United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

TAME KERNELS FOR BIQUADRATIC NUMBER FIELDS

Qin Yue¹

*Mathematics Department, Nanjing University of Aeronautics and Astronautics,
Nanjing Jiangsu 210016, People's Republic of China
and*

The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

Let $F = \mathbb{Q}(\sqrt{-d_1})$ and $E = \mathbb{Q}(\sqrt{-d_1}, \sqrt{d_2})$, d_1 and d_2 squarefree integers, be an imaginary field and a biquadratic field, respectively. Let S be the set consisting of all infinite primes, all dyadic primes and all finite primes which ramify in E . Suppose the 4-rank of the class group of F is zero and the S -ideal class group of F has odd order, we give the forms of all elements of order ≤ 2 in K_2O_E and use the Hurrelbrink and Kolster's method [5] to obtain the forms of all elements of order 4 in K_2O_E .

MIRAMARE – TRIESTE

August 2004

¹yueqin2@263.net

1 Introduction

Let $E = F(\sqrt{d})$ be a relative quadratic extension over a number field F , and let O_E and O_F be the rings of integers of E and F , respectively. Let S be the finite set of primes of F consisting of all dyadic primes, all finite primes that ramify in E and all infinite primes and let S_f be the set of finite primes in S . We will use the following notations for any abelian group A : ${}_2A$ denotes the subgroup of elements of order ≤ 2 of A ; $A(2)$ denotes the 2-Sylow subgroup of A ; $r_{2^n}(A)$ denotes the 2^n -rank of A .

For an arbitrary number field L with the ring of integers O_L , there is a surjective homomorphism

$$K_2O_L \rightarrow \bigoplus_{v \text{ real}} \mu_2 \rightarrow 0$$

induced by the Hilbert symbols at the real infinite primes. Hence its kernel denoted by $K_2O'_L$ is of index $2^{r_1(L)}$ in K_2O_L .

First we recall some results in [5]. Hurrelbrink and Kolster determined the structure of the 2-Sylow subgroups $K_2O'_E(2)$ and $K_2O_E(2)$, provided that $K_2O'_F(2) = 0$. If $K_2O'_F(2) = 0$, then F has only one dyadic prime and the S_2 -class group $C^{S_2}(F)$ has odd order, where S_2 is the set of all infinite primes and the dyadic prime. Let $Tr(E/F)$ denote the kernel of the transfer map (see [7])

$$Tr_{E/F} : K_2E \rightarrow K_2F.$$

In [5], there is a homomorphism ${}_2\phi_{E/F} : {}_2Tr(E/F) \rightarrow {}_2Br(F)/(-d, \Delta_F)$, where $(-d, \Delta_F)$ denotes the subgroup of the Brauer group of F generated by all classes $(-d, \delta)$ with δ in the Tate kernel Δ_F , satisfying $\{-1, \delta\} = 1$ in K_2F ; see [11].

Suppose ${}_2\phi_{F(i)/F}$ is trivial and the S -class group $C^S(F)$ has odd order, then there exists an exact sequence (see [5, Corollary 3.9])

$$0 \rightarrow ({}_2K_2O_E)^2 \cap Tr(E/F) \rightarrow {}_2K_2O_E \cap Tr(E/F) \xrightarrow{\psi_{E/F}} Br^{S_f}(E_0/F)/(-d, \Delta_F),$$

where $Br^{S_f}(E_0/F)$ consists of all classes of quaternion algebras of the form $(-d, a)$ with $a \in F^*$ which are locally trivial at primes in S_f . Hence we can use the above exact sequence to compute the 4-rank of $K_2O_E \cap Tr(E/F)$. In fact, if F is totally real or $K_2O'_F(2) = 0$, then ${}_2\phi_{F(i)/F}$ is trivial.

In the paper, we will always assume that $F = \mathbb{Q}(\sqrt{-d_1})$, $d_1 > 2$ a squarefree integer, is an imaginary quadratic field and $E = F(\sqrt{d_2})$, d_2 a squarefree integer, is a biquadratic field. Let $C(F)$ and $C^S(F)$ denote the class group of F and the S -ideal class group of F , respectively. We shall determine the structure of the 2-Sylow subgroup $K_2O_E(2)$ clearly, provided that $r_4(C(F)) = 0$ and $C^S(F)$ has odd order.

In section 2, we give the forms of all elements of order at most 2 in $K_2O_E \cap Tr(E/F)$ via solutions of systems of $\mathbb{F}_2$ -linear equations whose coefficient matrices consist of the Legendre

symbols (Theorem 2.1); further, we obtain the forms of all elements of order at most 2 in K_2O_E and calculate $r_2(K_2O_E)$ via the determination of the $\mathbb{F}_2$ -ranks of certain matrices of Legendre symbols (Theorem 2.2).

In section 3, we mainly use the results of section 2 and [5] to investigate $r_4(K_2O_E)$. We supply Proposition 3.1 and Proposition 3.2 such that we can use the preceding exact sequence even if F is an imaginary quadratic field with $r_4(C(F)) = 0$, which removes the restriction on a base field F in [5]. Hence we use it to compute $r_4(K_2O_E)$ (Theorem 3.2); we also give an algorithm to get elements of order 2 and 4 of K_2O_E via the determination of the $\mathbb{F}_2$ -ranks of certain matrices of local Hilbert symbols analogous to the Rédei's matrix (Corollary 3.1). We also illustrate the method through various examples.

In this paper, we express the forms of all elements of order at most 4 in K_2O_E by various $\mathbb{F}_2$ -matrices consisting of Legendre symbols and local Hilbert symbols analogous to Rédei matrix, but it has to be pointed out that this paper uses the Hurrelbrink and Kolster's method of relative quadratic extension [5] to investigate specially biquadratic number fields E and generalizes the results of Browkin, Schinzel [1] and Qin [9,10].

In this paper, we always assume that $F = \mathbb{Q}(\sqrt{-d_1})$ and $E = \mathbb{Q}(\sqrt{-d_1}, \sqrt{d_2})$, $d_1 > 2$ and d_2 squarefree integers, are an imaginary quadratic field and a biquadratic field, respectively.

2 2-rank

Let S be the finite set of primes in F consisting of all infinite primes, all dyadic primes, all finite primes ramifying in E . In this section, we will give elements of order ≤ 2 of K_2O_E , provided the 4-rank of the class group $C(F)$ is 0 and the S -ideal class group $C^S(F)$ has odd order.

Suppose $r_4(C(F)) = 0$, we observe the 2-Sylow subgroup of K_2O_F and the Tate kernel Δ_F .

Proposition 2.1 *Let $F = \mathbb{Q}(\sqrt{-d_1})$, $d_1 > 2$ a squarefree integer, be an imaginary quadratic number field and $r_4(C(F)) = 0$.*

(1) *If $d_1 \not\equiv \pm 1 \pmod{8}$ or $d_1 \equiv -1 \pmod{8}$ with $2 \notin NF$, then $r_4(K_2O_F) = 0$ and $\Delta_F = \{2, m\}F^{*2}$, $m|d_1$.*

(2) *If $d_1 \equiv 1 \pmod{8}$, then $r_4(K_2O_F) = 1$, $r_8(K_2O_F) = 0$ and $\Delta_F = \{2, m\}F^{*2}$, $m|d_1$.*

(3) *If $d_1 \equiv -1 \pmod{8}$ with $2 \in NF$, then $r_4(K_2O_F) = 0$ and $\Delta_F = \{2, m(u + \sqrt{-d_1})\}F^{*2}$, where $m|d_1$ and $u^2 - 2w^2 = -d_1$, $u, w \in \mathbb{N}$.*

Proof. Let $F_0 = \mathbb{Q}(\sqrt{d_1})$ be a real quadratic field versus $F = \mathbb{Q}(\sqrt{-d_1})$ and $C(F_0)$ the narrow class group of F_0 . We have the fact: $r_4(C(F_0)) \leq r_4(C(F)) \leq r_4(C(F_0)) + 1$; compare e.g. [12]. Since $r_4(C(F)) = 0$, $r_4(C(F_0)) = 0$, so $2 \notin NF$ if $d_1 \not\equiv -1 \pmod{8}$.

(1) If $d_1 \not\equiv \pm 1 \pmod{8}$ or $d_1 \equiv -1 \pmod{8}$ with $2 \notin NF$, then, by [17, theorem 4.1], $r_4(K_2O_F) = 0$, so $\Delta_F = \{2, m\}F^{*2}$, $m|d_1$, by [1] and [11].

(2) If $d_1 \equiv 1 \pmod{8}$, then $r_4(C(F)) = r_4(C(F_0)) = 0$ and $2 \notin NF$. By [12, theorem 1(i)], there is a positive divisor $n \equiv \pm 3 \pmod{8}$ of d_1 such that the equation $nz^2 = x^2 - d_1y^2$ is solvable

over $\mathbb{Z}$. By [14, theorem 4.1], $r_4(K_2O_F) = 1$, $r_8(K_2O_F) = 0$ and $1 \neq \{-1, n\} \in (K_2O_F)^2$. By [1] and $2 \notin NF$, we get $\Delta_F = \{2, m\}F^{*2}$, $m|d_1$.

(3) If $d_1 \equiv -1 \pmod{8}$ with $2 \in NF$, then $r_4(C(F)) = r_4(C(F_0)) = 0$, so the class $[\mathcal{D}] = 1$ in $C(F_0)$, where $\mathcal{D}$ is a dyadic ideal of F_0 . By [17, theorem 4.1 and 4.3] or [16], $r_4(K_2O_F) = 0$ and $m(u + \sqrt{-d_1}) \in \Delta_F, m|d_1$, $-d_1 = u^2 - 2w^2$, $u, w \in \mathbb{N}$. ■

Let S be a finite set of primes of F including all infinite primes and all dyadic primes. We will give a method to determine that the S -ideal class group $C^S(F)$ has odd order, provided $r_4(C(F)) = 0$. For convenience, we make a definition, which will always be applied for in this paper.

Definition 2.1 Let $F = \mathbb{Q}(\sqrt{-d_1})$ be an imaginary quadratic field. Set

$$(2.1) \quad S' = \{q_0 = 2, q_1, \dots, q_s, q_{s+1} \dots, q_{s+r}\},$$

where S' is the set of primes of $\mathbb{Q}$, odd primes $q_1, \dots, q_s$ are decomposable in F and odd primes $q_{s+1} \dots, q_{s+r}$ are inert in F . We define

$$S = \{ \text{all primes } Q \text{ of } F \mid Q|\infty \text{ or } Q|q_i \text{ for some } q_i \text{ in } S' \},$$

$$S_f = \{ \text{all primes } Q \text{ of } F \mid Q|q_i \text{ for some } q_i \text{ in } S' \}.$$

In fact, $|S_f| = 2s + r + s_2$, where s_2 is the number of dyadic primes of F .

In this paper, we will always assume that $p_1, \dots, p_t$ are all primes which ramify in F and $p_t = 2$ if $2|D_F$, where D_F is the discriminant of F . Let $M_F = (a_{ij})$ be the $t \times t$ Rédei's matrix of F with coefficients $a_{ij} \in \mathbb{F}_2$, where

$$(-1)^{a_{ij}} = \begin{cases} \left(\frac{p_i}{p_j}\right) & \text{if } i \neq j, \\ \left(\frac{D_F/p_i^*}{p_i}\right) & \text{if } i = j, \end{cases} \quad a_{ij} = \begin{cases} \left(\frac{p_i}{p_j}\right)' & \text{if } i \neq j, \\ \left(\frac{D_F/p_i^*}{p_i}\right)' & \text{if } i = j, \end{cases}$$

p_i^* the discriminant of p_i . Let M'_F be the $(t-1) \times t$ matrix by deleting the t -th row of M_F . It is clear that $r_4(C(F)) = 0$ if and only if $\text{rank} M_F = \text{rank} M'_F = t-1$.

Set the matrix over $\mathbb{F}_2$

$$(2.2) \quad M^S = \begin{pmatrix} \left(\frac{2}{p_1}\right)' & \left(\frac{q_1}{p_1}\right)' & \dots & \left(\frac{q_s}{p_1}\right)' \\ \vdots & \vdots & & \vdots \\ \left(\frac{2}{p_{t-1}}\right)' & \left(\frac{q_1}{p_{t-1}}\right)' & \dots & \left(\frac{q_s}{p_{t-1}}\right)' \end{pmatrix} \text{ or } \begin{pmatrix} \left(\frac{q_1}{p_1}\right)' & \dots & \left(\frac{q_s}{p_1}\right)' \\ \vdots & & \vdots \\ \left(\frac{q_1}{p_{t-1}}\right)' & \dots & \left(\frac{q_s}{p_{t-1}}\right)' \end{pmatrix},$$

where M^S is the $(t-1) \times s$ matrix only if $-d_1 \equiv 5 \pmod{8}$ and M^S is the $(t-1) \times (s+1)$ matrix otherwise.

Proposition 2.2 If $r_4(C(F)) = 0$, then the S -ideal class group $C^S(F)$ has odd order if and only if $\text{rank} M^S = t-1$.

Proof. Without loss of generality, suppose $-d_1 \not\equiv 5 \pmod{8}$. Take $Q_0 = \mathcal{D}$ a dyadic ideal and Q_j integral ideals of F over q_j , $j = 1, \dots, s+r$, respectively. Let $G = (\{[Q_j] | j = 0, 1, \dots, s\})$ be the subgroup of $C(F)$ generated by the ideal classes $[Q_j]$, $j = 0, 1, \dots, s$ (in fact: $[Q_j] = 1$, $j > s$, by q_j inert in F), then $C^S(F) = C(F)/G$.

If $\text{rank} M^S = t - 1$, then, without loss of generality, we assume that the beginning $t - 1$ columns of M^S are linear independent. Hence each column of M'_F can be linearly expressed by the beginning $t - 1$ columns of M^S over $\mathbb{F}_2$:

$$\begin{pmatrix} \left(\frac{p_i}{p_1}\right)' \\ \vdots \\ \left(\frac{p_i}{p_{t-1}}\right)' \end{pmatrix} = \begin{pmatrix} \left(\frac{q_{i_1} \cdots q_{i_j}}{p_1}\right)' \\ \vdots \\ \left(\frac{q_{i_1} \cdots q_{i_j}}{p_{t-1}}\right)' \end{pmatrix}.$$

Therefore, by Legendre theorem (see [17]), the Diophantine equation $p_i q_{i_1} \cdots q_{i_j} z^2 = x^2 + d_1 y^2$ has a nontrivial and relatively prime solution $(x, y, z) = (a, b, c)$ over $\mathbb{Z}$. Hence, in $C(F)$, $[P_i][Q_{i_1} \cdots Q_{i_j}] = [P_c]^2$, where P_i is the ambiguous ideal over p_i and P_c is a ideal over c . Since $r_4(C(F)) = 0$, there is an odd number k such that $[P_i] = [Q_{i_1} \cdots Q_{i_j}]^k \in G$. Hence each ambiguous ideal class $[P_i]$ belongs to G by the genus theory (see [4]).

Suppose that an ideal class $[X]$ of F satisfies $[X]^2 \equiv 1 \pmod{G}$. Since $r_4(C(F)) = 0$, there is an ambiguous ideal class $[P]$ such that $[P][X]$ has odd order in the class group $C(F)$. By $([P][X])^2 = [X]^2 \in G$, $[P][X] \in G$, so $[X] \in G$. Hence $C^S(F)$ has odd order.

Conversely, if $C^S(F)$ has odd order, then each ambiguous ideal class $[P_i]$ over p_i belongs to G . Hence $[P_i][Q_{i_1} \cdots Q_{i_j}] \in G^2 \subset C(F)^2$, so the Diophantine equation $p_i q_{i_1} \cdots q_{i_j} z^2 = x^2 + d_1 y^2$ has nontrivial solutions over $\mathbb{Z}$. By Legendre theorem (see [17]), the i -th column of M'_F is linearly expressed by columns of M^S over $\mathbb{F}_2$. Hence each column of M'_F can be linearly expressed by columns of M^S over $\mathbb{F}_2$. So $\text{rank} M^S = t - 1$ by $\text{rank} M'_F = t - 1$. ■

Let $E = \mathbb{Q}(\sqrt{-d_1}, \sqrt{d_2})$ be a biquadratic field. In this paper, we will assume that S defined as Definition 2.1 is the set of all infinite primes, all dyadic primes, and all finite primes which ramify in E . In fact, $d_2 = nq_1 \cdots q_{s+r}, n|2d_1$, is a squarefree integer. Suppose $r_4(C(F)) = 0$ and $C^S(F)$ has odd order, we will give all elements of order 2 of $K_2 O_E$.

For an arbitrary number field L with the ring of integers O_L we define

$$D_L = \{x \in L^* | v_P(x) \equiv 0 \pmod{2} \text{ for all finite primes } P \nmid 2\}.$$

We obtain an exact sequence

$$0 \rightarrow \tilde{\Delta}_L \rightarrow D_L/L^{*2} \rightarrow {}_2K_2 O_L \rightarrow 0,$$

where Δ_L is the Tate kernel of L and $\tilde{\Delta}_L = \Delta_L/L^{*2}$. Let

$$R(E/F) = \{x \in E^* | N_{E/F}(x) \in \Delta_F\}, D(E/F) = R(E/F) \cap D_E.$$

We define ϵ via

$$(2.3) \quad 2^\epsilon = [\Delta_F : \Delta_F \cap N_{E/F}(E^*)].$$

Let $Tr(E/F)$ denote the kernel of the transfer map $Tr_{E/F} : K_2E \rightarrow K_2F$. It is clear that there is an exact sequence

$$(2.4) \quad 0 \rightarrow \tilde{\Delta}_E \rightarrow D(E/F)/E^{*2} \rightarrow {}_2K_2O_E \cap Tr(E/F) \rightarrow 0.$$

Suppose $C^S(F)$ has odd order, by [5, lemma 2.4 and proposition 2.6], there is an exact sequence

$$(2.5) \quad 0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow U^S/(U^S)^2 \rightarrow D(E/F)/E^{*2} \rightarrow (\Delta_F \cap N_{E/F}(E^*)) / F^{*2} \rightarrow 0,$$

where U^S is the group of units of the ring of S -integers O_F^S . Hence

$$r_2(D(E/F)/E^{*2}) = r_1 + r_2 + |S_f| + r_2 + 1 - \epsilon - 1 = |S_f| + 2 - \epsilon,$$

where S_f consists of all finite primes in S , ϵ is defined as (2.3), $r_1 = 0$ and $r_2 = 1$ (r_1 and r_2 are the number of real and pairs of complex primes of F , respectively). By (2.4) and $r_2(\tilde{\Delta}_E) = 3$, we get the result of [5, remark 2.8]:

$$(2.6) \quad r_2(K_2O_E \cap Tr(E/F)) = |S_f| - 1 - \epsilon = 2s + r + s_2 - 1 - \epsilon.$$

Suppose $r_4(C(F)) = 0$ and $C^S(F)$ has odd order, then $\text{rank} M^S = t - 1$ by Proposition 2.2 and $2 \notin NF$ if $-d_2 \not\equiv 1 \pmod{8}$. In the following, we give the representative set of $D(E/F)/E^{*2}$.

Theorem 2.1 (1) *If $2|D_F$ and the rank of the following matrix is $t - 1$:*

$$(2.7) \quad M_1^S = \begin{pmatrix} \left(\frac{2}{p_1}\right)' & \left(\frac{q_1}{p_1}\right)' & \cdots & \left(\frac{q_{t-2}}{p_1}\right)' \\ \vdots & \vdots & & \vdots \\ \left(\frac{2}{p_{t-1}}\right)' & \left(\frac{q_1}{p_{t-1}}\right)' & \cdots & \left(\frac{q_{t-2}}{p_{t-1}}\right)' \end{pmatrix},$$

then the representative set of $D(E/F)/E^{*2}$ is

$$\{p_1, \dots, p_t = 2, q_2, \dots, q_{s+r}, a_{t-1} + \sqrt{-d_1}b_{t-1}, \dots, a_s + \sqrt{-d_1}b_s\} \cup T$$

where, for each q_i ($i = t-1, \dots, s$), there is $\{q_{i_1}, \dots, q_{i_j}\} \subset \{2, q_1, \dots, q_{t-2}\}$ such that $q_i q_{i_1} \cdots q_{i_j} z^2 = x^2 + d_1 y^2$ has a relatively prime solution $(x, y, z) = (a_j, b_j, c_j)$ over $\mathbb{Z}$, and

$$T = \begin{cases} \{e_1 + \sqrt{d_2}f_1, e_2 + \sqrt{d_2}f_2\} & \text{if } \epsilon = 0, \\ \{e_1 + \sqrt{d_2}f_1\} & \text{if } \epsilon = 1, \\ \emptyset & \text{if } \epsilon = 2, \end{cases}$$

with $e_j^2 - d_2 f_j^2 = \delta_j \in \Delta_F$, $e_j, f_j \in F^$.*

(2) *If $d_1 \equiv -1 \pmod{8}$, $2 \notin NF$ and the rank of the form (2.7) is $t - 1$, then the representative set of $D(E/F)/E^{*2}$ is*

$$\{p_1, \dots, p_t, 2, q_2, \dots, q_{s+r}, a_{t-1} + \sqrt{-d_1}b_{t-1}, \dots, a_s + \sqrt{-d_1}b_s\} \cup T.$$

(3) If $d_1 \equiv -5 \pmod{8}$ and the rank of the following matrix is $t-1$:

$$(2.8) \quad M_2^S = \begin{pmatrix} \left(\frac{q_1}{p_1}\right)' & \cdots & \left(\frac{q_{t-1}}{p_1}\right)' \\ \vdots & & \vdots \\ \left(\frac{q_1}{p_{t-1}}\right)' & \cdots & \left(\frac{q_{t-1}}{p_{t-1}}\right)' \end{pmatrix},$$

then the representative set of $D(E/F)/E^{*2}$ is

$$\{p_1, \dots, p_t, 2, q_2, \dots, q_{s+r}, a_t + \sqrt{-d_1}b_t, \dots, a_s + \sqrt{-d_1}b_s\} \cup T.$$

(4) If $d_1 \equiv -1 \pmod{8}$, $2 \in NF$, and the rank of the form (2.8) is $t-1$, then the representative set of $D(E/F)/E^{*2}$ is

$$\{p_1, \dots, p_t, 2, q_2, \dots, q_{s+r}, u + \sqrt{-d_1}, a_t + \sqrt{-d_1}b_t, \dots, a_s + \sqrt{-d_1}b_s\} \cup T,$$

where $-d_1 = u^2 - 2w^2$, $u, w \in \mathbb{N}$.

Note: in (2), (3) and (4), $a_i + \sqrt{-d_1}b_i$ and T are analogue to them in (1).

Proof. By the preceding discussion and the assumption, we get

$$r_2(D(E/F)/E^{*2}) = |S_f| + 2 - \epsilon = 2s + r + s_2 + 2 - \epsilon,$$

where s_2 is the number of Dyadic ideals in F .

(1) If $2|D_F$, then $s_2 = 1$. By $\text{rank} M_1^S = t-1$, each column $((\frac{q_i}{p_1})', \dots, (\frac{q_i}{p_{t-1}})')^T$ of M^S can be expressed by the columns of M_1^S , i.e.,

$$\begin{pmatrix} (\frac{q_i q_{i_1} \cdots q_{i_j}}{p_1})' \\ \vdots \\ (\frac{q_i q_{i_1} \cdots q_{i_j}}{p_{t-1}})' \end{pmatrix} = 0,$$

where $\{q_{i_1}, \dots, q_{i_j}\} \subset \{2, q_1, \dots, q_{t-1}\}$. By Legendre theorem (see [17]), the Diophantine equation $q_i q_{i_1} \cdots q_{i_j} z^2 = x^2 + d_1 y^2$ has a relatively prime solution $(x, y, z) = (a_i, b_i, c_i)$ over $\mathbb{Z}$. Hence we get $\{p_1, \dots, p_t, q_2, \dots, q_{s+r}, a_{t-1} + \sqrt{-d_1}b_{t-1}, \dots, a_s + \sqrt{-d_1}b_s\} \subseteq D(E/F)$ and such $2s+r+1$ elements must be distinct modulo E^{*2} each other.

On the other hand, suppose $\delta \in \Delta_F$, $\delta \notin F^{*2}$, is a norm from E^* , i.e., $\delta = N_{E/F}(\alpha)$, $\alpha \in E$, then clearly the valuation of α is even at all undecomposed non-dyadic finite primes and $v_P(\alpha) \equiv v_{P^\sigma}(\alpha) \pmod{2}$ at finite decomposed primes $P = \mathcal{P}\mathcal{P}^\sigma$ of F . Since $C^S(F)$ has odd order, we can find $f \in F^*$ such that $\alpha \cdot f \in D(E/F)$, so $\alpha \cdot f \in T$.

By the property of norm, it is verified that each product of such distinct elements does not belong to E^{*2} . Note the definition of ϵ in (2.3), then (1) holds.

In cases (2) and (4), $2 \nmid D$ and $s_2 = 2$; in case (3), $2 \nmid D$ and $s_2 = 1$. Similarly to case (1), we can prove them. ■

Remark 2.1 (1) By Theorem 2.1 and (2.4), we have given all elements of order at most 2 in $K_2O_E \cap \text{Tr}(E/F)$.

(2) Since $r_4(C(F)) = 0$, by Proposition 2.1 we get $\Delta_F = \{2, \delta\}F^{*2}$, so $2, \delta \in \Delta_E$. Suppose $2, \delta, 2\delta \notin E^{*2}$, then $\tilde{\Delta}_E = \{2, \delta, \delta'\}/E^{*2}$. Although $\Delta_E \subset D(E/F)$, it is difficult to find δ' from the representative set of $D(E/F)$ in Theorem 2.1.

About T in Theorem 2.1 (4), we have the following result.

Proposition 2.3 Let $-d_1 \equiv 1 \pmod{8}$, $2 \in NF$ and $u^2 - 2w^2 = -d_1, u, w \in \mathbb{N}$. Suppose $m(u + \sqrt{-d_1}) \in N_{E/F}(E^*) \cap \Delta_F, m|d_1$, then

(1) $q_j \equiv \pm 1 \pmod{8}$ for each odd prime divisor q_j of d_2 .

(2) If $-d_1 \equiv 9 \pmod{16}$, then $d_2 \equiv \pm 1 \pmod{8}$.

Proof. Let $u^2 - 2w^2 = -d_1, u, w \in \mathbb{N}, w \equiv 4 \pmod{8}$. Suppose $m(u + \sqrt{-d_1}) \in N_{E/F}(E^*) \cap \Delta_F$. By $m(u + \sqrt{-d_1}) \in \Delta_F$, in the field F the Hilbert symbol $(m(u + \sqrt{-d_1}), -1)_2 = 1$ and $m + u \equiv 2 \pmod{4}$, e.g. see [14]. By $m(u + \sqrt{-d_1}) \in N_{E/F}(E^*)$, in the field F the Hilbert symbol $(m(u + \sqrt{-d_1}), d_2)_2 = 1$ and $(m(u - \sqrt{-d_1}), d_2)_2 = 1$. Hence $(2m^2w^2, d_2)_2 = (2, d_2)_2 = 1$, so the local Hilbert symbol $\left(\frac{2, d_2}{Q_j}\right) = \left(\frac{2}{q_j}\right) = 1$ for each odd prime $Q_j|q_j$ of F . Hence $q_j \equiv \pm 1 \pmod{8}$ for each $q_j|d_2$.

If $-d_1 \equiv 9 \pmod{16}$, then $u \equiv \pm 3 \pmod{8}$. By $m + u \equiv 2 \pmod{4}$ and $m|d_1$, i.e. $m \equiv \pm 1 \pmod{8}$,

$$m \frac{u + \sqrt{-d_1}}{2} \equiv mu \equiv 5 \pmod{(2, \frac{u - \sqrt{-d_1}}{2})^3 = \mathcal{D}^3}.$$

Suppose that $m(u + \sqrt{-d_1}) \in N_{E/F}(E^*)$, then in the complete field $F_{\mathcal{D}}$, $\mathcal{D}|2$, the local Hilbert symbol

$$\left(\frac{m(u + \sqrt{-d_1}), d_2}{\mathcal{D}}\right) = \left(\frac{5, d_2}{\mathcal{D}}\right) = 1.$$

Hence $d_2 \equiv \pm 1 \pmod{8}$. ■

Although we know the forms of all elements of order ≤ 2 in $K_2O_E \cap \text{Tr}(E/F)$ from Theorem 2.1, we are interested in the forms of all elements of order ≤ 2 in K_2O_E .

Since $C^S(F)$ has odd order, we use the method of proving [5, proposition 2.1] to get the following result.

Lemma 2.1 If $x \in N_{E/F}(E^*)$, then $x \in D_F$ if and only if there is $\alpha \in D_E$ with $N_{E/F}(\alpha) = xf^2, f \in F^*$.

Proof. We only need to prove the necessary condition. Suppose $N_{E/F}(\alpha') = x, \alpha' \in E, x \in D_F$, then the valuation of α' is even at all undecomposed non-dyadic finite primes and $v_{\mathcal{P}}(\alpha') = v_{\mathcal{P}\sigma}(\alpha') \pmod{2}$ at finite decomposed primes $PO_E = \mathcal{P}\mathcal{P}^\sigma$. Since $C^S(F)$ has odd order, we find $f \in F^*$ such that $\alpha = \alpha'f \in D_E$ and $N_{E/F}(\alpha) = xf^2$. ■

Let $F = \mathbb{Q}(\sqrt{-d_1})$ and $E = F(\sqrt{d_2})$. Suppose m is a positive divisor of $2d_1$. We have that

$$m \in N_{E/F}(E^*)$$

$\iff$ the Hilbert symbol $(m, d_2)_2 = 1$ of F

$\iff$ the local Hilbert symbol $(\frac{m, d_2}{P}) = 1$ for each primes P of F .

By [8, theorem 5.4], for each finite non-dyadic prime P ,

$$\left(\frac{m, d_2}{P}\right) = ((-1)^{v_P(m)v_P(d_2)} m^{v_P(d_2)} d_2^{-v_P(m)})^{(NP-1)/2} \pmod{P}.$$

Hence, for each finite prime $P \notin S$, $(\frac{m, d_2}{P}) = 1$ and, for each finite non-dyadic prime $Q \in S$, $(\frac{m, d_2}{Q}) = (\frac{m}{Q}) = (\frac{m}{q})$, where Q is a prime over q . In the following, we find out all m with $(\frac{m, d_2}{Q}) = (\frac{m}{q}) = 1$ for all $Q \in S_f, Q|q$.

If $2|D_F$, i.e., $p_t = 2$. We define the system of homogeneous equations over $\mathbb{F}_2$

$$(2.9) \quad AX = 0,$$

where

$$(2.10) \quad A = \begin{pmatrix} \left(\frac{p_1}{q_1}\right)' & \dots & \left(\frac{p_t}{q_1}\right)' \\ \vdots & & \vdots \\ \left(\frac{p_1}{q_{s+r}}\right)' & \dots & \left(\frac{p_t}{q_{s+r}}\right)' \end{pmatrix}.$$

Let a non-zero vector $X = (x_1, \dots, x_t)^T$ over $\mathbb{F}_2$ be corresponding to $m = \prod_{x_i=1} p_i$. Hence each non-zero solution of the system (2.9) is 1-1 corresponding to $m \in N_{E/F}(E^*)$.

If $2 \nmid D_F$, i.e., $p_t \neq 2$. We also define the system of homogeneous equations over $\mathbb{F}_2$

$$(2.9'), \quad A'X = 0,$$

where

$$(2.10') \quad A' = \begin{pmatrix} \left(\frac{2}{q_1}\right)' & \left(\frac{p_1}{q_1}\right)' & \dots & \left(\frac{p_t}{q_1}\right)' \\ \vdots & \vdots & & \vdots \\ \left(\frac{2}{q_{s+r}}\right)' & \left(\frac{p_1}{q_{s+r}}\right)' & \dots & \left(\frac{p_t}{q_{s+r}}\right)' \\ \left(\frac{2, d_2}{\mathcal{D}}\right)' & \left(\frac{p_1, d_2}{\mathcal{D}}\right)' & \dots & \left(\frac{p_t, d_2}{\mathcal{D}}\right)' \end{pmatrix},$$

where $\mathcal{D}$ is a dyadic prime of F (in fact, we need the last row of A' only if $d_1 \equiv -1 \pmod{8}$). Hence each non-zero solution of the system (2.10') is 1-1 corresponding to $m \in N_{E/F}(E^*)$. Hence we get:

Lemma 2.2 *If $2|D_F$, then there is a non-zero solution $X = (x_1, \dots, x_t)^T$ of (2.9) if and only if $m \in N_{E/F}(E^*)$, where $m = \prod_{x_i=1} p_i$; if $2 \nmid D_F$ and $p_0 = 2$, then there is a non-zero solution $X = (x_0, x_1, \dots, x_t)^T$ of (2.9') if and only if $m \in N_{E/F}(E^*)$, where $m = \prod_{x_i=1} p_i$. ■*

Theorem 2.2 *Suppose A and A' are defined as (2.10) and (2.10'), respectively. If $2|D_F$, then*

$$r_2(K_2O_E) = 2s + r + s_2 + t - 3 - \text{rank}A;$$

if $2 \nmid D_F$, then

$$r_2(K_2O_E) = 2s + r + s_2 + t - 2 - \text{rank}A' + \varepsilon,$$

where $\varepsilon = 1$ if $m(u + \sqrt{-d_1}) \in N_{E/F}(E^*)$, $u^2 - 2w^2 = -d_1$, some $m|2d_1$, and $\varepsilon = 0$ otherwise.

Proof. Since $F = \mathbb{Q}(\sqrt{-d_1})$ and $E = \mathbb{Q}(\sqrt{-d_1}, \sqrt{d_2})$, by [5] there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & K_2O_E & \rightarrow & K_2E & \rightarrow & \bigoplus_{v \text{ fin.}} \overline{E}_v \rightarrow 0 \\ & & \downarrow Tr & & \downarrow Tr & & \downarrow \\ 0 & \rightarrow & K_2O_F & \rightarrow & K_2F & \rightarrow & \bigoplus_{v \text{ fin.}} \overline{F}_v \rightarrow 0. \end{array}$$

Hence there is a restrictive homomorphism of two elementary subgroups $Tr : {}_2K_2O_E \rightarrow {}_2K_2O_F$. Set $\overline{H} = Tr({}_2K_2O_E)$, then there is an exact sequence

$$0 \rightarrow {}_2(K_2O_E \cap Tr(E/F)) \rightarrow {}_2K_2O_E \xrightarrow{Tr} \overline{H} \rightarrow 0.$$

Therefore we have

$$(2.11) \quad {}_2K_2O_E = {}_2(K_2O_E \cap Tr(E/F)) \oplus H, Tr(H) = \overline{H}.$$

By (2.6), we only need to computer $r_2(H)$ in the following two cases.

Case 1: $2 \notin NF$. By [1], all elements of order ≤ 2 in K_2O_F are the forms: $\{-1, m\}$, where m are all positive divisors of $2d_1$, and $\Delta_F = \{2, m'\}F^{*2}$, where m' is a positive odd divisor of d_1 .

If $2|D_F$, i.e. $p_t = 2$, then, by Lemma 2.1 and 2.2, there is $\beta \in H$ with $Tr(\beta) = \{-1, m\} \in \overline{H}$ if and only if $m \in N_{E/F}(E^*)$ if and only if there is a non-zero solution of (2.9) corresponding to m . By the Tate kernel $\Delta_F = \{2, m'\}F^{*2}$ and the definition of ϵ in (2.3), we get

$$r_2(H) = t - \text{rank}A - (2 - \epsilon),$$

$$r_2(K_2O_E) = 2s + r + s_2 + t - 3 - \text{rank}A$$

If $2 \nmid D_F$ and $2 \notin NF$, then, similarly, we get

$$r_2(H) = t + 1 - \text{rank}A' - (2 - \epsilon),$$

$$r_2(K_2O_E) = 2s + r + s_2 + t - 2 - \text{rank}A'.$$

Moreover, let $\Delta_F = \{2, m'\}F^{*2}$, we can determine the value of ϵ from the system (2.9) or (2.9'). In fact, $m' \in N_{E/F}(E^*) \cap \Delta_E$ if and only if the column vector X corresponding to m' must be a solution of (2.9) or (2.9').

Case 2: $d_1 \equiv -1 \pmod{8}$ and $2 \in NF$. By [1] all elements of order ≤ 2 in K_2O_F are the forms: $\{-1, m\}$, $\{-1, m(u + \sqrt{d})\}$, where m are all positive divisors of d_1 and $u^2 - 2w^2 = -d_1, u, w \in \mathbb{N}$; by Proposition 2.1 $\Delta_F = \{2, m'(u + \sqrt{d})\}F^{*2}$, where m' is a positive odd divisor of d_1 .

First, suppose $m(u + \sqrt{-d_1}) \notin N_{E/F}(E^*)$ for each $m|2d_1$, then, by the preceding discussion, we get the same results as case 1. E.g., if there is some $q_i \equiv \pm 3 \pmod{8}$, then $m(u + \sqrt{-d_1}) \notin N_{E/F}(E^*)$ for each such m .

On the other hand, suppose $m'(u + \sqrt{-d_1}) \in N_{E/F}(E^*)$, then, by Proposition 2.2 each prime $q_i \equiv \pm 1 \pmod{8}$ and $\epsilon = 0$ in (2.3). By the preceding discussion, we get

$$r_2(H) = t + 1 - \text{rank} A' - 1.$$

Suppose $\epsilon = 1$ and $m''(u + \sqrt{-d_1}) \in N_{E/F}(E^*)$ for some $m'' \neq m'$, $m''|2d_1$, then $2 \in N_{E/F}(E^*)$ by Proposition 2.2 and

$$r_2(H) = t + 1 - \text{rank} A' - 1 + 1.$$

In the above two cases, we always have

$$r_2(K_2O_E) = 2s + r + s_2 + t - 2 - \text{rank} A' + \epsilon, \quad \epsilon = 1. \quad \blacksquare$$

We prove Theorem 2.2 by the method of construction, hence we can represent all elements of order ≤ 2 in K_2O_E .

Example 2.1 $F = \mathbb{Q}(\sqrt{-3 \times 7})$, $E = \mathbb{Q}(\sqrt{-3 \times 7}, \sqrt{5 \times 11 \times 13})$.

For $F = \mathbb{Q}(\sqrt{-3 \cdot 7})$, set $p_1 = 3, p_2 = 7, p_3 = 2$. There is the Rédei's matrix

$$M'_F = \begin{pmatrix} (\frac{D/p_1^*}{p_1})' & (\frac{p_2}{p_1})' & (\frac{p_3}{p_1})' \\ (\frac{p_1}{p_2})' & (\frac{D/p_2^*}{p_2})' & (\frac{p_3}{p_2})' \end{pmatrix} = \begin{pmatrix} (\frac{-7}{3})' & (\frac{7}{3})' & (\frac{2}{3})' \\ (\frac{3}{7})' & (\frac{-3}{7})' & (\frac{2}{7})' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Hence $r_4(C(F)) = 0$. It is clear that $\Delta_F = \{2, 7\}F^{*2}$. On the other hand, 5, 11 are decomposable in F and 13 is inert in F . Set $q_1 = 5, q_2 = 11, q_3 = 13$, there is a matrix as (2.2)

$$M^S = \begin{pmatrix} (\frac{2}{p_1})' & (\frac{q_1}{p_1})' & (\frac{q_2}{p_1})' \\ (\frac{2}{p_2})' & (\frac{q_1}{p_2})' & (\frac{q_2}{p_2})' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since $\text{rank} M^S = 2$, $C^S(F)$ has odd order. We also have a matrix as (2.10)

$$A = \begin{pmatrix} (\frac{p_1}{q_1})' & (\frac{p_2}{q_1})' & (\frac{p_3}{q_1})' \\ (\frac{p_1}{q_2})' & (\frac{p_2}{q_2})' & (\frac{p_3}{q_2})' \\ (\frac{p_1}{q_3})' & (\frac{p_2}{q_3})' & (\frac{p_3}{q_3})' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Hence $\text{rank} A = 2$, $2 \cdot 7 \in N_{E/F}(E^*) \cap \Delta_E$ and $r_2(H) = 0$ in (2.11), so $r_2(K_2O_E) = 2s + r + s_2 + t - 3 - \text{rank} A = 4$. By Theorem 2.1 and Theorem 2.2, we have the representative set of $D(E/F)/E^{*2}$:

$$\{3, 7, 2, 11, 13, 1 + \sqrt{-3 \cdot 7}\} \cup \{27 + \sqrt{5 \cdot 11 \cdot 13}\},$$

where $2 \cdot 11 \cdot 1^2 = 1^2 + 3 \cdot 7 \cdot 1^2$ and $2 \cdot 7 \cdot 1^2 = 27^2 - 5 \cdot 11 \cdot 13 \cdot 1^2$. Since $2, 7 \in \Delta_F \subset \Delta_E$, the representative set of ${}_2K_2O_E$ is

$$\{-1, 3\}, \{-1, 11\}, \{-1, 13\}, \{-1, 1 + \sqrt{-21}\}, \{-1, 27 + \sqrt{715}\},$$

where there is only one dependent relation. $\blacksquare$

Example 2.2 $F = \mathbb{Q}(\sqrt{-23})$, $E = \mathbb{Q}(\sqrt{-23}, \sqrt{47})$.

It is clear that $r_2(C(F)) = 0$ and $C^S(F)$ has odd order. Since $-23 = 3^2 - 2 \cdot 4^2$, $\Delta_F = \{2, 23(3 + \sqrt{-23})\}F^{*2}$. Also $20^2 + 23 \cdot 1^2 = 47 \cdot 3^2$, $P = (20 + \sqrt{-23}, 47)$ and $P' = (20 - \sqrt{-23}, 47)$ are prime ideals of F over 47, then the local Hilbert symbol $(\frac{23(3+\sqrt{-23})}{P}, 47) = 1$. Hence $2, 23(3 + \sqrt{-23}) \in \Delta_F \cap N_{E/F}(E^*)$, so $N_{E/F}(7 + \sqrt{47}) = 2$ and $N_{E/F}(\alpha) = 23(3 + \sqrt{-23})f^2$ for $\alpha \in D(E/F)$. By Theorem 2.1 and Theorem 2.2, $r_2(K_2O_E \cap Tr(E/F)) = 2 + 0 + 2 - 1 - \epsilon = 3$, $r_2(H) = 0$ in (2.11) and the representative set of ${}_2K_2O_E$ is

$$\{-1, 23\}, \{-1, 20 + \sqrt{-23}\}, \{-1, 7 + \sqrt{47}\}, \{-1, \alpha\},$$

where there is only one dependent relation. ■

3 4-rank

Let $F = \mathbb{Q}(\sqrt{-d_1})$ and $E = \mathbb{Q}(\sqrt{-d_1}, \sqrt{d_2})$ be an imaginary quadratic field and a biquadratic field, respectively. Let S be the finite set of primes in F consisting of all infinite primes, all dyadic primes, all finite primes ramifying in E . In this section, we will use the Hurrelbrink and Kolsters' results in [5] to investigate elements of order 4 of K_2O_E , provided the 4-rank of the class group $C(F)$ is 0 and the S -ideal class group $C^S(F)$ has odd order.

First, we describe those results of [5]. Let $Tr(E/F)$ is the kernel of a homomorphism $Tr : K_2E \rightarrow K_2F$ and let $\gamma = \frac{\alpha}{\alpha\sigma} \in Tr(E/F)$, then we define a homomorphism

$$\begin{aligned} \phi_{E/F} : Tr(E/F) &\rightarrow K_2F / \{-d_2, \Delta_F\} \cdot (K_2F)^2 \cong {}_2Br(F) / (-d_2, \Delta_F) \\ \gamma &\mapsto Tr(\alpha) \bmod \{-d_2, \Delta_F\} \cdot (K_2F)^2, \end{aligned}$$

where $(-d_2, \Delta_F)$ denotes the subgroup of the Brauer group of F generated by the classes $(-d_2, \varepsilon), \varepsilon \in \Delta_F$. Suppose $C^S(F)$ has odd order, there are exact sequences in [5, proposition 2.10, corollary 2.12]

$$0 \rightarrow Tr(E/F)^2 \rightarrow Tr(E/F) \xrightarrow{\phi_{E/F}} {}_2Br(F) / (-d_2, \Delta_F),$$

$$0 \rightarrow (K_2O_E)^2 \cap Tr(E/F) \rightarrow K_2O_F \cap Tr(E/F) \xrightarrow{\phi_{E/F}} {}_2Br^S(F) / (-d_2, \Delta_F).$$

If $\gamma = \{-1, t\} \in Tr(E/F), t \in F^*$, then $\phi_{E/F}(\gamma) = \overline{(-d_2, t)}$ by [5, proposition 2.13].

To compute the homomorphism $\phi_{E/F}$ easier, we define another homomorphism in [5]. Let $E_1 = E = F(\sqrt{d_2}) \neq E_2 = F(i), E_3 = E_0 = F(\sqrt{-d_2})$. Suppose $x \in R(E/F) = N_{E/F}^{-1}(\Delta_F)$ and $y \in F(i)$, satisfying $N_{E/F}(x) = N_{F(i)/F}(y)$. We call y *admissible* for x if

$$Tr_{F(i)/F}(\{i, y\}) = 1 \text{ and } t_{E/F}(x) + t_{F(i)/F}(y) \neq 0,$$

where $t_{E/F}$ and $t_{F(i)/F}$ denote the trace maps from E and $F(i)$ to F , respectively. In this case we also call (x, y) an *admissible pair*. Moreover, provided that each element $x \in R(E/F)$ has

an admissible element y , we define the homomorphism

$$\begin{aligned}\psi_{E/F} : {}_2\text{Tr}(E/F) &\rightarrow \text{Br}(E_0/F)/(-d_2, \Delta_F), \\ \{-1, x\} &\mapsto (-d_2, t_{E/F}(x) + t_{F(i)/F}(y)),\end{aligned}$$

where $\text{Br}(E_0/F)$ consists of all classes of quaternion algebras of the form $(-d_2, a)$ with $a \in F^*$. Hence, by [5, theorem 3.7 and corollary 3.9] and $D(E/F) = R(E/F) \cap D_E$ in section 2, there is a result:

Lemma 3.1 *Suppose $C^S(F)$ has odd and each element $x \in D(E/F)$ has an admissible element $y \in F(i)$. Then there is a exact sequence*

$$0 \rightarrow {}_2K_2O_E \cap \text{Tr}(E/F) \cap K_2O_E^2 \rightarrow {}_2K_2O_E \cap \text{Tr}(E/F) \xrightarrow{\psi_{E/F}} {}_2\text{Br}^S(E_0/F)/(-d_2, \Delta_F).$$

■

In the section, we will use Lemma 3.1 to compute the 4-rank of K_2O_E . Hence we need to investigate the condition of Lemma 3.1. In fact, if $\phi_{F(i)/F}$ is trivial, then there is always an admissible $y \in F(i)$ for $x \in R(E/F)$; specially, if $K_2O'_F(2) = 0$, then ${}_2\phi_{F(i)/F}$ is trivial, compare e.g. [2, proposition 2.17].

Let $F = \mathbb{Q}(\sqrt{-d_1})$ and $E = \mathbb{Q}(\sqrt{-d_1}, \sqrt{d_2})$ be an imaginary quadratic field and a biquadratic field, respectively. We have a quite general result:

Proposition 3.1 *Suppose that $r_4(C(F)) = 0$, $2 \notin NF$ and $C^S(F)$ has odd order. Then each element $x \in D(E/F)$ has an admissible element $y \in F(i)$.*

Proof. By Proposition 2.1, the Tate kernel $\Delta_F = \{2, m\}F^{*2}, m|d_1$. If $x \in D(E/F)$ and $N_{E/F}(x) = 2 \in \Delta_F$, then $y = 1 + i$ or $-1 + i \in F(i)$ satisfies $\text{Tr}_{F(i)/F}(\{i, y\}) = 1$ and $t_{E/F}(x) + t_{F(i)/F}(y) \neq 0$, so y is admissible for x . We only need to find an admissible element y for each element $x \in D(E/F)$, $N_{E/F}(x) = m$ or $2m \in \Delta_F$.

Case 1: $d_1 \not\equiv \pm 1 \pmod{8}$. By $r_4(C(F)) = 0$, Proposition 2.1 and [17, theorem 4.3], $r_4(K_2O_F) = 0$, $\Delta_F = \{2, m\}F^{*2}$ and $[P_m] = 1$ in the narrow class group $C(F_0)$ of $F_0 = \mathbb{Q}(\sqrt{d_1})$. Without loss of generality, we assume that there are integers $a, b \in \mathbb{N}$ such that

$$m = a^2 - d_1b^2.$$

Take $\alpha = a + (-\sqrt{-d_1}b)i \in F(i)$, then

$$\text{Tr}_{F(i)/F}\{i, \alpha\} = \text{Tr}_{F(i)/F}\{\alpha/a, \sqrt{-d_1}b/a\} = \{m/a^2, \sqrt{-d_1}b/a\}.$$

Note that $a^2 - d_1b^2 = m \in \Delta_F$ and $\text{Tr}_{F(i)/F}\{i, \alpha\}^2 = \{-1, m\} = 1$, then

$$(3.12) \quad \beta = \left\{ \frac{m}{a^2}, \frac{\sqrt{-d_1}b}{a} \right\} = \{-1, t\} \in K_2O_F$$

By $d_1 \not\equiv \pm 1 \pmod{8}$ and F imaginary,

$$\beta \in K_2 F^2.$$

Hence there is an element $\alpha' \in F(i)$ such that $t = N_{F(i)/F}(\alpha')$. We get such $y = t^{-1} \alpha \alpha'^2 \in F(i)$ with $Tr_{F(i)/F}(\{i, y\}) = 1$ and $N_{F(i)/F}(y) = m$. Hence (x, y) is an admissible pair.

Case 2: $d_1 \equiv 1 \pmod{8}$. By $r_4(C(F)) = 0$ and the process of proving Proposition 2.1 (2), we get that $r_4(C(F_0)) = 0$ and the ambiguous class $[P_n] = 1$ in $C(F_0)$, where $C(F_0)$ is the narrow class group of $F_0 = \mathbb{Q}(\sqrt{d_1})$ and $n \equiv \pm 3 \pmod{8}$ is a positive divisor of d_1 . Hence, without loss of generality, $n = a^2 - d_1 b^2$, $a, b \in \mathbb{N}$. Since the rank of Rédei $t \times t$ matrix M_{F_0} of F_0 is equal to $t - 1$ and $d_1 = p_1 \cdots p_t \equiv 1 \pmod{8}$, the system of equations

$$M_{F_0} X = \begin{pmatrix} \left(\frac{2}{p_1}\right)' \\ \vdots \\ \left(\frac{2}{p_t}\right)' \end{pmatrix}$$

is solvable over $\mathbb{F}_2$. Hence there must be a positive divisor $m \equiv \pm 1 \pmod{8}$ of d_1 such that the Diophantine equation

$$2mz^2 = x^2 - d_1 y^2$$

is solvable over $\mathbb{Z}$, see e.g. [17]. Hence $\{-1, m\} \in (K_2 O_F)^2$, so $m \in \Delta_F$ by Proposition 2.1 (2). Moreover, the ideal class $[\mathcal{D}][P_m] \in C(F_0)^2$, where $\mathcal{D}$ is a dyadic ideal and P_m is an ambiguous ideal over m . Since $r_4(C(F_0)) = 0$, $[\mathcal{D}][P_m]$ must be of odd order k , so $([\mathcal{D}][P_m])^k = [\mathcal{D}]^k [P_m] = 1$. Hence there are $a, b \in \mathbb{N}$ satisfying that

$$2^k m = a^2 - d_1 b^2.$$

By the process of proving case 1, we also find $y \in F(i)$ satisfying $Tr_{F(i)/F}(\{i, y\}) = 1$ and $N_{F(i)/F}(y) = m$. Hence (x, y) is an admissible pair.

Case 3: $-d_1 \equiv 1 \pmod{8}$ with $2 \notin NF$. By Proposition 2.1 $\Delta_F = \{2, m\} F^{*2}$, where $m \equiv 1 \pmod{4}$ is a positive divisor of d_1 . In the following, we prove that $m, 2m \notin N_{E/F}(E^*)$. We assume that primes $q_0 = 2, q_1, \dots, q_{t-2}$ satisfy that $\text{rank } M_1^S = t - 1$ in (2.7). Set $Q_i | q_i$, $i = 1, \dots, t - 2$.

Suppose that $m \in N_{E/F}(E^*)$, i.e., the Hilbert symbol $(m, d_2)_2 = 1$. By Minkowski-Hasse theorem, each local Hilbert symbol $(\frac{m, d_2}{Q_i}) = (\frac{m}{q_i}) = 1$, $i = 1, \dots, t - 2$. Since $m \equiv 1 \pmod{4}$, each Jacobi symbol

$$\left(\frac{q_i}{m}\right) = 1, \quad i = 1, \dots, t - 2.$$

Set $m' = d_1 m \equiv -1 \pmod{4}$, $1 = (\frac{-d}{q_i}) = (\frac{m}{q_i})(\frac{-m'}{q_i}) = (\frac{-m'}{q_i})$, $i = 1, \dots, t - 2$, hence the Jacobi symbol

$$\left(\frac{q_i}{m}\right) = 1, \quad i = 1, \dots, t - 2.$$

Moreover, by $d_1 = mm' \equiv -1 \pmod{8}$,

$$\left(\frac{2}{m'}\right) = \left(\frac{2}{m}\right).$$

It is contradictory to $\text{rank} M_1^S = t - 1$ in (2.7).

Suppose that $2m \in N_{E/F}(E^*)$, i.e., the Hilbert symbol $(2m, d_2)_2 = 1$. the local Hilbert symbols $\left(\frac{2m, d_2}{Q_i}\right) = \left(\frac{2m}{q_i}\right) = 1, i = 1, \dots, t - 2$. Since $m \equiv 1 \pmod{4}$, the Jacobi symbols

$$\left(\frac{q_i}{m}\right) = \left(\frac{2}{q_i}\right), i = 1, \dots, t - 2.$$

Set $m' = d_1/m \equiv -1 \pmod{4}$, $1 = \left(\frac{-4d_1}{q_i}\right) = \left(\frac{2m}{q_i}\right)\left(\frac{-2m'}{q_i}\right) = \left(\frac{-2m'}{q_i}\right), i = 1, \dots, t - 2$, hence the Jacobi symbol

$$\left(\frac{q_i}{m'}\right) = \left(\frac{2}{q_i}\right), i = 1, \dots, t - 2.$$

Moreover, by $d_1 = mn' \equiv -1 \pmod{8}$,

$$\left(\frac{2}{m}\right) = \left(\frac{2}{m'}\right).$$

It is also contradictory with $\text{rank} M_1^S = t - 1$ in (2.7).

Therefore $m, 2m \notin N_{E/F}(E^*)$, where $\Delta_F = \{2, m\}F^{*2}$. ■

Proposition 3.2 *Suppose that $r_4(C(F)) = 0$, $C^S(F)$ has odd order and $-d_1 \equiv 1 \pmod{8}$ with $2 \in NF$.*

(1) *If $-d_1 \equiv 1 \pmod{16}$, then each element $x \in D(E/F)$ has an admissible element $y \in F(i)$.*

(2) *If $-d_1 \equiv 9 \pmod{16}$ and $m(u + \sqrt{-d_1}) \in \Delta_F \cap N_{E/F}(E^*)$, $m|d_1, u^2 - 2w^2 \in \Delta_F$, then there is not an admissible pair (x, y) such that $N_{E/F}(x) = N_{F(i)/F}(y) = m(u + \sqrt{-d_1})$, $x \in D(E/F)$, $y \in F(i)$.*

Proof. Since $r_4(C(F)) = 0$ and $-d_1 \equiv 1 \pmod{8}$ with $2 \in NF$, by Proposition 2.1 $r_4(C(F_0)) = 0$ for $F_0 = \mathbb{Q}(\sqrt{d_1})$ and $\Delta_F = \{2, m(u + \sqrt{-d_1})\}F^{*2}$, where $-d_1 = u^2 - 2w^2, u, w \in \mathbb{N}, m|d_1$, and $u + m \equiv 2 \pmod{4}$. In the following, we will find an element $y \in F(i)$ such that $N_{F(i)/F}(y) = m(u + \sqrt{-d_1})$ and $\text{Tr}_{E/F}(\{i, y\}) = 1$.

By $r_4(C(F_0)) = 0$ and $2 \in NF$, the dyadic class $[\mathcal{D}]$ in $C(F_0)$ is trivial. Also $-d_1 = u^2 - 2w^2 = 2(u + w)^2 - (u + 2w)^2, u, w \in \mathbb{N}$, and

$$2(u + w)(u + \sqrt{-d_1}) = w^2 + (u + w + \sqrt{-d_1})^2.$$

Set $\alpha_1 = w + (u + w + \sqrt{-d_1})i = (u + 2w - \sqrt{d_1}) + (1 + i)(u + w) \in F(i)$. In the field $F_0 = \mathbb{Q}(\sqrt{d_1})$, by $(u + 2w)^2 - d_1 = 2(u + w)^2$

$$(u + 2w - \sqrt{d_1}) = \mathcal{D}P_{u+w}^2,$$

where $\mathcal{D}$ is the dyadic ideal and P_{u+w} is an integer ideal over $u + w$. Hence, in the class group $C(F_0)$, $[\mathcal{D}][P_{u+w}]^2 = [P_{u+w}]^2 = 1$ and $[\overline{P}_{u+w}]^2 = 1$, where $\overline{P}_{u+w}$ is the conjugate ideal of P_{u+w} . Hence there is an ambiguous class $[P_m]$ such that $[P_m][\overline{P}_{u+w}] = 1$, so there is the integral element $\alpha_2 = a + b\sqrt{d_1} \in O_{F_0}$ such that

$$(a + b\sqrt{d_1}) = P_m \overline{P}_{u+w} \text{ and } a^2 - d_1 b^2 = m(u + w).$$

Set $\alpha = \alpha_1\alpha_2 = e + fi \in F(i)$ and set p an odd prime, P a prime of F over p and $\mathcal{P}$ a prime of $F(i)$ over P . Then $v_P(N_{F(i)/F}(\alpha))/2 + v_{\mathcal{P}}(\alpha) \equiv 0 \pmod{2}$, for all odd primes P of F . Thus, by [6, proposition 1.5] or [2],

$$\begin{aligned}\beta &= \text{Tr}_{F(i)/F}\{i, \alpha\} = \left\{ \frac{2m(u + \sqrt{-d_1})(u + w)^2}{e^2}, -\frac{f}{e} \right\} \\ &= \text{Tr}_{F(i)/F}\{i, \alpha_1\} \text{Tr}_{F(i)/F}\{i, \alpha_2\} \\ &= \left\{ \frac{2(u + w)(u + \sqrt{-d_1})}{w^2}, -\frac{u + w + \sqrt{-d_1}}{w} \right\} \left\{ \frac{m(u + w)}{a^2}, -\frac{\sqrt{-d_1}b}{a} \right\} \in K_2O_F, \\ \beta^2 &= \{-1, m(u + \sqrt{-d_1})\} = 1,\end{aligned}$$

so

$$(3.13) \quad \beta = \text{Tr}_{F(i)/F}\{i, \alpha\} = \{-1, t\} \in K_2O_F.$$

For all non-dyadic finite primes P , the local Hilbert symbol $\eta_P(\beta) = 1$. Since F has two dyadic primes $\mathcal{D}_1, \mathcal{D}_2$, $t \in N_{F(i)/F}(F(i)^*)$ if and only if the Hilbert symbol $\eta_{\mathcal{D}_1}(\beta) = 1$. We need to compute the value of the local Hilbert symbol $\eta_{\mathcal{D}_1}(\beta)$ in a dyadic prime $\mathcal{D}_1$.

If $-d_1 \equiv 1 \pmod{16}$, then we assume that $-d_1 = u^2 - 2w^2$, $u, w \in \mathbb{N}$, $w \equiv 4 \pmod{8}$, and $u \equiv \pm 1 \pmod{8}$ (see [3, 14]). Set $\mathcal{D}_1 = (2, \frac{u - \sqrt{-d_1}}{2})$. Since $\frac{u - \sqrt{-d_1}}{2} \cdot \frac{u + \sqrt{-d_1}}{2} = \frac{w^2}{2} \equiv 0 \pmod{8}$,

$$\frac{u + \sqrt{-d_1}}{2} = u - \frac{u - \sqrt{-d_1}}{2} \equiv u \pmod{\mathcal{D}_1^3}$$

and

$$m \cdot \frac{u + \sqrt{-d_1}}{2} \equiv mu \equiv 1 \pmod{\mathcal{D}_1^3}$$

by $m \cdot (u + \sqrt{-d_1}) \in \Delta_F$, $m + u \equiv 2 \pmod{4}$. In the complete field $F_{\mathcal{D}_1} \cong \mathbb{Q}_2$, $N_{F(i)/F}(\alpha/2) = m \cdot \frac{u + \sqrt{-d_1}}{2} \cdot (u + w)^2 \equiv 1 \pmod{\mathcal{D}_1^3}$. Hence

$$\eta_{\mathcal{D}_1}(\text{Tr}\{i, \alpha\}) = \eta_{\mathcal{D}_1}\left(\left\{ \frac{2m(u + \sqrt{-d_1})(u + w)^2}{e^2}, -\frac{f}{e} \right\}\right) = 1.$$

Therefore, in the case, $\beta = \{-1, t\} \in (K_2F)^2$, i.e., $t \in N_{F(i)/F}(F(i)^*)$. By the process of proving Proposition 3.1, we find $y \in F(i)$ such that $N_{F(i)/F}(y) = m(u + \sqrt{-d_1})$ and $\text{Tr}_{F(i)/F}\{i, y\} = 1$. Therefore every element $x \in D(E/F)$ has an admissible element $y \in F(i)$.

If $-d_1 \equiv 9 \pmod{16}$, then we assume that $-d_1 = u^2 - 2w^2$, $u, w \in \mathbb{N}$, $w \equiv 4 \pmod{8}$, and $u \equiv \pm 3 \pmod{8}$. Hence

$$m \cdot \frac{u + \sqrt{-d_1}}{2} \equiv mu \equiv 5 \pmod{\mathcal{D}_1^3}.$$

In the complete field $F_{\mathcal{D}_1} \cong \mathbb{Q}_2$, $m \cdot \frac{u + \sqrt{-d_1}}{2} \cdot (u + w)^2 \equiv 5 \pmod{\mathcal{D}_1^3}$ and $\frac{f}{e} \equiv 2 \text{ or } 6 \pmod{\mathcal{D}_1^3}$ by $e^2 + f^2 = 2m(u + \sqrt{-d_1})(u + w)^2$. By [13, p.252 table]

$$\eta_{\mathcal{D}_1}(\text{Tr}\{i, \alpha\}) = \eta_{\mathcal{D}_1}(5, -\frac{e}{f}) = -1.$$

Hence $\beta = \text{Tr}\{i, \alpha\} \notin (K_2F)^2$, so there is not any element $y \in F(i)$ such that $N_{F(i)/F}(y) = m(u + \sqrt{-d_1})$ and $\text{Tr}\{i, y\} = 1$. ■

In the following we compute 4-rank of K_2O_E for the biquadratic field $E = \mathbb{Q}(\sqrt{-d_1}, \sqrt{d_2})$.

Theorem 3.1 Suppose $r_4(C(F)) = 0$, $C^S(F)$ have odd order and $d_1 \not\equiv 1 \pmod{8}$. Then $r_4(K_2O_E) = r_2({}_2K_2O_E \cap K_2O_E^2 \cap \text{Tr}(E/F))$.

Proof. Since $r_4(C(F)) = 0$ and $d_1 \not\equiv 1 \pmod{8}$, $r_4(K_2O_F) = 0$ by Proposition 2.1, i.e. $K_2O_F(2)$ is elementary. There is a homomorphism $\text{Tr} : K_2O_E(2) \rightarrow K_2O_F(2)$ for two 2-Sylow subgroups, so there is an exact sequence

$$0 \rightarrow K_2O_E(2) \cap \text{Tr}(E/F) \rightarrow K_2O_E(2) \xrightarrow{\text{Tr}} \text{Tr}(K_2O_E(2)) \rightarrow 0.$$

By $\text{Tr}(K_2O_E(2)) \subset K_2O_F(2)$ elementary, ${}_2K_2O_E \cap K_2O_E^2 \subset K_2O_E(2) \cap \text{Tr}(E/F)$. Then

$$r_4(K_2O_E) = r_2({}_2K_2O_E \cap K_2O_E^2) = r_2({}_2K_2O_E \cap K_2O_E^2 \cap \text{Tr}(E/F)). \blacksquare$$

Remark 3.1 If $d_1 \equiv 1 \pmod{8}$, then by Proposition 2.1 there is a positive divisor $n \equiv \pm 3 \pmod{8}$ of d_1 such that $\beta \in K_2O_F$, $\beta^2 = \{-1, n\} \neq 1$. Suppose there is not any $\beta \in K_2O_E$ of order 2 such that $\text{Tr}(\beta) = \{-1, n\}$, then we get the same result as Theorem 3.1. Suppose there is $\beta \in K_2O_E$ of order 2 such that $\text{Tr}(\beta) = \{-1, n\}$, then $r_4(K_2O_E) - r_2({}_2K_2O_E \cap K_2O_E^2 \cap \text{Tr}(E/F)) = 0$ or 1. In fact, we can determine the above conditions by the method of proving Theorem 2.1.

By Proposition 3.1, Proposition 3.2, Theorem 3.1 and [5, corollary 3.9], we get

Theorem 3.2 Suppose $r_4(C(F)) = 0$, $C^S(F)$ has odd order and the assumption of Proposition 3.2(2) does not hold, then there is an exact sequence

$$0 \rightarrow {}_2K_2O_E \cap K_2O_E^2 \cap \text{Tr}(E/F) \rightarrow {}_2(K_2O_F \cap \text{Tr}(E/F)) \xrightarrow{\psi_{E/F}} {}_2Br^S(E_0/F)/(-d_2, \Delta_F).$$

Moreover, if $d_1 \not\equiv 1 \pmod{8}$, then

$$r_4(K_2O_E) = 2s + r + s_2 - 1 - \epsilon - r_2(\text{Im}\psi_{E/F}). \blacksquare$$

It is clear that this result with the representative set of $R(E/F)$ and the action of $\psi_{E/F}$ yields a computational approach to the 4-rank of K_2O_E . For convenience, we describe the algorithm of $r_4(K_2O_E)$ in detail (see [5, P173]).

Let $Q_0, Q'_0, Q_1, \dots, Q_{s+r}$ denote the primes in S_f with Q_0, Q'_0 dyadic primes (note $Q_0 = Q'_0$ if $d_1 \not\equiv -1 \pmod{8}$). Let $[x_1], [x_2], \dots, [x_l]$ denote a basis of the cokernel of the map

$$(\Delta_F \cap N_{E/F}(E^*)) / F^{*2} \rightarrow D(E/F) / E^{*2}.$$

Since $[d_2] \notin \Delta_F / F^{*2}$, the above map is injective and $l = |S_f| = 2s + r + s_2$. We form the $(2s + r + s_2) \times (2s + r + s_2)$ matrix over $\mathbb{F}_2$ by replacing the 1's by 0's and the -1's by 1's

$$M_{E/F} = ((-d_2, t_{E/F}(x_i) + t_{F(i)/F}(y_i))'_{Q_j}),$$

where $y_i \in F(i)$ is admissible for x_i . By [5, lemma 3.8] the local Hilbert symbols $(-d_2, t_{E/F}(x_i) + t_{E/F}(y_i))_P = 1$ for all finite primes $P \notin S$. Hence by the quadratic reciprocity law $\text{rank} M_{E/F} \leq 2s + r + s_2 - 1$. By the definition of ϵ and Theorem 3.2 $r_2(\text{Im}\psi_{E/F}) = \text{rank} M_{E/F} - \epsilon$ and $r_2(K_2O_E \cap \text{Tr}(E/F)) = |S_f| - 1 - \epsilon$. By Theorem 3.2 and [5, lemma 5.1], we get

Corollary 3.1 *The assumption is as Theorem 3.2.*

$$(1) \ r_2(2K_2O_E \cap K_2O_E^2 \cap \text{Tr}(E/F)) = 2s + r + s_2 - 1 - \text{rank}M_{E/F}.$$

$$(2) \text{ If } d_1 \not\equiv 1 \pmod{8}, \text{ then } r_4(K_2O_E) = 2s + r + s_2 - 1 - \text{rank}M_{E/F}. \blacksquare$$

We now illustrate the method through some examples.

Example 3.1 *Let $F = \mathbb{Q}(\sqrt{-3 \times 7})$, $E = \mathbb{Q}(\sqrt{-3 \times 7}, \sqrt{5 \times 11 \times 13})$. Describe the structure of the 2-Sylow subgroup of K_2O_E .*

In Example 2.1, we know that $r_4(C(F)) = 0$, $C^S(F)$ has odd order, $\Delta_F = \{2, 7\}F^{*2}$ and the representative set of $D(E/F)/E^{*2}$. Since 5, 11 are decomposed in F , 13 is inert in F , $s = 2$, $r = 1$, $s_2 = 1$ and $2 \times 7 \in \Delta_F \cap N_{E/F}(E^*)$, let $x_1 = 2, x_2 = 3, x_3 = 11, x_4 = 13, x_5 = 1 + \sqrt{-21}, x_6 = 27 + \sqrt{715}$ and let $Q_0|2, Q_1|5, Q_2|5, Q_3|11, Q_4|11, Q_5|13$. There is the 6×6 matrix

$$M_{E/F} = \begin{pmatrix} (-d_2, x_1)'_{Q_0} & (-d_2, x_1)'_{Q_1} & * & (-d_2, x_1)'_{Q_3} & * & (-d_2, x_1)'_{Q_5} \\ (-d_2, x_2)'_{Q_0} & (-d_2, x_2)'_{Q_1} & * & (-d_2, x_2)'_{Q_3} & * & (-d_2, x_2)'_{Q_5} \\ (-d_2, x_3)'_{Q_0} & (-d_2, x_3)'_{Q_1} & * & (-d_2, x_3)'_{Q_3} & * & (-d_2, x_3)'_{Q_5} \\ (-d_2, x_4)'_{Q_0} & (-d_2, x_4)'_{Q_1} & * & (-d_2, x_4)'_{Q_3} & * & (-d_2, x_4)'_{Q_5} \\ (-d_2, x_5)'_{Q_0} & (-d_2, x_5)'_{Q_1} & * & (-d_2, x_5)'_{Q_3} & * & (-d_2, x_5)'_{Q_5} \\ (-d_2, x_6)'_{Q_0} & (-d_2, x_6)'_{Q_1} & * & (-d_2, x_6)'_{Q_3} & * & (-d_2, x_6)'_{Q_5} \end{pmatrix}$$

$$= \begin{pmatrix} * & 1 & 1 & 1 & 1 & 1 \\ * & 1 & 1 & 0 & 0 & 0 \\ * & 0 & 0 & 1 & 1 & 1 \\ * & 1 & 1 & 1 & 1 & 0 \\ * & 0 & 1 & 1 & 1 & * \\ * & * & * & 0 & 1 & * \end{pmatrix}.$$

In the following, we verify the value of $(-d_2, x_i)_{Q_j}$ of $M_{E/F}$.

(1) In the 5-th row of $M_{E/F}$, since $(\frac{22}{5}) = -1$, $(-d_2, 1 + \sqrt{-21})_{Q_3} = 1$ and $(-d_2, 1 + \sqrt{-21})_{Q_4} = -1$. Note $(-d_2, 1 + \sqrt{-21}) = (2 \cdot 5 \cdot 13 \cdot (1 - \sqrt{-21}), 1 + \sqrt{-21})$, we get other values.

(2) In the 6-th row of $M_{E/F}$, $x_6 = 27 + \sqrt{715} \in E$ and $N_{E/F}(x_6) = 14$, hence we need to find an x_6 -admissible element $y \in F(i)$. Since $7 \cdot 2^2 = 7^2 - 21 \cdot 1^2$ and $\alpha = 7/2 + i\sqrt{-21}/2 \in F(i)$, by (3.12) $\beta = \text{Tr}\{i, \alpha\} = \{4/7, \sqrt{-21}/7\} = \{28, \sqrt{-21}\} \in K_2O_F$ and $\beta^2 = 1$, in fact, $\beta = 1$ (see [15]). Then $y = \alpha(-1 - i)$ is admissible for x_6 and $x'_6 = t_{E/F}(x_6) + t_{E/F(i)}(y) = 47 - \sqrt{-21}$. Hence $N_{E/F}(x'_6) = 2230$ and $(\frac{2230}{11}) = -1$, so we get the values of the 6-th row.

Therefore $\text{rank}M_{E/F} = 5$ and $r_4(K_2O_F) = 0$. From the matrix $M_{E/F}$, we know that $\psi_{E/F}(\{-1, 3 \cdot 11\}) = 1$, so $\Delta_E = \{2, 7, -11\}E^{*2}$. $\blacksquare$

By Proposition 3.2, we can not use our method to investigate the 4-rank of K_2O_E in Example 2.2.

Example 3.2 *Let $F = \mathbb{Q}(\sqrt{-47})$, $E = \mathbb{Q}(\sqrt{-47}, \sqrt{-23})$. Describe the structure of the 2-Sylow subgroup of K_2O_E .*

It is clear that $r_2(C(F)) = 0$, $C^S(F)$ has odd order and 23 is inert in F . Since $-47 = 9^2 - 2 \cdot 8^2$, $\Delta_F = \{2, 9 + \sqrt{-47}\}F^{*2}$. By $20^2 + 23 \cdot 1^2 = 47 \cdot 3^2$, $\{-1, 20 + \sqrt{-23}\} \in K_2O_E$ and $Tr_{E/F}(\{-1, 20 + \sqrt{-23}\}) = \{-1, -1\}$, so $r_2(H) = 1$ in (2.8) of Theorem 2.2. It is clear that $2, 9 + \sqrt{-47} \in \Delta_F \cap N_{E/F}(E^*)$, i.e., $N_{E/F}(3 + \sqrt{-23}) = 2 \cdot 2^4$ and

$$x = \frac{43 - \sqrt{-47}}{12} + \frac{\sqrt{-23}(5 + \sqrt{-47})}{12}, \quad N_{E/F}(x) = 9 + \sqrt{-47}.$$

By Theorem 2.1 and Theorem 2.2, $r_2(K_2O_E \cap Tr(E/F)) = 0 + 1 + 2 - 1 - \epsilon = 2$, $r_2(K_2O_E) = 3$ and the representative set of ${}_2K_2O_E$ is

$$\{-1, 47\}, \{-1, 3 + \sqrt{-23}\}, \{-1, x\}, \{-1, 20 + \sqrt{-23}\},$$

where there is only one dependent relation.

In the following, we use Theorem 3.2 to investigate the 4-rank of K_2O_E . Let $Q_0|2, Q'_0|2$ and $Q_1|23$ and let $x_1 = 47, x_2 = (3 + \sqrt{-23})/4, x_3 = x$. We know that $y_2 = -1 - i \in F(i)$ is x_2 -admissible and we find an x_3 -admissible element. By $2(9 + \sqrt{-47}) = 8^2 + (1 + \sqrt{-47})^2$, Let $y'_3 = 8 + i(1 + \sqrt{-47}) \in F(i)$, then

$$\beta = Tr\{i, y'_3\} = \left\{ \frac{2(9 + \sqrt{-47})}{8^2}, -\frac{1 + \sqrt{-47}}{8} \right\} = \{1 + \sqrt{-47}, 4\}.$$

Since $\beta^2 = 1, \beta \in K_2O_F$ and $\beta \in K_2F^2$, $\beta = 1$ by $r_4(K_2O_F) = 0$ and $C(F)$ of odd order. Set $y_3 = -y'_3(1 + i)/2 = -\frac{7 - \sqrt{-47}}{2} - i\frac{9 + \sqrt{-47}}{2} \in F(i)$, then y_3 is x_3 -admissible. Hence $x'_2 = t_{E/F}(x_2) + t_{E/F}(y_2) = \frac{-1}{2}$ and $x'_3 = t_{E/F}(x_3) + t_{F(i)/F}(y_3) = \frac{1 + 5\sqrt{-47}}{6}$, so we get

$$M_{E/F} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Hence $\text{rank} M_{E/F} = 2$, $r_4(K_2O_E) = 0$ and $\Delta_E = \{2, -3 - \sqrt{-23}, 9 + \sqrt{-47}\}E^{*2}$. ■

Acknowledgments. The research was partially supported by NNSF grant of China (10371054).

References

- [1] J.Browkin and A.Schinzel, *On Sylow 2-subgroups of K_2O_F for quadratic fields F* , J. reine angew. Math.331 (1982) 101-113.
- [2] P.E.Conner and J.Hurrelbrink, *On the 4-rank of K_2O_F* , Can. J. Math. 41 (1989), 932-960.
- [3] P.E.Conner and J.Hurrelbrink, *On the 4-rank of the tame kernel $K_2(\mathcal{O})$ in positive definite terms*, J. of Number Th. 88, 263-282 (2001).
- [4] E.Hecke, *Lecture on the Theory of Algebraic Numbers*, GTM 77, Springer-Verlag, 1981.
- [5] J.Hurrelbrink and M.Kolster, *Tame kernels under relative quadratic extensions and Hilbert symbols*, J. reine angew. Math. 499(1998), 145-188.
- [6] M.Kolster, *The structure of the 2-Sylow subgroup of $K_2(O)$* , I, Comment. Math. Helv. 61 (1986), 576-588.
- [7] J.Milnor, *Introduction to Algebraic K-theory*, Ann. of Math. Stud. 72, Princeton Univ. Press, 1971.
- [8] J.Neukirch, *Class Field Theory*, Springer, Berlin, 1986.
- [9] H.Qin, *The 2-Sylow subgroups of the tame kernel of imaginary quadratic fields*, Acta Arith., 69 (1995), 153-169.
- [10] –, *The 4-rank of K_2O_F for real quadratic fields*, Acta Arith. 72 (1995), 323-333.
- [11] J.Tate, *Relations between K_2 and Galois cohomology*, Invent. Math., 36(1976), 257-274.
- [12] Tsuyoshi Uehara, *On the 4-rank of the narrow ideal class group of a quadratic field*, J. of Number Th. 31, 167-173(1989).
- [13] E.Weiss, *Algebraic Number Theory*, McGraw-Hill Book Company.Inc, 1963.
- [14] Q.Yue, *Elements of order 4 of the Hilbert kernel in quadratic number fields*, Acta Arith. 97 (2001), no.4, 295-311.
- [15] –, *The structure of certain K_2O_F* , J. of Mathematical Research & Exposition (Chinese), Vol.21,No.1 (2001), 1-6.
- [16] –, *Dyadic ideal, class group, tame kernel in quadratic number fields*, J. Pure and Applied Algebra, Vol 166 No.1-2 (2002), 229-238.
- [17] Q.Yue and K.Feng, *The 4-rank of the tame kernel versus the 4-rank of the narrow class group in quadratic number fields*, Acta Arith. 96(2000), no.2, 155-165.