connected and the trajectory spreads out over this system of intersecting layers. The resonance along whose stochastic layer the motion goes on at a given time is called guiding resonance [18]. In this case the motion along the other resonance layers is negligible. These resonances are known as driving resonances [18]. This behaviour takes place for time units of τ as large as 10^5 as suggested by Fig. 7. The variable τ has been rescaled by dividing by 4π .

We believe that this behaviour is the result of mainly two effects. The first is the dynamical trap effect. A dynamical trap is a domain in phase space where a trajectory can spend arbitrarily long finite time, performing almost regular dynamics, despite the fact that the full trajectory is chaotic [30]. Typically, Hamiltonian systems are characterized by different trapping domains that belong to boundaries of certain resonances [30]. The existence of traps is a robust effect observed in different experimental situations such as in two-dimensional time-periodic flows and the standard map [30]. In contrast to dynamical traps which can occur already in two-dimensional maps, the second effect arises for a Hamiltonian system with three or more degrees of freedom. In such Hamiltonian systems, a dense set of resonant tori can be preserved. Generically, the stable and unstable manifolds of each preserved torus intersect transversally yielding a homoclinic tangle. Moreover, the homoclinic tangle of each torus becomes intertwined with the homoclinic tangles of nearby tori leading to the possibility that orbits starting near any one of these tori may diffuse along this dense set of homoclinic tangles in a chaotic fashion [31]. This phenomenon is known as Arnold diffusion, although what Arnold described in his 1964 article was a system with two and a half degrees of freedom [18].

As we decrease the negative parameter δ_3 further, the oscillation amplitudes of N_m expand slightly leading, for a given parameter δ_3 , to the onset of a stronger stochastic motion in a large region of the energy shell as shown in Fig. 2c, Fig. 5b,d and Fig. 6. This larger region appears clearly on the projections in the planes (N_n, N_m) . Moreover, we will see that when SSI takes place, in spite of the small change of δ_3 , the largest Liapunov exponent increases several times.

This qualitative change can be understood as the overlap of certain resonances which belong to different regions of phase space. When $\delta_3 = -0.0065$, the motion along the set of resonance layers is isolated from the region displaying SSI provided that the initial conditions are in the self-trapping chaotic region. This suggests that the self-trapping chaotic region is kept isolated from the SSI region due to the existence of barriers to transport. These barriers are known to exist in systems with three or more degrees of freedom [31]. Instead, for $\delta_3 = -0.007$, these barriers no longer operate due to the overlapping of certain resonances and, therefore, the orbit can explore both the region of self-trapping and that of SSI as observed in Fig. 6. Moreover, the short lifetime of the orbit in the self-trapping region, when $\delta_3 = -0.007$, suggests that the measure of the stochastic trajectories in the region of SSI is much larger than that in the self-trapping region. The solutions with initial conditions in the SSI region for which $\delta = -0.007$ but with the parameter $\delta_3 = -0.0065$ remain mostly in this region displaying SSI between the population inversions $N_1 - N_5$, $N_2 - N_4$ and $N_7 - N_6$.

To have a quantitative measure of the stochasticity, we consider the time series of $\tau_{i+1} - \tau_i$ as well as that of the exponential rate of divergence of the linearized system between two consecutive Poincaré sections W_i . The time series $\tau_{i+1} - \tau_i$ when $\delta_3 = -0.0065$ is narrow, sharply bounded and has time intervals with some regularity, as shown in Fig. 9a. These intervals are typically generated by the motion around a given resonance torus. Instead, when $\delta_3 = -0.007$ there are basically no signs of such a regularity. In fact, the time series looks quite noisy as shown in Fig. 9b. The probability distribution function (PDF) of these Poincaré cycles $\tau_{i+1} - \tau_i$ is sharply peaked and bounded when $\delta_3 = -0.0065$, while the core of this PDF when $\delta_3 = -0.007$ has to a good extent an exponential dependence. These PDF are shown in Fig. 9c. We stress that this dependence of the PDF core indicates that these Poincaré cycles have, to a good degree, a Markovian character. In other words, in case we have a perfect Markovian process [32], the probability that the trajectory continues until $\tau^{**} > \tau^*$ is independent of the past duration $\tau^* - \tau_i > 0$. Here, τ_i is the last time that the trajectory intersected the Poincaré surface of section. The prevalent Markovian character of this time series implies basically small memory. That is precisely what we show below by calculating the autocorrelation function (ACF) $C(\Delta i)$ for these time series, where $C(\Delta i) = \frac{\sum_{i=1}^{R-S} (H_i - \langle H \rangle)(H_{i+\Delta i} - \langle H \rangle)}{\sum_{i=1}^{R-S} (H_i - \langle H \rangle)^2}$. In this equation, H_i and $\langle H \rangle$ are the time series and its sample average respectively. Δi is the time lag, R is the number of data, $\Delta i < S$ and $R \gg S$ [18]. The ACF $C(\Delta i)$ of $\tau_{i+1} - \tau_i$ for the time series of Fig. 9a has a slow non exponential decay, when $\delta_3 = -0.0065$, as a result of the presence of intervals with some regularity. Instead, the ACF $C(\Delta i)$ has a fast decay which implies a small memory for the time series of Fig. 9b, as indicated before on the basis of the aforementioned exponential dependence. The plots of these ACF $C(\Delta i)$ are shown in Fig. 9d. However, there are, besides some fitting arguments, at least two reasons that make the time series of Fig. 9b somewhat different from a Markovian process. The first are the temporal jumps to the self-trapping region, where the motion is highly correlated. The second is the existence of very long

time intervals $\tau_{i+1} - \tau_i$ that appear as rare events. The latter arises in the tails of our PDF and, typically, have power law dependence [33]. These are not shown in the core of the PDF of Fig. 9c since its probability is negligible, at least during our time series consisting of 10^5 time units. It should be underlined that, in spite of this quasi Markovian behaviour, certain pairs of condensates can synchronize their symbolic dynamics. This is related to phase locking, since lack of the latter inhibits the presence of SSI [20]. In the theory of quasi-random dynamical systems, the trajectory is qualitatively similar to a Markovian random process [18]. An almost Markovian behaviour was also found in the statistics of the length of symbolic signals [20], which represent, in the present context, the duration of positive or negative inversion between pairs of condensates. The PDF of these lengths is also, to a good extent, described by an exponential PDF.

To verify the stochasticity of the motion, let us consider the largest Liapunov exponent. The latter is the time average of the exponential rate of divergence of two initially close trajectories W_i [18]. W_i is calculated between two consecutive Poincaré sections. In Fig. 10a, where $\delta_3 = -0.0065$, we see that W_i is localized in a small interval while in Fig. 10b, for which $\delta_3 = -0.007$, W_i is spread over a larger interval. The PDF of W_i are given in Fig. 10c. The trend towards convergence of the largest Liapunov exponents Λ_1 , when $\delta_3 = -0.0065$ and $\delta_3 = -0.007$, to their associated limits is appreciated in Fig. 10d. The Liapunov exponent Λ_1 when $\delta_3 = -0.0065$ is almost six times smaller than that when $\delta_3 = -0.007$. To understand this, let us point out that in the chaotic self-trapping regime there is some regularity, as appreciated in Fig. 7 and Fig. 8. Here, the trajectories wander several hundreds units of time in the neighborhood of a given resonant torus. In contrast, when $\delta_3 = -0.007$, the system behaves almost like a Markovian process. We can have an estimate for W_i as the product of some averaged, over each time interval $\tau_{i+1} - \tau_i$, finite-time Liapunov exponent [23] by the time interval $\tau_{i+1} - \tau_i$. As a result, since the PDF of $\tau_{i+1} - \tau_i$ has an exponential core, the PDF core of W_i has to have roughly a similar shape, as shown in Fig. 10c. When SSI takes place, there is a single positive Liapunov exponent whose magnitude is much larger than that of the other positive exponents [20]. The two conservation laws of our autonomous Hamiltonian system imply that we have at least four Liapunov exponents which are equal to zero. A similar relationship between the PDF of W_i and that of $\tau_{i+1} - \tau_i$ takes place when $\delta_3 = -0.0065$.

6 Numerical Continuation of the Stationary Solutions

We have carried out a numerical continuation study of the stationary solutions for the case of a ring with seven condensates. It was found that the continued stationary solution remains stable as the parameter δ_3 is changed from the stationary solution at $\delta_3 = 0$ to the first bifurcation at $\delta_3^{(1)} = -0.02399$. At this bifurcation, two different branches intersect. One of these branches is shown in Fig. 11a where for $\delta_3^{(12)} = -0.028$, the stationary solution is unstable, whereas for $\delta_3^{(13)}$,

the solution is linearly stable. The points at $\delta_3^{(12)}$ and $\delta_3^{(12)}$ are close to the bifurcation point at $\delta_3^{(1)}$. The points at $\delta_3^{(1)}$, $\delta_3^{(12)}$ and $\delta_3^{(13)}$ are labeled 1, 12, and 13, in Fig. 11. The other branch is shown in Fig. 11b. On this more complex branch we consider the bifurcation located between the points at $\delta_3^{(6)} = -0.02999$ and $\delta_3^{(7)} = -0.03299$. At the two unstable points at $\delta_3^{(12)}$ and $\delta_3^{(7)}$, two eigenvalues have left the unit circle through the point (1,0), *i.e.* in a pitchfork bifurcation. The period of the stationary solutions, $2\pi/\lambda$, along the two bifurcation branches remains constant. In the neighborhood of the stable stationary solutions, just before the bifurcations, the dynamics is quasiperiodic. Instead, just after the bifurcation point, initial conditions in the vicinity of the unstable stationary solution trigger the onset of the SSI dynamics.

The dependence of the amplitudes $\sqrt{N_m}$ on the the index m for the unstable stationary solution at $\delta_3^{(7)} = -0.03299$ is given in Fig.12 a. In this figure we can see that the amplitude dependence is not symmetric with respect to the on-site defect at the index $m = 3$, as in the case of Fig. 1a. In spite of that, we can see that, as shown by Fig. 12b and Fig. 12d, SSI takes place to a good degree in the continuous flow. Moreover, at the Poincaré section, we find that $|N_1 - N_5| \sim 10^{-3}$, $|N_2 - N_4| \sim 10^{-3}$ and $|N_6 - N_7| \sim 10^{-3}$. This is shown in Fig. 12c. That is, these population inversions at $\mu = 0$ are about a hundred times smaller than their maximum magnitude. A qualitatively similar behaviour was observed for $\delta_3 = -0.007$. Quite a different story is, however, the case for which $\delta_3^{(12)} = -0.028$. In spite of the fact that the dependence of the amplitudes $\sqrt{N_m}$ on the the index m is symmetric, as shown in Fig. 13a, the behaviour of SSI has been somehow degraded as suggested by Fig. 13b, Fig. 13c and Fig. 13d. In fact, the aforementioned population inversions at $\mu = 0$ are only about ten times smaller than their maximum magnitude, as suggested by Fig.13c, and do not change sign in a synchronous way with the accuracy of the previous cases when $\delta_3^{(7)} = -0.03299$ and $\delta_3 = -0.007$.

The time series of the exponential rates of divergence of two initially close trajectories between two consecutive Poincaré sections W_i for the unstable stationary solutions at $\delta_3^{(7)}$ and $\delta_3^{(12)}$ are shown in Fig. 14a and Fig. 14c respectively. These figures suggest that the positive Liapunov exponent Λ_1 for $\delta_3^{(12)}$ is larger than that for $\delta_3^{(7)}$. Moreover, the cores of the PDF for the Poincaré cycles $\tau_{i+1} - \tau_i$ when $\delta_3^{(7)}$ and $\delta_3^{(12)}$ are shown in Fig. 14b and Fig. 14d respectively. Both have roughly an exponential dependence which accounts for the rough Markovian behaviour in these time series of $\tau_{i+1} - \tau_i$. We underline that we have also seen rare events which do not fit in the exponential cores of these PDF. These events appear to have negligible probability.

7 Phase Locking and Local Interference

In the tight binding approximation, which is valid when the power of the periodic external laser field is high enough, besides the number of particles in each site N_m , the relative phases $\theta_m - \theta_{m+1}$ of neighboring condensates are relevant quantities. In fact, these determine the robustness of a constructive interference pattern [22, 34]. We show below that phase locking takes place in our

model, *i.e.* $|\theta_m - \theta_{m+1}| < 2\pi$ for all m [23]. A single parameter, such as the Kuramoto order parameter [20] is useful to characterize the extent of phase locking. It should be underlined that lack of a good degree of phase locking inhibits the presence of SSI, as discussed elsewhere [20]. A good interference pattern arises, however, whenever $|\theta_m - \theta_{m+1}| \ll 1$. That is precisely what we will show next.

In Fig. 15a, we can appreciate that in the quasiperiodic regime, where $\delta_3 = -0.005$, $|\theta_m - \theta_{m+1}| \sim 10^{-2}$ for $m = 1, 2$. This is similar in the other pairs of neighboring condensates. Notice that these phase differences are not locked to the same value, as reported in a previous study [6]. It is remarkable, however, that when chaos arises, in the form of self-trapping or SSI, $|\theta_m - \theta_n| \ll 1$ in spite of fact that nearby trajectories diverge exponentially fast. Indeed, as suggested by Fig. 15b for which $\delta_3 = -0.0065$, $|\theta_m - \theta_n| < 10^{-2}$. Instead, in Fig. 15c, where $\delta_3 = -0.007$, $|\theta_m - \theta_n| < 10^{-1}$. As a result, the value of the Hamiltonian H in Eq. 3 can be approximated at any time by

$$H_1 = \sum_{m=1}^M (\sqrt{N_m} - \sqrt{N_{m+1}})^2 - N_m^2 - \delta_m N_m. \quad (7)$$

Defining the parameter $\alpha_1 \equiv \frac{H_1 - H}{H}$, we find that $0 < \alpha_1 < 10^{-3}$ when $\delta_3 = -0.0065$ and $0 < \alpha_1 < 10^{-2}$ when $\delta_3 = -0.007$.

We can make use of the definition of the Poincaré section, where $\mu = 0$, to find that $N = 2N_1 + 2N_2 + 2N_7 + N_3$ is conserved. Moreover, Fig. 4a,c and Fig. 5a,c suggest that we can replace at the Poincaré section $N_4 \rightarrow N_2$, $N_5 \rightarrow N_1$ and $N_6 \rightarrow N_7$ in the expression for H_1 . As a result, we get the following simplified expression for the value of H at the Poincaré section

$$H_2 = 2(N - N_7 - 2\sqrt{N_1 N_2} - 2\sqrt{N_2 N_3} - 2\sqrt{N_1 N_7}) - 2N_1^2 - 2N_2^2 - 2N_7^2 - N_3^2 - \delta_3 N_3. \quad (8)$$

N_7 can be eliminated using, at the Poincaré section, the identity $N = 2N_1 + 2N_2 + 2N_7 + N_3$. Defining now the parameter $\alpha_2 \equiv \frac{H_2 - H}{H}$, we find that $|\alpha_2| < 3 \times 10^{-4}$ when $\delta_3 = -0.0065$ and $|\alpha_2| < 6 \times 10^{-3}$ when $\delta_3 = -0.007$. In Eq. (8) H_2 is a two-dimensional surface embedded in a three-dimensional space (N_1, N_2, N_3) provided that H_2 is constant. Since H_2 is preserved to a good extent, we can qualitatively explain why the spectrum of the Liapunov exponents, when SSI occurs, has approximately a single positive Liapunov exponent [20]. The relative small variations of H_2 suggest that the largest expansion and smallest contraction rates of the related linearized map goes along this two-dimensional surface. Other expansions and contractions are much smaller. In fact, the remaining six positive Liapunov exponents are negligible when compared to the largest one [20]. We underline, however, that, by no means, this is the dynamics of a two-dimensional map. We have shown evidence of motion along different resonance layers, which we believe are partially due to Arnold diffusion.

One of the manifestations of the wave nature of Bose-Einstein condensates is the observation of interference when the condensed and initially separated clouds of atoms are allowed to overlap [11]. This is carried out in the experiments by turning off, after different evolution times,

both the magnetic and optical traps. Therefore, upon expansion the condensates overlap and interfere, and if the effects of particle interactions in the overlap region can be neglected, the particle density at any point on the central axis of the ring of atoms, which is perpendicular to the plane of the ring, is proportional to

$$\Sigma(\tau) = \left| \sum_{m=1}^M \psi_m \right|^2. \quad (9)$$

If $\Sigma(\tau)$ is measured just after the traps are turned off at τ , we can have an idea of the dynamics by considering several evolution times τ of the system. Since the phase differences satisfy $|\theta_m - \theta_{m+1}| \ll 1$, we can approximate $\Sigma(\tau)$ as follows,

$$\Sigma_1(\tau) = (\sum_{m=1}^M \sqrt{N_m})^2. \quad (10)$$

Indeed, $\Sigma(\tau)$ approximates $\Sigma_1(\tau)$ with a good accuracy. Moreover, we further assume that the measurement times τ , where the traps are turned off, are distributed uniformly over a large enough span of time for many replicas of the same experiment having only different τ . In this case the probability distribution function (PDF) of Σ , which we label as $P(\Sigma)$, will have a relatively small width when SSI takes place. This is the case when $\delta_3 = -0.007$ and $\delta_3 = \delta_3^{(7)}$. However, when SSI occurs to a smaller degree, such as when $\delta_3 = \delta_3^{(12)}$, the PDF $P(\Sigma)$ will have a broader width with larger tails. Here, $|\theta_m - \theta_{m+1}| \sim 1$. Taking into account the physical parameters K , ρ and Δ_m in Eq. (1) according to the article of Cataliotti *et al.* [10], we find the estimate $\Delta_3 \sim 0.00045 E_R$ to generate SSI, since $K \sim 0.07 E_R$ [10]. E_R stands for the recoil energy. This value of Δ_3 is realistic in experiments with microtraps, where also the regime of weakly coupled condensates can be achieved with suitable interwell barriers [13].

8 Conclusions and Discussion

We have studied a new family of chaotic solutions in a system consisting of a ring of weakly coupled BEC described by the DNLS. The initial conditions are determined by the resonances of a suitable Hamiltonian map. The number of condensates equals the periodicity of these resonances. The onset of the chaotic solutions has been systematically considered by changing the on-site defect and, also, by studying a suitable Poincaré map of the system related to the symmetry of the solutions. This allowed us to gain further insight and new information on the dynamics of this system. In particular, the consideration of return maps of successive Poincaré cycles has been useful to describe the dynamics of our system which, to our knowledge, represents the first example of the usefulness of these return maps to study the evolution of many-dimensional Hamiltonian systems.

For a small negative on-site defect, the system undergoes first a transition from the quasiperiodic regime to a chaotic self-trapping regime as the defect $\delta_3 < 0$ is decreased. Next, a transition from the chaotic self-trapping regime to a chaotic regime displaying synchronization of symbolic

information (SSI), among the populations inversions of different pairs of condensates, occurs as $\delta_3 < 0$ is decreased further. We have shown that the change of the parameter $\delta_3 < 0$, to reach the chaotic self-trapping regime, induces an effective displacement of the initial conditions from the central resonance corresponding to a given $\delta_3 < 0$. Once the chaotic self-trapping regime sets in, it is characterized by relatively small fluctuations of the populations of the condensates. Moreover, the dynamics consists mostly of a sequence of motions along the stochastic layers of different resonances as suggested by the return maps of successive Poincaré cycles. We attribute this behaviour to mainly two effects. The first is the dynamical trap effect, where the trajectory can spend arbitrarily long finite time, performing almost regular dynamics. This effect can occur already in two-dimensional area-preserving maps. The second effect is Arnold diffusion, where the connection between the stochastic resonance layers arises as a result of the number of degrees of freedom of our model. In contrast, the SSI regime is characterized by large fluctuations of the populations of the condensates which display small memory effects as suggested by the time series of the Poincaré cycles, which remind of an almost Markovian process. However, the coherence of the system, related to the good degree of phase locking, is a necessary condition for the onset of SSI. In this regime, the system has basically a single positive Liapunov exponent. This is related to an approximate recurrent mapping, at the Poincaré section, on a two-dimensional manifold. We have also carried out a numerical continuation study of the stationary solutions as the on-site defect changes. Two families of stationary solutions having the same period arise. One of these families is symmetric with respect to the position of the on-site defect. We find that a pitchfork bifurcation on the branch of the asymmetric solutions triggers the onset of the SSI dynamics. Moreover, the pitchfork bifurcation on the symmetric branch brings about, to a smaller degree, a SSI dynamics. Finally, we have seen that the presence of phase locking in the SSI regime induces a sharp distribution of the particle density along an spatial axis of symmetry of the ring of BEC.

The present study suggests that the interesting and complex dynamics of the SSI regime can be a good candidate for experimental verification since, it appears to be a robust effect. In contrast, the chaotic self-trapping regime seems to be less robust since it appears in a relatively small parameter range. The latter regime is more typical whenever $\delta_3 > 0$ [20]. Moreover, the usefulness of the return maps and the time series of the Poincaré cycles indicates that these can give further insight in the dynamics of equations like the DNLSE and other multidimensional conservative systems.

Acknowledgements. We would like to thank Felix Izrailev for his interest in this article and his useful comments. We would also like to thank Ennio Arimondo, Aranya Bhattacharjee and M. Saba for interesting discussions. This work was supported by CONACYT-México and the Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy. This work was done within the framework of the Associateship Scheme of the Abdus Salam ICTP.

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Figure 1: Plot of stationary solution amplitude $\sqrt{N_m}$ versus condensate index m . (a) resonance of period seven and $\Gamma = 2.5$ and (b) resonance of period six and $\Gamma = 2.6$.

Figure 2: Plot of N_m versus time τ for (a) $\delta_3 = -0.005$, (b) $\delta_3 = -0.0065$ and (c) $\delta_3 = -0.007$. The labels (1),(2),(3) and (7) are the condensate indices. (d) Distance σ versus $-\delta_3$. The variable τ has been rescaled by dividing by 4π .

Figure 3: Plot of μ versus time τ for (a) $\delta_3 = -0.005$, (b) $\delta_3 = -0.0065$ and (c,d) $\delta_3 = -0.007$. The variable τ has been rescaled by dividing by 4π .

Figure 4: Plot of $N_2 - N_4$ versus $N_1 - N_5$ when $\delta = -0.0065$ for (a) the Poincaré section and (b) continuous time. (c) Plot of $N_7 - N_6$ versus $N_1 - N_5$ when $\delta = -0.0065$ for the Poincaré section. (d) Plot of N_m versus time τ_i at the Poincaré section when $\delta_3 = -0.0065$. The labels (1),(2),(3) and (7) are the condensate indices.

Figure 5: Plot of $N_2 - N_4$ versus $N_1 - N_5$ when $\delta = -0.007$ for (a) the Poincaré section and (b) continuous time. Plot of $N_7 - N_6$ versus $N_1 - N_5$ when $\delta = -0.007$ for (c) the Poincaré section and (d) continuous time.

Figure 6: Plot of N_m versus time τ_i when $\delta_3 = -0.007$ at the Poincaré section. (a) N_1 , (b) N_2 , (c) N_3

Figure 7: Return map of the Poincaré cycles, that is $\tau_{i+2} - \tau_{i+1}$ versus $\tau_{i+1} - \tau_i$ when $\delta_3 = -0.0065$ at the Poincaré section for (a) $2700 < i < 3900$, (b) $8580 < i < 8670$, (c) $6600 < i < 7100$ and (d) $5100 < i < 5380$.

Figure 8: (a) Plot of the time series of $\tau_{i+1} - \tau_i$ for $12000 < i < 26000$ when $\delta_3 = -0.0065$. (b) $\tau_{i+2} - \tau_{i+1}$ versus $\tau_{i+1} - \tau_i$ for the time series of Fig8. a.

Figure 9: Plot of $\tau_{i+1} - \tau_i$ versus the Poincaré section index i when (a) $\delta_3 = -0.0065$ and (b) $\delta_3 = -0.007$. (c) Plot of $\log_{10} P(\tau_{i+1} - \tau_i)$ for $\delta_3 = -0.0065$ (continuous line) and $\delta_3 = -0.007$ (dashed line). (d) Plot of the ACF $C(\Delta i)$ for $\delta_3 = -0.0065$ (continuous line) and $\delta_3 = -0.007$ (dots).

Figure 10: Plot of W_i versus time τ_i for (a) $\delta_3 = -0.0065$ and (b) $\delta_3 = -0.007$. (c) Plot of $\log_{10} P(W_i)$ for $\delta_3 = -0.0065$ (continuous line) and $\delta_3 = -0.007$ (dashed line). (d) Evolution of the finite-time largest Liapunov exponent Λ_1 towards its limit when time goes to infinity for (1) $\delta_3 = -0.0065$ and (2) $\delta_3 = -0.007$.

Figure 11: Plot of the norm of the stationary solution $NORM$, with constant period, versus δ_3 for (a) the symmetric branch and (b) the asymmetric branch. Both branches start from point with label (1).

Figure 12: (a) Plot of stationary solution amplitude $\sqrt{N_m}$ versus condensate index m when $\delta_3 = \delta_3^{(7)}$. (b) Continuous time series for $N_1 - N_5$ versus time τ (continuous line), $N_2 - N_4$ versus time τ (dotted line) and $N_7 - N_6$ versus time τ (dashed line). (c) Plot of $N_2 - N_4$ versus $N_1 - N_5$ when $\delta = \delta_3^{(7)}$ for the Poincaré section. (d) Same as (c) but for continuous time.

Figure 13: Same as Fig. 12 but when $\delta_3 = \delta_3^{(12)}$.

Figure 14: Plot of W_i versus time τ_i for (a) $\delta_3 = \delta_3^{(7)}$ and (c) $\delta_3 = \delta_3^{(12)}$. Plot of $\log_{10} P(\tau_{i+1} - \tau_i)$ for (b) $\delta_3 = \delta_3^{(7)}$ and (d) $\delta_3 = \delta_3^{(12)}$.

Figure 15: Plot of $\theta_3 - \theta_2$ versus $\theta_2 - \theta_1$ for the Poincaré section when (a) $\delta = -0.007$, (b) $\delta = \delta_3^{(7)}$, and (c) $\delta = \delta_3^{(12)}$. (d) Plot of the PDF $P(\Sigma)$ for $\delta = -0.007$ (continuous line), $\delta = \delta_3^{(7)}$ (dotted line) and $\delta = \delta_3^{(12)}$ (dashed line).