Abstract.

The present review is devoted to the muon magnetic moment and its role in supersymmetry phenomenology. Analytical results for the leading supersymmetric one- and two-loop contributions are provided, numerical examples are given and the dominant $\tan \beta \text{sign}(\mu)/M_{\text{SUSY}}^2$ behaviour is qualitatively explained. The consequences of the Brookhaven measurement are discussed. The $2\sigma$ deviation from the Standard Model prediction implies preferred ranges for supersymmetry parameters, in particular upper and lower mass bounds. Correlations with other observables from collider physics and cosmology are reviewed. We give, wherever possible, an intuitive understanding of each result before providing a detailed discussion.
1 Introduction

The magnetic moment of the muon is one of the most precisely measured and calculated quantities in elementary particle physics. It has been measured recently at Brookhaven National Laboratory to a precision of 0.54 parts per million (ppm) [1–6]. The Standard
Model (SM) theory prediction has reached a comparable level. Thanks to this fantastic precision the comparison of theory and experiment is not only a sensitive test of all SM interactions but also of possible new physics at the electroweak scale, which is typically suppressed by a factor \((m_\mu/M_{\text{new ph.}})^2\) and can be expected to contribute at the ppm-level.

Indeed, the experimental result [3] published in 2001 showed a 4 ppm deviation with a statistical significance of almost 3\(\sigma\) from the SM prediction. This caused a lot of enthusiasm and could be nicely explained by a variety of new physics scenarios at the weak scale, in particular by weak-scale supersymmetry (SUSY). In the meantime several errors in the SM prediction have been corrected, but due to smaller error bars on the theoretical and the experimental side, the current 2 ppm deviation still has a significance of more than 2\(\sigma\) (if the SM prediction is based on \(e^+e^-\) data, see below). Of course, this deviation does not rule out the SM. However, it is intriguing that SUSY, which is widely regarded as one of the most compelling ideas for physics beyond the SM, would naturally lead to a deviation of the observed magnitude.

In the present review we describe the rich interplay between SUSY and the magnetic moment of the muon. In section 2 we present the analytical results for the SUSY contributions at the one- and two-loop level and discuss their main features. In section 3 the numerical values of the SUSY contributions as functions of the SUSY parameters are analyzed. In section 4 we discuss the implications of the experimental result for the possible values of SUSY parameters and the correlations with other observables. In the remainder of this introduction we review the status of the comparison of experiment and SM theory and provide some necessary background on SUSY and on the theory of magnetic moments.

1.1. Comparison of experiment and Standard Model theory

The magnetic moment \(\vec{\mu}\) of a particle with mass \(m\) and charge \(e\) is related to the particle spin \(\vec{S}\) by the gyromagnetic ratio \(g\):

\[
\vec{\mu} = g \left( \frac{e}{2m} \right) \vec{S}.
\]

At tree level, QED predicts the exact result \(g = 2\) for elementary spin-\(\frac{1}{2}\)-particles such as electron and muon. Quantum effects from QED loop diagrams, from strong or weak interactions, or from hypothetical new particles lead to a deviation

\[
a = \frac{1}{2}(g - 2),
\]

the so-called anomalous magnetic moment. The theoretical prediction of the anomalous magnetic moment of a lepton with mass \(m\) is dominated by the one-loop QED contribution, the famous Schwinger term \(\alpha/2\pi\) [7], followed by higher-order QED and strong interaction effects. Loop contributions from heavy particles with mass \(M\) are generally suppressed by a factor \(m^2/M^2\) as explained below in section 1.3. Therefore, the anomalous magnetic moment of the muon is a factor \((m_\mu/m_e)^2 \approx 40,000\) more sensitive to such contributions than the one of the electron.
The anomalous magnetic moment of the muon $a_\mu$ has been determined to an unprecedented precision of 0.54 parts per million (ppm) at the Brookhaven $g-2$ experiment E821 [1–6]:

$$a_\mu^{\exp} = 11659\,208.0\, (6.3) \times 10^{-10}.$$  \hspace{1cm} (3)

This is the first magnetic moment measurement that is sensitive to effects from physics at the electroweak scale, and it has the potential to constrain and discriminate between models of physics beyond the Standard Model (SM).

Inspired by the success of this experiment, the SM theory prediction of $a_\mu$ has been considerably refined and scrutinized in the last few years (see e.g. [8–12] for recent reviews and references). The SM prediction can be decomposed into QED, hadronic and weak contributions. The QED contribution is the largest, but it is theoretically well under control. The weak contributions are the smallest SM contributions and amount to $15.4(0.2) \times 10^{-10}$, but they are relevant at the current level of experimental sensitivity. Their structure and numerical value are comparable to the ones of potential new physics contributions.

Currently, the hadronic contributions are the main source of the SM theory uncertainty. They can be decomposed according to the hadronic subdiagrams into vacuum polarization and light-by-light scattering contributions. The hadronic vacuum polarization contributions can be inferred from experimental data on the hadronic $e^+e^-$ annihilation cross section using a dispersion relation. Thus ultimately the precision of these contributions is related to the precision of experimental data, which is constantly increasing due to ongoing measurements at Novosibirsk, $B$ factories and the $\phi$ factory DAΦNE. The alternative method of determining the hadronic vacuum polarization from data on hadronic $\tau$ decays suffers from theory uncertainties that are difficult to assess and is not in perfect agreement with the $e^+e^-$-based results. The current relative accuracy of these contributions is about 1%, corresponding to $7.2 \times 10^{-10}$ error in $a_\mu$. The hadronic light-by-light contributions cannot be related to experimental data and are notoriously difficult to evaluate. Early evaluations had a sign error, identified in [13,14], which led to a seemingly large deviation between the experimental result published in 2001 and the then current SM prediction [3]. Current estimates vary between $8.6(3.5) \times 10^{-10}$ [15] and $13.6(2.5) \times 10^{-10}$ [16].

For the purpose of the present review we will use the SM theory prediction given in [11], based on the most recent $e^+e^-$-based evaluations of the hadronic contributions [15,17] (see [18] for another recent evaluation, leading to a similar result):

$$a_\mu^{\text{SM}} = 11659\,184.1\,(7.2)^{\text{Vac.Pol.}}\,(3.5)^{\text{LBL}}\,(0.3)^{\text{QED/weak}} \times 10^{-10}$$  
$$= 11659\,184.1\,(8.0) \times 10^{-10}.$$  \hspace{1cm} (4)

Thus there is a deviation of about 2 ppm between the experimental result and the SM theory prediction:

$$\Delta a_\mu(\exp - \text{SM}) = 23.9\,(9.9) \times 10^{-10}.$$  \hspace{1cm} (5)

In order to appreciate the result further, we put it into historical context and compare it to the situation after the first measurement of $a_\mu$ at CERN in the 1970’s,
with the result [19]

\[ a_{\mu}^{\text{exp,1978}} = 11\,659\,240\,(85) \times 10^{-10}. \]  

(6)

This experiment was the first to be sensitive to hadronic contributions to \( a_{\mu} \), so it was interesting to compare its result to the theory prediction without hadronic contributions and to the hadronic contributions individually,

\[ \Delta a_{\mu}(\text{[exp,1978]} - \text{[SM without had]}) = 720\,(85) \times 10^{-10}, \]  

(7)

\[ a_{\mu,\text{had,1978}} = 667\,(81) \times 10^{-10}, \]  

(8)

where the central values and errors from Ref. [19] have been used. Without including the hadronic contributions there was a gap of more than 8\( \sigma \) between theory and experiment, but this gap was beautifully closed by the hadronic contributions. Hence the CERN experiment confirmed the existence of hadronic contributions to \( a_{\mu} \) and the correctness of the SM with a high significance.

Likewise, the new Brookhaven experiment is the first to be sensitive to weak interaction effects, and it is instructive to compare its result to the SM prediction without weak contributions and the weak contributions individually,

\[ \Delta a_{\mu}(\text{[exp]} - \text{[SM without weak]}) = 39.3\,(9.9) \times 10^{-10}, \]  

(9)

\[ a_{\mu,\text{weak}} = 15.4\,(0.2) \times 10^{-10} \]  

(10)

Again, without including the weak contributions there is a gap of about 4\( \sigma \), which establishes the existence of contributions beyond the QED and hadronic effects. However, in this case the gap (9) is about 2.5 times larger than the SM weak contribution (10).

The deviation (5) or the difference between (9) and (10) has a significance of about 2.4\( \sigma \). This is clearly not sufficient to prove the existence of physics beyond the SM. The deviation could be due to e.g. statistical fluctuations in the experimental result of \( a_{\mu} \) itself or the experimental data leading to the prediction of \( a_{\mu,\text{Vac.Pol.}} \), or the imperfect understanding of \( a_{\mu,\text{LBL}} \), or a combination of these effects.

Nevertheless, the result is tantalizing in view of new physics at the weak scale [20]. The existence of new physics at the weak scale has been suspected long before the \( a_{\mu} \) measurement of (3), for reasons related e.g. to the naturalness problem of the SM Higgs sector, grand unification, or cosmology and dark matter. Generically, new physics at a scale \( M \) can be expected to contribute at the order \( m_{\mu}^2/M^2 \) to \( a_{\mu} \), up to some numerical prefactors. For \( M \sim M_W \) such a contribution could easily amount to 2 ppm, and this is just the magnitude of the observed deviation (5). In the following we focus on supersymmetry as a particularly well-motivated and predictive idea for physics beyond the SM.

1.2. Relevant properties of the MSSM

The deviation between experimental and theoretical value of \( a_{\mu} \) could be due to contributions from supersymmetry. SUSY at the electroweak scale is one of the most
Table 1. The field/particle content of the MSSM. Only 2nd generation (s)leptons and 3rd generation (s)quarks are listed explicitly. The mass eigenstates corresponding to the electroweak gauge and Higgs bosons and their superpartners are indicated.

<table>
<thead>
<tr>
<th>(s)leptons</th>
<th>(s)quarks</th>
<th>Higgs</th>
<th>gauge bosons</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_\mu$, $\mu_R$, ...</td>
<td>$t_L$, $t_R$, $b_R$, ...</td>
<td>$H_1$, $H_2$</td>
<td>$B^\mu$, $W^{\alpha \mu}$, $G^{\alpha \mu}$</td>
</tr>
<tr>
<td>$\bar{\nu}_\mu$, $\bar{\mu}_L$, ...</td>
<td>$\bar{t}_L$, $\bar{t}_R$, $\bar{b}_R$, ...</td>
<td>$\tilde{H}_1$, $\tilde{H}_2$</td>
<td>$\tilde{B}$, $\tilde{W}^\alpha$, $\tilde{g}$</td>
</tr>
</tbody>
</table>

compelling ideas of physics beyond the SM (see e.g. [21–25] for reviews). SUSY is the unique symmetry that relates fermions and bosons in relativistic quantum field theories. It eliminates the quadratic divergences associated with the Higgs boson mass and thus stabilizes the weak scale against quantum corrections from ultra-high scales. SUSY at the weak scale also automatically leads to gauge coupling unification, and the lightest SUSY particle (LSP) can be neutral and stable and constitutes a natural candidate for cold dark matter. Moreover, in contrast to many other scenarios for physics beyond the SM, the minimal supersymmetric standard model (MSSM) is a weakly coupled, renormalizable gauge theory [26], such that quantum effects are computable and well-defined, and it has survived many non-trivial electroweak precision tests [27].

The MSSM is the appropriate framework for a general discussion of $a_\mu$ and SUSY. The unknown supersymmetry breaking mechanism is parametrized in terms of a set of in principle arbitrary soft SUSY breaking parameters. Specific models of supersymmetry breaking can be accommodated within the MSSM by suitable restrictions on these parameters.

The MSSM as the minimal supersymmetric extension of the SM contains all SM particles and corresponding SUSY partners, see table 1. In addition it also contains a second Higgs doublet with associated SUSY partner; hence the MSSM can actually be regarded as the SUSY version of the two-Higgs-doublet model (THDM). Two Higgs doublets $H_{1,2}$ of opposite hypercharge $\mp 1$ are required for the cancellation of chiral gauge anomalies caused by the corresponding Higgsinos. Thus the field content of the MSSM comprises the THDM fields, including five physical Higgs bosons, see (31)–(33) below, scalar SUSY partners of each chiral SM fermion, called sfermions $\tilde{f}_{L,R}$, Higgsino doublets $\tilde{H}_{1,2}$, and $U(1)$, SU(2) and SU(3) gauginos (called bino, winos and gluinos) $\tilde{B}$, $\tilde{W}^{\pm,3}$, $\tilde{g}$. Right-handed (s)neutrinos as well as non-vanishing neutrino masses are not relevant for this review and are ignored.

Two central MSSM parameters that are of particular importance for $a_\mu$ are related
to the two Higgs doublets. The first of these is the ratio of the two vacuum expectation values,
\[ \tan \beta = \frac{v_2}{v_1}. \]  
(11)

SUSY and gauge invariance require that the doublet \( \mathcal{H}_1 \) gives masses to down-type fermions, while \( \mathcal{H}_2 \) gives masses to up-type fermions. As a result, e.g. the top- and bottom-Yukawa couplings in the MSSM are enhanced by factors \( 1/\sin \beta \) and \( 1/\cos \beta \), respectively. In order to avoid non-perturbative values of these Yukawa couplings, \( \tan \beta \) is commonly restricted to the range between about 1 and 50. High values \( \tan \beta = \mathcal{O}(50) \) lead to similar top and bottom Yukawa couplings and are therefore favoured by the idea of top–bottom Yukawa coupling unification [28].

The second important parameter relating the two Higgs doublets is the \( \mu \)-parameter, which appears in the MSSM Lagrangian in the terms
\[ \mu \tilde{H}_1 \tilde{H}_2 - \mu F_{\mathcal{H}_1} \mathcal{H}_2 - \mu F_{\mathcal{H}_2} \mathcal{H}_1 + h.c. \]  
(12)

The first term describes a Higgsino mass term, while in the other terms \( F_{\mathcal{H}_1,2} \) are auxiliary fields whose elimination gives rise to interactions of \( \mathcal{H}_{1,2} \) with sfermions of the opposite type compared to the Yukawa couplings, e.g. to \( \mathcal{H}_1^0 \tilde{f}_R \tilde{f}_R^\dagger \) and \( \mathcal{H}_2^0 \tilde{\mu}_L \tilde{\mu}_L^\dagger \).

In addition the MSSM contains a large number of parameters that parametrize soft SUSY breaking. Except where explicitly stated we will restrict the number of these parameters by neglecting generation mixing in the sfermion sectors. Furthermore, we restrict ourselves to the case of \( R \)-parity conservation, since \( R \)-parity violating interactions have not much impact on \( a_\mu \).

The SUSY particles of particular importance to \( a_\mu \) are the smuons, muon-sneutrino, gauginos and Higgsinos since they appear in the SUSY one-loop contributions. At higher order, also other sectors of the MSSM become relevant, most notably the third generation squarks and the Higgs sector. Since in the MSSM all particles of equal quantum numbers can mix and these mixings have an important influence on the \( a_\mu \)-prediction, we briefly discuss the mixing of the individual sectors in the following.

The sfermions \( \tilde{f}_{L,R} \) for each flavour can mix, and the mass matrices corresponding to the \( \tilde{f}_L, \tilde{f}_R \) basis read
\[ M_f^2 = \begin{pmatrix} M_{LL}^2 & m_f X_f^* \\ m_f X_f & M_{RR}^2 \end{pmatrix}, \]  
(13)

where
\[ M_{LL}^2 = m_f^2 + m_{L,f}^2 + M_Z^2 \cos 2\beta (I_f^3 - Q_f s_W^2), \]  
(14)
\[ M_{RR}^2 = m_f^2 + m_{R,f}^2 + M_Z^2 \cos 2\beta Q_f s_W^2, \]  
(15)
\[ X_f = A_f - \mu^* \{ \cot \beta, \tan \beta \} \]  
(16)

with \( \{ \cot \beta, \tan \beta \} \) for up- and down-type sfermions, respectively. \( m_f, I_f^3 \) and \( Q_f \) denote the mass, weak isospin and electric charge of the corresponding fermion; \( s_W^2 = \sin^2 \theta_W = 1 - M_W^2/M_Z^2 \), where \( \theta_W \) denotes the weak mixing angle and \( M_{W,Z} \) the \( W \) and \( Z \) boson masses. The quantities \( A_f \) are soft SUSY breaking parameters.
for trilinear interactions of sfermions with Higgs bosons of the form $\tilde{f}_L \tilde{f}_R H$. The remaining entries $m_{L,R}$ of the diagonal elements are governed by the five independent soft SUSY-breaking parameters for each generation:

$$
m_{L,i} = m_{L,\tilde{b}} \equiv M_{Q3}, \quad m_{R,i} = m_{R,\tilde{t}} \equiv M_{U3},$$

$$
m_{L,\tilde{\mu}} = m_{L,\tilde{\nu}_\mu} \equiv M_{L2}, \quad m_{R,\tilde{\mu}} \equiv M_{R2},$$

(17) (18)

We have given these relations for 3rd generation squarks and 2nd generation sleptons as these are most important for our purposes. Analogous formulas hold for the other generations. The mass matrices can be diagonalized by unitary matrices $U_{\tilde{f}}$ in the form

$$U^f M^2_{\tilde{f}} U^{f\dagger} = \text{diag}(m^2_{\tilde{f}_1}, m^2_{\tilde{f}_2}),$$

(19)

and sfermion mass eigenstates can be defined by

$$\begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} = U^f \begin{pmatrix} \tilde{f}_L \\ \tilde{f}_R \end{pmatrix}.$$

(20)

The mass of the sneutrino $\tilde{\nu}_\mu$ is given by

$$m^2_{\tilde{\nu}_\mu} = m^2_{L,\tilde{\mu}} + \frac{1}{2} M_Z^2 \cos 2\beta$$

(21)

and similar for the other generations.

The superpartners of the charged gauge and Higgs bosons also mix, and the mass and mixing terms can be easiest expressed in terms of the Weyl spinor combinations $\psi^+ = (\tilde{W}^+, \tilde{H}_1^+, \tilde{H}_2^+)$ and $\psi^- = (\tilde{W}^-, \tilde{H}_1^- \tilde{H}_2^-)$. The mass term for these fields is given by $\psi^+ X \psi^+ + h.c.$ with the mass matrix

$$X = \begin{pmatrix} M_2 & M_W \sqrt{2} \sin \beta \\ M_W \sqrt{2} \cos \beta & \mu \end{pmatrix},$$

(22)

where $M_2$ is the soft SUSY breaking parameter corresponding to the SU(2) gaugino mass. This matrix can be diagonalized using two unitary matrices $U,V$ in the form

$$U^* X V^{-1} = \text{diag}(m_{\chi_1^\pm}, m_{\chi_2^\pm}).$$

(23)

Similarly, the mass matrix $Y$ corresponding to the superpartners of the neutral gauge and Higgs bosons in the basis $\psi^0 = (\tilde{B}, \tilde{W}^3, \tilde{H}_1^0, \tilde{H}_2^0)$ is given by

$$Y = \begin{pmatrix} M_1 & 0 & -M_Z s_W \cos \beta & M_Z s_W \sin \beta \\ 0 & M_2 & M_Z c_W \cos \beta & -M_Z c_W \sin \beta \\ -M_Z s_W \cos \beta & M_Z c_W \cos \beta & 0 & -\mu \\ M_Z s_W \sin \beta & -M_Z c_W \sin \beta & -\mu & 0 \end{pmatrix},$$

(24)

where $M_1$ is a U(1) gaugino (bino) soft SUSY breaking parameter and $c_W = M_W/M_Z$. It can be diagonalized with the help of one unitary matrix $N$ in the form

$$N^* Y N^{-1} = \text{diag}(m_{\chi_1^0}, \ldots, m_{\chi_5^0}).$$

(25)

The mass eigenstates corresponding to the charged and neutral gauginos and Higgsinos are called charginos and neutralinos, and they are related to the interaction eigenstates.
by
\[ \chi^+ = V_{ij} \psi^+_j, \quad \chi^- = U_{ij} \psi^-_j, \quad \chi^0 = N_{ij} \psi^0_j, \]
(26)
(27)
(28)
The gluinos do not mix, and their tree-level mass is given by \(|M_3|\) in terms of the SU(3) gaugino mass parameter \(M_3\).

The SUSY parameters \(\mu, A_f, M_{1,2,3}\) can be complex. However, not all complex phases can appear in observables. The only physical phases of the MSSM (beyond the phase in the CKM matrix) are the ones of the combinations
\[ \mu A_f, \quad \mu M_{1,2,3}. \]
(29)
Hence the frequently adopted convention that \(M_2\) is real and positive constitutes no restriction. Below, we will adopt this convention only in section 4 and remain general elsewhere.

After spontaneous symmetry breaking the two MSSM Higgs doublets lead to 5 physical Higgs bosons and 3 unphysical Goldstone bosons. Parametrizing the two doublets in the form
\[ \mathcal{H}_1 = \begin{pmatrix} v_1 + \frac{1}{\sqrt{2}} (\phi^0_1 - i\chi^0_1) \\ -\phi^-_1 \end{pmatrix}, \quad \mathcal{H}_2 = \begin{pmatrix} v_2 + \frac{1}{\sqrt{2}} (\phi^0_2 + i\chi^0_2) \end{pmatrix}, \]
(30)
the mass eigenstates are given by
\[ \begin{pmatrix} H^0 \\ h^0 \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \phi^0_1 \\ \phi^0_2 \end{pmatrix}, \]
(31)
\[ \begin{pmatrix} G^0 \\ A^0 \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} \chi^0_1 \\ \chi^0_2 \end{pmatrix}, \]
(32)
\[ \begin{pmatrix} G^\pm \\ H^\pm \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} \phi^\pm_1 \\ \phi^\pm_2 \end{pmatrix}, \]
(33)
where the mixing angle \(\alpha\) is related to \(\beta\) and the mass \(M_A\) of the CP-odd scalar \(A^0\) by
\[ \tan 2\alpha = \tan 2\beta \frac{M_A^2 + M_Z^2}{M_A^2 - M_Z^2}, \quad -\frac{\pi}{2} < \alpha < 0. \]
(34)
The physical Higgs degrees of freedom are the light and heavy CP-even scalars \(h^0, H^0\), the CP-odd scalar \(A^0\) and the charged Higgs bosons \(H^\pm\). For \(M_A \gg M_Z\) the masses of \(H^0\) and \(H^\pm\) are of the order \(M_A\). The three unphysical Goldstone bosons \(G^{0,\pm}\) are eaten to give masses to the \(Z\) and \(W^\pm\) bosons.

For the discussion of two-loop contributions to \(a_\mu\) with Higgs exchange it is important to note that the muon receives its mass from the doublet \(\mathcal{H}_1\). In the case that \(M_A \gg M_Z\) and \(\tan \beta\) is large the heavy Higgs bosons \(H^0, A^0\) and \(H^\pm\) are predominantly composed of \(\mathcal{H}_1\)-components. In this case they have larger couplings to the muon than the light Higgs boson \(h^0\).
1.3. Magnetic moment of the muon and chirality flips

The magnetic moment is a property of the muon in presence of an electromagnetic field. In quantum field theory it is related to the muon–photon vertex function \( \Gamma_{\mu \bar{\mu} A'}(p, -p', q) \), which has the covariant decomposition

\[
\bar{u}(p') \Gamma_{\mu \bar{\mu} A'}(p, -p', q) u(p) = e \bar{u}(p') \left[ \gamma_{\mu} F_V(q^2) + (p + p')_{\rho} F_M(q^2) + \ldots \right] u(p) \tag{35}
\]

for on-shell momenta \( p, p' \) and spinors \( u, \bar{u} \) that satisfy the Dirac equation. The charge renormalization condition implies \( F_V(0) + 2m_{\mu} F_M(0) = 1 \), and rearranging (35) leads to a term

\[
\frac{i e}{2m_{\mu}} [1 - 2m_{\mu} F_M(0)] \sigma_{\rho\nu} q^\nu \tag{36}
\]

for \( q^2 \to 0 \) in the covariant decomposition. This term describes the interaction of the muon dipole moment with a magnetic field, and the corresponding gyromagnetic ratio is given by \( g = 2[1 - 2m_{\mu} F_M(0)] \), or equivalently

\[
a_{\mu} = -2m_{\mu} F_M(0). \tag{37}
\]

In practical calculations it is often useful to extract the form factor \( F_M \) before evaluating the full vertex function with the help of projection operators \([29, 30]\).

It is important to understand where the generic behaviour \( a_{\mu}^{\text{SUSY}} \propto m_{\mu}^2/M_{\text{SUSY}}^2 \) comes from and how it can be modified by additional factors.† The \( 1/M_{\text{SUSY}}^2 \)-behaviour for SUSY masses of order \( M_{\text{SUSY}} \) reflects the decoupling properties of SUSY. The \( m_{\mu}^2 \)-behaviour of \( a_{\mu} \), or equivalently the relation \( F_M \propto m_{\mu} \) is related to chiral symmetry, and crucial modifications by additional factors are possible. The form factor \( F_M \) in (35) corresponds to a chirality-flipping interaction between the left- and right-handed muon. If the MSSM were invariant under the discrete chiral transformation

\[
\begin{pmatrix}
\nu_{\mu} \\
\mu_L \\
\mu_R \\
\tilde{\nu}_{\mu} \\
\tilde{\mu}_L \\
\tilde{\mu}_R
\end{pmatrix}
\rightarrow
\begin{cases}
+\begin{pmatrix}
\nu_{\mu} \\
\mu_L \\
-\mu_R \\
\tilde{\nu}_{\mu} \\
\tilde{\mu}_L \\
-\tilde{\mu}_R
\end{pmatrix}, \\
-\begin{pmatrix}
\nu_{\mu} \\
\mu_L \\
-\mu_R \\
\tilde{\nu}_{\mu} \\
\tilde{\mu}_L \\
-\tilde{\mu}_R
\end{pmatrix}
\end{cases}
\tag{38}
\]

of the left-handed doublets and right-handed singlets, \( F_M \) and thus \( a_{\mu} \) would vanish in the MSSM. \( F_M \) is proportional to \( m_{\mu} \) because the invariance of the MSSM under (38) is broken by the muon mass, or more precisely by all terms in the MSSM Lagrangian that are proportional to the muon Yukawa coupling. In each Feynman diagram that contributes to \( a_{\mu} \), the \( \mu \)-chirality has to be flipped by one of these terms. The main possibilities for the chirality flip are illustrated in figure 1 and are the following:

- at a \( \mu \)-line through a muon mass term, contributing a factor \( m_{\mu} \),
- at a Yukawa coupling of \( H_1 \) to \( \mu_R \) and \( \mu_L \) or \( \nu_{\mu} \), contributing a factor \( y_{\mu} \),
- at a \( \tilde{\mu} \)-line, corresponding to the transition \( \tilde{\mu}_L - \tilde{\mu}_R \), contributing a factor \( m_{\mu} X_{\mu} \approx m_{\mu} \tan \beta \mu \) for large \( \tan \beta \) from the smuon mass matrix,
- at a SUSY Yukawa coupling of a Higgsino to \( \mu \) and \( \tilde{\mu} \) or \( \tilde{\nu}_{\mu} \), contributing a factor \( y_{\mu} \).

† Logarithmic factors like \( \log M_{\text{SUSY}}/m_{\mu} \), which can appear at the two-loop level, are disregarded here.
The muon Yukawa coupling $y_\mu$ is given by

$$y_\mu = \frac{m_\mu}{v_1} = \frac{m_\mu g_2}{\sqrt{2} M_W \cos \beta},$$

(39)

where $g_2 = e / s_W$, and is thus enhanced by the factor $1 / \cos \beta \approx \tan \beta$ for large $\tan \beta$ compared to its SM value. Hence, while all four possibilities are proportional to the muon mass $m_\mu$, the last three are enhanced by a factor $\tan \beta$ for large $\tan \beta$. SUSY contributions to $a_\mu$ that make use of these enhanced chirality flips are themselves enhanced compared to the generic estimate $m_\mu^2 / M_{\text{SUSY}}^2$.

2. Analytic results

In a theory with unbroken SUSY, the gyromagnetic ratio of a charged fermion is exactly 2 and thus $a_\mu = 0$ [31, 32], that is the SUSY contribution exactly cancels the SM contribution. In the MSSM, SUSY is softly broken, and SUSY contributions to $a_\mu$ were studied already in the early 1980’s [33–39], when it was still conceivable that SUSY particles were significantly lighter than the $W$ and $Z$ bosons. Although very light SUSY particles are now experimentally excluded and SUSY contributions to $a_\mu$ are suppressed as $1 / M_{\text{SUSY}}^2$, it is still possible that these contributions are significant. In the following we first discuss the generic size of the SUSY one- and two-loop contributions on a more intuitive level; then we present the analytic results for the most important contributions.

2.1. How large can the SUSY contributions be?

Before presenting the full SUSY one-loop contributions to $a_\mu$ [33–44], it is instructive to discuss the dominant parameter dependence on an intuitive level and to obtain useful estimates. As shown in section 1.3 the contributions from SUSY particles of a generic mass $M_{\text{SUSY}}$ are of the order $m_\mu^2 / M_{\text{SUSY}}^2$, and hence suppressed by a factor $M_W^2 / M_{\text{SUSY}}^2$ compared to the SM electroweak contributions.

However, it has been observed in [37] and further stressed and discussed in [42–44] that the SUSY contributions can be significantly enhanced if $\tan \beta$ is large. Moreover, for large $\tan \beta$ the sign of the one-loop contributions is mainly determined by the sign of the $\mu$-parameter introduced in (12). We will see here that not only the one-loop but also the leading two-loop contributions behave in this way.
\( m_\mu^2 \tan \beta \mu M_2 F(\mu, M_2, m_\tilde{\mu}_L) \)

\( m_\mu^2 \tan \beta \mu M_1 F(M_1, m_\tilde{\mu}_{L,R}) \)

\( m_\mu^2 \tan \beta \mu M_1 F(\mu, M_1, m_\tilde{\mu}_R) \)

\( m_\mu^2 \tan \beta \mu M_2 F(\mu, M_2, M_{H_1}) \)

\( m_\mu^2 \tan \beta \frac{m_{\tilde{\mu}_R}}{M_W} (m_\tilde{\mu}_i X_i) F(m_{\tilde{\mu}_{L,R}}, M_{H_1^0}) \)

**Figure 2.** Five sample mass-insertion diagrams. Vertices and mass insertions are denoted by dots, and the interaction eigenstates corresponding to each line are displayed explicitly. The external photon has to be attached in all possible ways to the charged internal lines. The one-loop diagrams (C), (N1), (N2) have been discussed also in [44]. The loop functions \( F \) in the results are different in the five cases and depend on different masses.
It is easiest to understand the leading behaviour with the help of diagrams that are written in terms of interaction eigenstates, where the insertions of mass and mixing terms and chirality flips are explicitly shown \[44\]. The five diagrams in figure 2 exemplify the main enhancement mechanisms. The basic reason for the tan $\beta$-enhancement is the fact that the muon Yukawa coupling in the MSSM is larger by a factor $1/\cos\beta \approx \tan\beta$ for large $\tan\beta$ than its SM counterpart. This Yukawa coupling enters the diagrams in figure 2 in the vertices where the muon chirality is flipped, i.e. in the couplings of the muon to the Higgsino or Higgs boson in cases $(C,N2,C_2L,\tilde{t}_2L)$, and in the $\tilde{\mu}_L-\tilde{\mu}_R$ transition, given by $(M_\tilde{\mu})_{12}$, in case $(N1)$.

The second important parameter entering all five diagrams is the $\mu$-parameter, which governs the mixing between the two Higgs doublets. In all cases, the enhancement due to this mixing can be traced back to the fact that $H_2$ has the larger vacuum expectation value and strongly couples to top quarks, while only $H_1$ couples to muons.

In diagrams $(C,N2,C_2L)$ the $\mu$-parameter enters via the Higgsino $\tilde{H}_1-\tilde{H}_2$ transitions. These transitions enhance the diagrams because the following $\tilde{H}_2$-gaugino transitions are by a factor $v_2 : v_1 = \tan\beta$ larger than $\tilde{H}_1$-gaugino transitions. In diagram $(N1)$ $\mu$ enters via the dominant part of the smuon mixing. This mass insertion is obtained from the $F$-term $F_{H_1}H_2$, see (12), by replacing $H_2$ by its large vacuum expectation value. Finally, in diagram $(\tilde{t}_2L)$ the dominant part of the Higgs–stop coupling originates from $F_{H_2}H_1$ and thus enables $H_1$ to couple with the top-Yukawa coupling.

The remaining mass insertions in the diagrams provide additional factors of the gaugino mass $M_{1,2}$ and stop mixing parameter $X_t$. They are necessary in order to obtain an even number of $\gamma$-matrices in the fermion line and in order to connect $\tilde{t}_L$ and $\tilde{t}_R$, respectively. As an illustration, the relevant factors of diagram $(C)$ are given by

$$y_\mu X_{22} X_{12} X_{22} = \frac{m_\mu}{v_1} \mu (g_2 v_2) M_2 = g_2 m_\mu \tan\beta \mu M_2.$$  \hspace{1cm} (40)

Combining the enhancement factors of all diagrams leads to the estimates given in figure 2. They all have a similar form,

$$a_\mu^{(C,C_2L;N1,N2)} \propto m_\mu^2 \tan\beta \mu M_{2,1} F,$$  \hspace{1cm} (41)

$$a_\mu^{(\tilde{t}_2L)} \propto m_\mu^2 \tan\beta \frac{\mu m_t}{M_W^2} (m_t X_t) F,$$  \hspace{1cm} (42)

where $M_{2,1}$ corresponds to $(C,C_2L)$ and $(N1,N2)$, respectively. The loop functions $F$ are different in the five cases, depend on the masses appearing in the respective diagrams and generally behave as $F \propto M^{-4}_{\text{SUSY}}$ for large SUSY masses. In these formulas one power of $m_\mu$ is due to the $a_\mu - F_M$ relation (37) and gauge couplings have been suppressed.

Therefore all leading one- and two-loop contributions are approximately linear in $\tan\beta$, and their sign is given by the sign of $\mu$, together with the sign of $M_{1,2}$ or $X_t$. Generally, all diagrams are suppressed by two powers of the SUSY mass scale. Hence the basic behaviour of diagrams $(C,C_2L,N1,N2)$ is given by

$$a_\mu^{(C,C_2L;N1,N2)} \propto \frac{m_\mu^2}{M_{\text{SUSY}}^2} \tan\beta \text{sign}(\mu M_{2,1})$$  \hspace{1cm} (43)
if all SUSY masses are set equal to a common scale $M_{\text{SUSY}}$. However, it is important to keep in mind that the relevant SUSY masses are different in the five diagrams.

In particular, diagrams (N1) and ($\tilde{t}_2$) are special because they increase linearly with $\mu$, while the all other diagrams are suppressed for large $\mu$ by their $\mu$-dependent loop functions $F$. Likewise, only the one-loop diagrams are sensitive to the smuon and sneutrino masses. If these are large, the one-loop diagrams can be suppressed and the two-loop diagrams can become dominant. The chargino-loop diagram ($C_{2L}$) can be large if the chargino and Higgs masses are small.

The discussion of the stop-loop diagram ($\tilde{t}_2$) is complicated by the fact that the seemingly linear dependence on $(m_t X_t)$ is cut off by the requirement that both stop mass eigenvalues are positive. This diagram is largest for maximal stop mixing, i.e. if $(m_t X_t)$ is large but both eigenvalues are positive and $m_{\tilde{t}_1} \ll m_{\tilde{t}_2}$, and if $m_{\tilde{t}_1}$ and the Higgs boson mass are small. In this case, diagram ($\tilde{t}_2$) has the behaviour

$$a_{\mu}^{(\tilde{t}_2)} \propto \frac{m_{\mu}^2}{M_{\text{SUSY}}^2} \frac{\mu m_t}{M_W^2} \tan \beta \text{ sign}(X_t),$$

where $M_{\text{SUSY}}$ denotes here the common mass scale of the appearing Higgs boson and the lightest stop. Thus the diagram is linearly enhanced by large $\mu$, and its sign is determined by sign$(X_t)$. In the following subsections we will provide the exact analytical formulas for all these diagrams and also derive the numerical prefactors in the proportionailities (43) and (44).

2.2. One-loop contributions

Each diagram that contributes to $a_{\mu}$ contains one line carrying the $\mu$-lepton number. This fact allows to divide the MSSM one-loop diagrams into two classes:

(a) SM-like diagrams, where the $\mu$-lepton number is carried only by $\mu$ and/or $\nu_\mu$.
(b) SUSY diagrams, where the $\mu$-lepton number is carried also by $\tilde{\mu}$ and/or $\tilde{\nu}_\mu$.

The diagrams of the first class involve only SM-particles, and they are essentially identical in the SM and the MSSM. The only non-identical diagrams involve two couplings of physical SM or MSSM Higgs bosons to the muon line. Owing to the additional suppression factor $m_{\mu}^2/M_W^2$ such diagrams are entirely negligible both in the SM and the MSSM.

Therefore the SUSY one-loop contribution, i.e. the difference between $a_{\mu}$ in the MSSM and the SM, is given entirely by the diagrams of the second class. They are displayed in figure 3 and involve either a chargino–sneutrino or a neutralino–smuon loop. In contrast to the diagrams in figure 2 they are written in terms of interaction eigenstates, which is more appropriate for an exact evaluation. The diagrams have been evaluated in Refs. [33–36] with various restrictions on the masses and mixings. These restrictions have been dropped in Refs. [37–40], and exact results have been derived. Later, more comprehensive and general evaluations of these diagrams have been presented in the context of particular supersymmetric models [41–43] and the
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Figure 3. The two SUSY one-loop diagrams, written in terms of mass eigenstates. The external photon line has to be attached to the charged internal lines.

unconstrained MSSM [44] (see also [45, 46] for related results on weak dipole moments in the MSSM). We present the general result in the form given in [47]:

\[ a_{\mu}^{\text{SUSY,1L}} = a_{\mu}^{\chi^0} + a_{\mu}^{\chi^\pm}, \tag{45} \]

with

\[ a_{\mu}^{\chi^0} = \frac{m_\mu}{16\pi^2} \sum_{i,m} \left\{ -\frac{m_\mu}{12m_{\tilde{\mu}m}^2} (|n_{im}^L|^2 + |n_{im}^R|^2) F_1^N(x_{im}) + \frac{m_{\tilde{\mu}}^0}{3m_{\tilde{\mu}m}^2} \text{Re}[n_{im}^L n_{im}^R] F_2^N(x_{im}) \right\}, \tag{46} \]

\[ a_{\mu}^{\chi^\pm} = \frac{m_\mu}{16\pi^2} \sum_k \left\{ \frac{m_\mu}{12m_{\tilde{\nu}\mu}^2} (|c_k^L|^2 + |c_k^R|^2) F_1^C(x_k) + \frac{2m_{\tilde{\nu}m}^0}{3m_{\tilde{\nu}m}^2} \text{Re}[c_k^L c_k^R] F_2^C(x_k) \right\}, \tag{47} \]

where \( i = 1 \ldots 4 \) and \( k = 1, 2 \) denote the neutralino and chargino indices, \( m = 1, 2 \) denotes the smuon index, and the couplings are given by

\[ n_{im}^L = \frac{1}{\sqrt{2}} (g_1 N_{i1} + g_2 N_{i2}) U_{m1}^{\bar{\mu}} \bar{U}_{m2}^{\mu} - y_\mu N_{i3} U_{m1}^{\bar{\mu}} \], \tag{48} \]

\[ n_{im}^R = \sqrt{2} g_1 N_{i1} U_{m2}^{\bar{\mu}} + y_\mu N_{i3} U_{m1}^{\bar{\mu}} \], \tag{49} \]

\[ c_k^L = -g_2 V_{k1}, \tag{50} \]

\[ c_k^R = y_\mu U_{k2}. \tag{51} \]

The kinematic variables are defined as the mass ratios \( x_{im} = m_{\chi_{i}^0}/m_{\tilde{\mu}m}^2, \) \( x_k = m_{\chi_k^\pm}/m_{\tilde{\nu}m}^2, \) and the loop functions are given by

\[ F_1^N(x) = \frac{2}{(1-x)^4} [1 - 6x + 3x^2 + 2x^3 - 6x^2 \log x], \tag{52} \]

\[ F_2^N(x) = \frac{3}{(1-x)^3} [1 - x^2 + 2x \log x], \tag{53} \]

\[ F_1^C(x) = \frac{2}{(1-x)^3} [2 + 3x - 6x^2 + x^3 + 6x \log x], \tag{54} \]

\[ F_2^C(x) = \frac{3}{(1-x)^3} [-3 + 4x - x^2 - 2 \log x], \tag{55} \]

normalized such that \( F_1^\alpha(1) = 1. \) The U(1) and SU(2) gauge couplings are given by \( g_{1,2} = e/\{c_W, s_W\}, \) such that the one-loop contributions are of the order \( \alpha = e^2/(4\pi). \)

A class of large two-loop logarithms can be taken into account by the replacement \( \alpha \rightarrow \alpha(M_{\text{SUSY}}) \) (see later for more details).

For discussing the one-loop contributions \( a_{\mu}^{\chi^0,\pm} \) it is noteworthy that the terms linear in \( m_{\chi^0,\pm} \) are not enhanced by a factor \( m_{\chi^0,\pm}/m_{\mu} \) compared to the other terms. Rather,
these terms involve either an explicit factor of the muon Yukawa coupling $y_\mu$ or of the combination $U_{m_1}^*_\mu U_{m_2}^\mu/m_\mu^2$, which in turn is proportional to $(M^2_\mu)_{12}$ and thus to $y_\mu$. Hence, all terms are of the same basic order $m^2_\mu/M^2_{SUSY}$, and the terms linear in $m_{\chi^0,\pm}$ are enhanced merely by a factor $\tan \beta$ from the muon Yukawa coupling.

It is instructive to close this subsection by deriving a simple approximation of (46), (47) for large $\tan \beta$ and the case that all SUSY mass parameters in the smuon, chargino and neutralino mass matrices are equal to a common scale $M_{SUSY}$. In this case only the terms linear in $m_{\chi^0,\pm}$ have to be considered, and the loop functions $F_j(x)$ can be approximated by a Taylor series around $x = 1$. For example, the factors $m_{\chi^\pm}^k e_k^R \epsilon^R \omega^C (x_k)$ appearing in $a_{\mu}^{\chi^\pm}$ can be approximated as

$$-g_2 y_\mu \sum_k U_{k2} V_{k1} m_{\chi^\pm}^k \left( \frac{7}{4} - \frac{3 m_{\chi^\pm}^2}{4 m_{\nu_{\mu}}^2} \right) \approx \frac{3 g_2 y_\mu X_{22}(X^\dagger)_{21} X_{11}}{m_{\nu_{\mu}}^2} \approx \frac{3 g_2 y_\mu}{4} \text{sign}(\mu M_2) X_{12}. \quad (56)$$

Here terms that are suppressed by $1/\tan \beta$ or $M_W/M_{SUSY}$ have been neglected. Note that the factors on the right-hand side correspond directly to the mass-insertion diagram (C) in figure 2 and the approximation (43). The factors appearing in $a_{\mu}^{\chi^0}$ can be similarly approximated. Inserting these approximations, one obtains [44]

$$a_{\mu}^{\chi^0} = \frac{g_1^2 - g_2^2}{192\pi^2} \frac{m_{\mu}^2}{M_{SUSY}^2} \text{sign}(\mu M_2) \tan \beta \left[ 1 + \mathcal{O}\left( \frac{1}{\tan \beta}, \frac{M_W}{M_{SUSY}} \right) \right], \quad (57)$$

$$a_{\mu}^{\chi^\pm} = \frac{g_2^2}{32\pi^2} \frac{m_{\mu}^2}{M_{SUSY}^2} \text{sign}(\mu M_2) \tan \beta \left[ 1 + \mathcal{O}\left( \frac{1}{\tan \beta}, \frac{M_W}{M_{SUSY}} \right) \right], \quad (58)$$

where real parameters and equal signs of $M_1$ and $M_2$ have been assumed.

### 2.3. Two-loop contributions

It is useful to classify the MSSM two-loop diagrams similar to the one-loop diagrams, into

(a) two-loop corrections to SM one-loop diagrams, where the $\mu$-lepton number is carried only by $\mu$ and/or $\nu_\mu$.

(b) two-loop corrections to SUSY one-loop diagrams, where the $\mu$-lepton number is carried also by $\tilde{\mu}$ and/or $\tilde{\nu}_\mu$.

The first class contains in particular SM-like diagrams with an insertion of a loop of SUSY particles, e.g. of $\tilde{t}$, $\tilde{b}$ or $\chi^\pm$. Such diagrams are particularly interesting since they constitute SUSY two-loop contributions that involve other particles and have a completely different parameter dependence than the SUSY one-loop contributions. Most importantly, these two-loop contributions can be large even if $a_{\mu}^{SUSY,1L}$ is suppressed. The contributions of this class are exactly known [48, 49].

The SUSY two-loop contributions of the second class involve the same particles as the SUSY one-loop contributions (possibly plus additional ones). Hence they can be expected to have a similar parameter dependence as $a_{\mu}^{SUSY,1L}$. The contributions of this class are known in the approximation of leading QED-logarithms [50].
2.3.1. Two-loop corrections to SM one-loop diagrams  

The MSSM two-loop contributions of the first class can be decomposed into a SM- and SUSY-part,  

\[ a_\mu^{\text{SM,2L}} + a_\mu^{\text{SUSY,2L(a)}}, \]  

where \( a_\mu^{\text{SM,2L}} \) denotes the SM two-loop contributions. The genuine SUSY contributions of this class can be split into four parts:  

\[ a_\mu^{\text{SUSY,2L(a)}} = a_\mu^{\chi,2L} + a_\mu^{\tilde{f},2L} + a_\mu^{\text{SUSY,ferm,2L}} + a_\mu^{\text{SUSY,bos,2L}}. \]  

The first two terms correspond to diagrams involving a closed chargino/neutralino or sfermion loop, respectively. These diagrams are further categorized according to the particles coupling to the muon line,  

\[ a_\mu^{X,2L} = a_\mu^{(XVH)} + a_\mu^{(XVV)} + a_\mu^{(XVG)}, \quad X = \chi, \tilde{f}. \]  

Diagrams where one gauge boson and one physical Higgs boson couple to the muon line are denoted as \((XVH)\) with \(V = \gamma, W, Z\) and \(H = h^0, H^0, A^0, H^\pm\). Diagrams where only gauge bosons or unphysical Goldstone bosons couple to the muon are denoted as \((XVV)\), \((XVG)\).‡ Sample diagrams are shown in figure 4.

The remaining two terms in (60) correspond to diagrams involving only SM- or two-Higgs-doublet model particles and no SUSY particles. These diagrams are different in the MSSM and the SM due to the additional Higgs bosons and the modified Higgs boson couplings. \( a_\mu^{\text{SUSY,ferm,2L}} \) denotes the difference between the MSSM- and SM-evaluation of the diagrams involving a SM fermion (i.e. quark or lepton) loop; likewise, \( a_\mu^{\text{SUSY,bos,2L}} \) denotes the corresponding difference of the diagrams without fermion loop, the so-called bosonic contributions. Sample diagrams are shown in figure 5.

The SUSY two-loop diagrams can be conveniently evaluated by first applying a large mass expansion [51], where the muon mass is treated as small and all other masses as large. This results in a separation of scales, and all remaining integrals are of one of two types. One type are one-scale two-point integrals with external momentum \( p^2 = m_\mu^2 \) and all internal masses being either zero or equal to \( m_\mu \). The other type are integrals where all internal masses are heavy but the external momentum can be neglected. All these integrals and the corresponding prefactors can be evaluated analytically [48, 49].

In addition to the genuine two-loop diagrams, one-loop counterterm diagrams have to be evaluated. These contain renormalization constants corresponding to charge, mass, and tadpole renormalization, which are defined in the on-shell renormalization scheme [26, 52, 53].

The diagrams of classes \((XVH)\), \(X = \chi, \tilde{f}, f\) can be calculated in an alternative way. In these so-called Barr-Zee diagrams [54], a closed loop generates an effective \(\gamma-V-H\) vertex, and this vertex can be evaluated first by a one-loop computation. By inserting the result and performing the second loop integral one obtains a simple integral representation for the full two-loop diagram. Barr-Zee diagrams were first considered because they give rise to important contributions to electric dipole moments

‡ Diagrams of the form \((XHH)\), \((XHG)\) etc. in which two Higgs or Goldstone bosons couple to the muon line are suppressed by an additional muon Yukawa coupling and can be neglected.
in extensions of the SM (see e.g. Refs. [54–56]). The contributions from particular Barr-Zee diagrams of the classes \((f \gamma A^0)\), \((f \gamma H)\), \((\tilde{f} \gamma H)\), \((\tilde{f} W^\pm H^\mp)\) to \(a_\mu\) were considered in Refs. [57–60], respectively.

In Refs. [48, 49] the numerical results of all two-loop contributions in (60) were compared and analyzed in detail, taking into account that the SUSY parameters are constrained by experimental bounds on \(b\)-decays, \(M_h\) and other quantities. It turned out that the numerical values of the various contributions is very different:

- The by far largest contributions are the ones from the photonic Barr-Zee diagrams \((\chi \gamma H)\) and \((\tilde{f} \gamma H)\), where \(H\) denotes the neutral physical Higgs bosons \(h^0, H^0, A^0\). As explained in section 2.1 they are enhanced by a factor \(\tan \beta\) and, in the case of the sfermion loop diagrams, by the potentially large Higgs–sfermion coupling. They can have values up to

\[
a_\mu^{(\chi \gamma H)}, a_\mu^{(\tilde{f} \gamma H)} \sim \mathcal{O}(10) \times 10^{-10}.
\]
Barr-Zee diagrams with Z or W± exchange have a similar parameter dependence but are typically smaller by a factor of about 3–5.

- The diagrams of classes (χVV) and (fVV) and the corresponding Goldstone diagrams (χVΓ) and (fVΓ) involve no enhanced muon–Higgs Yukawa coupling and thus no tan β-enhancement, and they do not involve any other enhancement factors. Their numerical impact is tiny. For SUSY masses larger than 100 GeV these contributions are smaller than $0.1 \times 10^{-10}$.

- The genuine SUSY contributions to the SM-like diagrams $a_{\mu}^{\text{SUSY,ferm,2L}}$ and $a_{\mu}^{\text{SUSY,bos,2L}}$ depend only on tan β and $M_A$ and are small. Only for $M_A < 200$ GeV they can reach $10^{-10}$, but for larger $M_A$ they are typically below $0.5 \times 10^{-10}$.

Hence, for the purpose of the present review, we only present the analytical result for the dominant contributions from the photonic Barr-Zee diagrams with physical Higgs bosons. They can be written as

$$a_{\mu}^{(\gamma H)} = \frac{\alpha^2 m_{\mu}^2}{8\pi^2 M_W^2 s_W^2} \sum_{k=1,2} \left[ \text{Re}[\lambda_\mu^{A_0} \lambda_{\chi_k}^{A_0}] f_{PS}(m_{\chi_k^+}^2/M_A^2) \right. $$

$$+ \sum_{S=h^0, H^0} \text{Re}[\lambda_\mu^{S} \lambda_{\chi_k}^{S}] f_{S}(m_{\chi_k^+}^2/M_S^2) \right],$$

$$a_{\mu}^{(f\gamma H)} = \frac{\alpha^2 m_{\mu}^2}{8\pi^2 M_W^2 s_W^2} \sum_{f=\ell, b, t} \sum_{i=1,2} \left[ \sum_{S=h^0, H^0} (N_f Q^2_f) \text{Re}[\lambda_\mu^{S} \lambda_{\chi_k}^{S}] f_{f}(m_{\chi_k^+}^2/M_S^2) \right].$$

The Higgs–muon and Higgs–chargino coupling factors are given by

$$\lambda_{\mu}^{(h^0, H^0, A^0)} = \begin{pmatrix} -s_\alpha & c_\alpha \\ c_\beta & c_\beta \end{pmatrix},$$

$$\lambda_{\chi_k}^{(h^0, H^0, A^0)} = \frac{\sqrt{2} M_W}{m_{\chi_k^+}} \left( U_{k1} V_{k2} \{c_\alpha, s_\alpha, -c_\beta \} + U_{k2} V_{k1} \{-s_\alpha, c_\alpha, -s_\beta \} \right).$$

In the Higgs–fermion couplings we neglect terms that are subleading in tan β and that give rise to negligible contributions to $a_\mu$:

$$\lambda_{\ell_i}^{(h^0, H^0)} = \frac{2m_t}{m_{\ell_i}^2 c_\beta} (-\mu^* \{s_\alpha, -c_\alpha \} + A_i \{c_\alpha, s_\alpha \} \ (U_{i1}^T)^* U_{i2}^T, $$

$$\lambda_{b_i}^{(h^0, H^0)} = \frac{2m_b}{m_{b_i}^2 c_\beta} (-\mu^* \{c_\alpha, s_\alpha \} + A_b \{-s_\alpha, c_\alpha \} \ (U_{i1}^T)^* U_{i2}^T, $$

$$\lambda_{t_i}^{(h^0, H^0)} = \frac{2m_t}{m_{t_i}^2 c_\beta} (-\mu^* \{c_\alpha, s_\alpha \} + A_t \{-s_\alpha, c_\alpha \} \ (U_{i1}^T)^* U_{i2}^T.$$

The loop integral function $f_{PS}$ can be given either as a one-dimensional integral or in terms of dilogarithms:

$$f_{PS}(z) = z \int_0^1 \frac{dz \log \left( \frac{x(1-x)}{z} \right)}{x(1-x) - z} = \frac{2z}{y} \left[ \text{Li}_2 \left( 1 - \frac{1+y}{2z} \right) - \text{Li}_2 \left( 1 - \frac{1+y}{2z} \right) \right],$$

with $y = \sqrt{1-4z}$. Note that $f_{PS}(z)$ is real and analytic even for $z \geq 1/4$. The other loop functions are related to $f_{PS}$ as

$$f(z) = (2z - 1)f_{PS}(z) - 2z(2 + \log z),$$

$$f_s(z) = (2z - 1)f_{PS}(z) - 2z(2 + \log z),$$
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Figure 6. Sample two-loop diagrams contributing to $a_{\mu,2L(b)}^{\text{SUSY}}$, i.e. involving a SUSY one-loop diagram. The external photon can be attached to all charged internal lines. (a) shows a diagram with additional photon loop, giving rise to large QED-logarithms. (b) shows a diagram of the class computed in [63]. (c) shows a diagram with an additional fermion/sfermion loop.

$$f_f(z) = \frac{z}{2} \left[ 2 + \log z - f_{PS}(z) \right].$$  \hfill (72)

We remark that useful numerical estimates for the leading two-loop contributions can be obtained by taking into account only the $\tan \beta$-enhanced terms in the couplings and by approximating the loop functions. In the case of the sfermion-loop contributions, simple approximations for the most important Barr-Zee diagrams ($\tilde{t}HV$, $\tilde{b}HV$) with stop or sbottom loops can be derived [48]. The approximation for ($\tilde{t}HV$) agrees with the estimate (44) discussed in section 2.1, and the approximation for ($\tilde{b}HV$) has a similar form. In the case of the chargino/neutralino-loop diagrams the parameter dependence is simpler, and the diagrams with $W$- and $Z$-exchange can be included in the approximation [49]. We will collect all these approximations below in equations (81)–(83).

2.3.2. Two-loop corrections to SUSY one-loop diagrams

The two-loop corrections to SUSY one-loop diagrams can be decomposed into two pieces,

$$a_{\mu,2L(b)}^{\text{SUSY}} = c_s^{\text{SUSY,2L(b)}} \log \frac{M_{\text{SUSY}}}{m_{\mu}} + c_0^{\text{SUSY,2L(b)}}.$$  \hfill (73)

The first piece contains the large logarithm of the ratio $M_{\text{SUSY}}/m_{\mu}$, where $M_{\text{SUSY}}$ is the generic SUSY mass scale, and the second piece contains at most small logarithms of ratios of different SUSY masses. Sample diagrams are shown in figure 6. The diagrams all involve the same particles and the same couplings as the SUSY one-loop diagrams (possibly plus additional ones). Hence the overall parameter dependence of $a_{\mu,2L(b)}^{\text{SUSY}}$ and of $a_{\mu,1L}^{\text{SUSY}}$ can be expected to be similar, up to the additional two-loop suppression of $a_{\mu,2L(b)}^{\text{SUSY}}$. The large logarithm is the most relevant enhancement factor.

The decomposition (73) is analogous to the one of the bosonic two-loop contributions in the SM, which have been evaluated in [49,61,62]. In the case of the SM, the term enhanced by $\log M_Z/m_{\mu}$ is roughly a factor 10 larger than the non-logarithmic piece.

In the case of the MSSM, the logarithmic term is known [50], but for the non-logarithmic remainder only a particular subclass of diagrams has been computed recently.
In the following we first discuss the logarithmic term, which can be assumed to be dominant like in the SM, and then we describe the computation of \cite{63}.

The large logarithms in (73) are QED-logarithms and arise from two-loop diagrams that involve a SUSY one-loop diagram and an additional photon loop. The loop integrals of such diagrams have an infrared singularity in the limit $m_\mu \to 0$ and therefore give rise to terms $\propto \log m_\mu$. As shown by \cite{50}, the appropriate framework to evaluate these logarithms efficiently is the framework of effective field theories.

The relevant effective field theory is obtained from the MSSM by integrating out all fields of mass $\geq M_{\text{SUSY}}$ and retaining only the muon and photon. All further light or heavy SM fields are irrelevant in this analysis and can be ignored. The resulting theory is QED with additional higher-dimensional terms, described by

$$L_{\text{eff}} = -2\sqrt{2} G_\mu \sum_i C_i(\mu) O_i$$

in the notation of \cite{50}. The $O_i$ are higher-dimensional operators. The analysis of \cite{50} shows that in the MSSM, like in many new physics models, only one higher-dimensional operator has to be considered, namely the one corresponding to the muon anomalous magnetic moment,

$$H_\mu = -\frac{e}{16\pi^2} m_\mu \bar{\mu} \sigma^{\nu \rho} \mu F_{\nu \rho}.$$  \hspace{1cm} (75)

The prefactors $C_i(\mu)$ are renormalization-scale dependent Wilson coefficients, which can be determined by matching the effective theory to the full MSSM at the high scale $M_{\text{SUSY}}$. Determining the Wilson coefficient $C_{H_\mu}(M_{\text{SUSY}})$ thus corresponds to the one-loop computation of $a_\mu^{\text{SUSY}}$.

By construction, the large logarithms are identical in the full MSSM and the effective theory. However, in the effective theory the logarithms can be obtained simply from the one-loop renormalization-group running of the Wilson coefficient $C_{H_\mu}(\mu)$ from $\mu = M_{\text{SUSY}}$ down to $\mu = m_\mu$. This running is described by

$$C_{H_\mu}(\mu) = C_{H_\mu}(M_{\text{SUSY}}) - \gamma(H_\mu, H_\mu) \frac{\alpha(\mu)}{4\pi} \log \frac{M_{\text{SUSY}}}{\mu} C_{H_\mu}(M_{\text{SUSY}}).$$ \hspace{1cm} (76)

where $\gamma(H_\mu, H_\mu)$ is the anomalous dimension of $H_\mu$. On a diagrammatic level, the correspondence of this formula to the two-loop computation in the full MSSM is easy to see. In the MSSM, the logarithms arise from diagrams like the one in figure 6 (a). Corresponding diagrams in the effective theory are obtained by contracting the insertion of the SUSY one-loop diagram to a point. The resulting diagrams are one-loop contributions to $H_\mu$, involving the effective vertex $H_\mu$. Their UV-divergence, and thus their $\log \mu$-terms, determine the anomalous dimension $\gamma(H_\mu, H_\mu)$.

The value of the anomalous dimension is

$$\gamma(H_\mu, H_\mu) = 16.$$ \hspace{1cm} (77)

As a result, the QED-logarithms in the two-loop contributions to $a_\mu^{\text{SUSY}}$ are given by

$$a_\mu^{\text{SUSY,2L(b)}} = -\frac{4\alpha}{\pi} \log \frac{M_{\text{SUSY}}}{m_\mu} a_\mu^{\text{SUSY,1L}} + c_0^{\text{SUSY,2L(b)}}.$$ \hspace{1cm} (78)
This logarithmic correction is negative, and it amounts to $-7\% \ldots -9\%$ of the SUSY one-loop contributions for $M_{\text{SUSY}}$ between 100 and 1000 GeV. This result can be compared to the case of the bosonic SM two-loop contributions, where the logarithms amount to $-19\%$ of the SM electroweak one-loop result.

As mentioned before, the non-logarithmic terms in $a_{\mu,2L}^{\text{SUSY}}$, $c_0^{\text{SUSY,2L}}$, are not known so far. A first evaluation of a subclass of diagrams has been carried out in [63]. The considered diagrams involve only sleptons, charginos and neutralinos, and only topologies as in figure 6 (b) are taken into account that contain no self-energy subdiagrams. These diagrams constitute a finite contribution to $c_0^{\text{SUSY,2L}}$. However, the result of [63] can only be viewed as intermediate because there are more diagrams, e.g. containing self-energy subdiagrams or $W$- or $Z$-exchange, that would involve exactly the same coupling constants, such that non-trivial cancellations could be possible.

Nevertheless, the investigation of [63] provides a first insight to the possible values of the remaining two-loop contributions. The numerical values found in [63] are surprisingly large. In a range of SUSY parameters with SUSY masses of the order 300...500 GeV, the values are mostly below $10^{-10}$ but can become up to $2 \times 10^{-10}$, which is significantly larger than the corresponding non-logarithmic terms of the bosonic SM two-loop contributions.

2.4. Summary of known contributions and error estimate

To summarize, the SUSY contributions to $a_{\mu}$ up to the two-loop level, i.e. the difference of $a_{\mu}$ in the MSSM and the SM, are given by

$$
a_{\mu,2L}^{\text{SUSY}} = a_{\mu,1L}^{\text{SUSY}} \left( 1 - \frac{4\alpha}{\pi} \log \frac{M_{\text{SUSY}}}{m_\mu} \right) + a_{\mu}^{(\chi^+H)} + a_{\mu}^{(f^+H)} + a_{\mu}^{(\chi^{(W,Z)H})} + a_{\mu}^{(f^{(W,Z)H})} + a_{\mu,\text{form,2L}}^{\text{SUSY}} + a_{\mu,\text{bos,2L}}^{\text{SUSY}} + \ldots, \quad (79)
$$

where the terms in the first line have been given analytically in (45), (63), (64), (78).

Note that the discussion of the two-loop QED-logarithms and equation (76) also show that the one-loop result should be parametrized in terms of the running $\alpha(M_{\text{SUSY}})$. In practice, it is sufficiently accurate to approximate $\alpha(M_{\text{SUSY}})$ by $\alpha(M_Z) = 1/127.9$, and we define $M_{\text{SUSY}}$ in the logarithm as the mass of the lightest charged SUSY particle.

For many applications it should be sufficient to take into account the terms in the first line. The explicitly written terms in the second line are known, and they can be up to $O(1) \times 10^{-10}$, but in the largest part of the MSSM parameter space they are much smaller. The dots denote the known but negligible contributions of the type $(\chi V V)$, $(f V V)$, the contributions evaluated in [63], and the remaining unknown two-loop contributions.

Handy approximations for the dominant terms are given by

$$
a_{\mu,1L}^{\text{SUSY}} \approx 13 \times 10^{-10} \left( \frac{100 \text{ GeV}}{M_{\text{SUSY}}} \right)^2 \tan \beta \text{ sign}(\mu m_2), \quad (80)
$$

$$
a_{\mu}^{(\chi V H)} \approx 11 \times 10^{-10} \left( \frac{\tan \beta}{50} \right) \left( \frac{100 \text{ GeV}}{M_{\text{SUSY}}} \right)^2 \text{ sign}(\mu m_2), \quad (81)
$$
The first two are valid if all SUSY masses are approximately equal (note that the relevant masses are different in the two cases), and the third and fourth are valid if the stop/sbottom mixing is large and the relevant stop/sbottom and Higgs masses are of similar size. The result for the $\tilde{t}\gamma H$ contribution has not been here discussed before, but it can be understood in the same way as the $\tilde{b}\gamma H$ result [48].

In the following we list the missing contributions and estimate the theory error of the SUSY prediction (79).

- Two-loop QED-corrections beyond the leading logarithm (78). The leading-log approximation does not exactly fix the scale $M_{\text{SUSY}}$ in the logarithm and in $\alpha(M_{\text{SUSY}})$ (the latter appears in the one-loop result). The exact form of the logarithms and of the non-logarithmic terms can only be found by a complete computation of the two-loop diagrams with a SUSY one-loop diagram and additional photon exchange. The error of the approximation (78) can be estimated by varying $M_{\text{SUSY}}$ in the range 100...1000 GeV to about 2% of the SUSY one-loop contributions. If the SUSY contributions to $a_\mu$ are the origin of the observed deviation (5), they are certainly smaller than roughly $5 \times 10^{-10}$, and then this error is below $1 \times 10^{-10}$.

- Further electroweak and SUSY two-loop corrections to SUSY one-loop diagrams. These corrections include two-loop diagrams similar to figure 6 (a) but with $W$-, $Z$-, Higgs- instead of photon-exchange, and like in figure 6 (b) with purely SUSY particles in the loops. Given the result for the subclass evaluated in [63], we assign an error of $\pm 2 \times 10^{-10}$ to these diagrams. Note that this is a factor of 10 larger than the known result of the corresponding non-logarithmic bosonic two-loop contributions in the SM [49, 61, 62].

- Two-loop corrections to SUSY one-loop diagrams with fermion/sfermion-loops (see figure 6 (c) for an example). This class of diagrams involves in particular top/stop- and bottom/sbottom-loops, which are enhanced by the large 3rd generation Yukawa couplings. We estimate the numerical value of these diagrams to be less than $\pm 0.5 \times 10^{-10}$ for not too light SUSY masses for the following reasons. SUSY relates these diagrams to SM diagrams with top/bottom loops, which amount to about $0.6 \times 10^{-10}$ but are not suppressed by possibly heavy SUSY masses. SUSY also relates the fermion/sfermion-loop diagrams to pure sfermion-loop diagrams such as $a_\mu^{(\tilde{f}\gamma H)}$, see (82), (83). However, this relation should be most accurate for rather small $A_t$, $A_b$ and $\mu$, since the fermion/sfermion-loop diagrams are not enhanced by making these parameters large. In that case, the approximations (82), (83) lead to values below $0.5 \times 10^{-10}$.

- Three-loop contributions. In general, three-loop contributions can be expected to
be significantly smaller than the two-loop contributions. Two potential exceptions are three-loop diagrams that correspond to the two-loop contributions of the types \((\chi \gamma H), (\tilde{f} \gamma H)\) with subloop-corrections to the Higgs-boson masses or the \(b\)-quark Yukawa coupling. It is well-known that the one-loop corrections to the Higgs-boson masses, in particular to \(M_h\), and to \(y_b\) can be very large. Hence, in cases where the diagrams with \(h\)-exchange and/or sbottom loop are very large, the missing three-loop contributions could amount to \(\mathcal{O}(1) \times 10^{-10}\). Fortunately, however, the influence of the lightest Higgs boson mass and \(y_b\) on the \((\chi \gamma H), (\tilde{f} \gamma H)\) diagrams is small in the largest part of the MSSM parameter space. Hence we neglect the theory error associated with the missing three-loop contributions.

To summarize, we estimate the theory error associated with (79) to

\[
\delta a_{\mu}^{\text{SUSY}} (\text{unknown}) = 0.02 \ a_{\mu}^{\text{SUSY},1L} + 2.5 \times 10^{-10},
\]

where the errors associated with the individual classes of missing diagrams have been added linearly. If \(a_{\mu}^{\text{SUSY}}\) is approximated by only the first line of (79), the error increases by the neglected contributions in the second line. An upper limit of these can be well approximated by [48, 49]

\[
\delta a_{\mu}^{\text{SUSY}} (2\text{nd line}) = 0.3 \ (a_{\mu}^{(\chi \gamma H)} + a_{\mu}^{(\tilde{f} \gamma H)}) + 0.3 \times 10^{-10}.
\]

It should be noted that the error estimate is deliberately conservative. The later numerical analysis, see e.g. table 2, shows that often already the known two-loop contributions are much smaller than \(10^{-10}\). In these cases, it is reasonable to assume that the theory error due to the unknown higher-order corrections is also much smaller than the estimate (84). In any case, the theory error of the SUSY contributions is smaller than both the current SM theory error and the experimental uncertainty.

3. Numerical behaviour of the SUSY contributions

The SUSY contributions to \(a_{\mu}\) depend on the MSSM parameters in a complicated way. In this section we analyze the parameter dependence of the numerical results in detail. For each of the one- and two-loop contributions to \(a_{\mu}\) we will describe which parameters are more and which are less influential and explain the dominant behaviour. We discuss the parameter choices for which each contribution can become particularly large and the numerical values that can be expected for various typical parameter choices. Finally, we provide the values of the SUSY contributions to \(a_{\mu}\) for the benchmark “SPS” reference points [64].

3.1. One-loop contributions

The generic behaviour of the SUSY one-loop contributions to \(a_{\mu}\) is well described by (80). The suppression by \(1/M_{SUSY}^2\), the enhancement \(\propto \tan \beta\) and the dependence on the sign of \(\mu\) has been explained in section 2.1 using mass insertion diagrams. This generic \(\tan \beta \text{sign}(\mu)\) behaviour is illustrated by figure 7, which shows \(a_{\mu}^{\text{SUSY,1L}}\) for various values
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of the four mass parameters, and $m_{\tilde{L},\tilde{R}}$ for various values of $\tan \beta$ and $\mu > 0$ (panel (a)), $\mu < 0$ (panel (b)). The smaller values of $\tan \beta$ correspond to smaller $|a^\mu_{SUSY,1L}|$. The preferred 1σ region (5) is indicated in grey.

Figure 7. $a^\mu_{SUSY,1L}$ as a function of a common mass scale $M_{SUSY} = |\mu| = M_2 = m_{\tilde{L},\tilde{R}}$ for various values of $\tan \beta$ and $\mu > 0$ (panel (a)), $\mu < 0$ (panel (b)). The smaller values of $\tan \beta$ correspond to smaller $|a^\mu_{SUSY,1L}|$. The preferred 1σ region (5) is indicated in grey.

We visualize the behaviour of $a^\mu_{SUSY,1L}$ for $\tan \beta = 50$ in this four-dimensional parameter space with the help of figure 8. In this figure, $M_2$ is chosen to be the smallest of the four mass parameters, and $\mu$, $m_{\tilde{L},\tilde{R}}$, $m_{\tilde{R},\tilde{R}}$ are independently varied in the range $M_2 \ldots 5M_2$. The left and right panels correspond to the choices $m_{\tilde{L},\tilde{R}} = M_2$ (panel (a)) and $m_{\tilde{L},\tilde{R}} = 5M_2$ (panel (b)). The different colours correspond to the values $\mu = M_2$ (dark blue) and $\mu = 5M_2$ (light yellow). In all regions, the right-handed smuon mass is varied between $m_{\tilde{R},\tilde{R}} = (1 \ldots 5)M_2$. Keeping these parameter ratios fixed, $M_2$ is varied, and the resulting $a^\mu_{SUSY,1L}$ is plotted as a function of the mass $M_{LOSP}$ of the lightest

3.1.1. Dependence on mass parameters

In general, $a^\mu_{SUSY,1L}$ depends on $\tan \beta$ and six mass parameters $\mu$, $M_{1,2}$, $m_{\tilde{L},\tilde{R}}$, $m_{\tilde{R},\tilde{R}}$, $A_\mu$. In the following we will first concentrate on the behaviour of $a^\mu_{SUSY,1L}$ as a function of the four parameters

$$\mu, \ M_2, \ m_{\tilde{L},\tilde{R}}, \ m_{\tilde{R},\tilde{R}}$$

and fix $M_1$ by the GUT relation $M_1/M_2 = 5g_1^2/3g_2^2 \approx 1/2$ and set $A_\mu = 0$.

We visualize the behaviour of $a^\mu_{SUSY,1L}$ for $\tan \beta = 50$ in this four-dimensional parameter space with the help of figure 8. In this figure, $M_2$ is chosen to be the smallest of the four mass parameters, and $\mu$, $m_{\tilde{L},\tilde{R}}$, $m_{\tilde{R},\tilde{R}}$ are independently varied in the range $M_2 \ldots 5M_2$. The left and right panels correspond to the choices $m_{\tilde{L},\tilde{R}} = M_2$ (panel (a)) and $m_{\tilde{L},\tilde{R}} = 5M_2$ (panel (b)). The different colours correspond to the values $\mu = M_2$ (dark blue) and $\mu = 5M_2$ (light yellow). In all regions, the right-handed smuon mass is varied between $m_{\tilde{R},\tilde{R}} = (1 \ldots 5)M_2$. Keeping these parameter ratios fixed, $M_2$ is varied, and the resulting $a^\mu_{SUSY,1L}$ is plotted as a function of the mass $M_{LOSP}$ of the lightest
observable SUSY particle \[\mu\] defined here as \(\min(m_{\tilde{\mu}_1}, m_{\tilde{\chi}_1^\pm}, m_{\tilde{\chi}_2^0})\).

The two panels show that the \(\mu\)-dependence is quite different for small and for large \(m_{L,\tilde{\mu}}\). This intricate interplay between the parameters has been first studied in [44], where it has been shown that the mass-insertion diagrams of figure 2 provide an intuitive understanding of the parameter-dependence. In general, the chargino-diagram (C) dominates due to its large numerical prefactor in (58). In special parameter regions, the diagrams (N1) with bino-exchange and (N2) with \(\tilde{\mu}_R\)-exchange can become important.

- **Panel (a), dark blue:** In the simple case \(\mu = m_{L,\tilde{\mu}} = M_2\), the chargino diagram (C) dominates. There is hardly any suppression for heavy \(m_{R,\tilde{\mu}}\), which enters only via the smaller neutralino diagrams. Therefore the corresponding dark blue region in panel (a) is very narrow.

- **Panel (a), light yellow:** If \(\mu\) is increased, the chargino and most neutralino diagrams are suppressed. For large \(\mu\) the largest contribution can come from the bino-diagram (N1), which is linear in \(\mu\). However, (N1) is only large if both left- and right-handed smuons are light. The large spread of the yellow region in panel (a) corresponds mainly to this \(m_{R,\tilde{\mu}}\)-dependence of (N1). The upper border corresponds to small \(m_{R,\tilde{\mu}}\) (i.e. to \(\mu = 5M_2\), \(m_{L,\tilde{\mu}} = m_{R,\tilde{\mu}} = M_2\)) and leads to values of \(a_\mu^{\text{SUSY,1L}}\) that are even larger than the ones in the blue region. This shows that the largest \(a_\mu^{\text{SUSY,1L}}\) for a given value of \(M_{\text{LOSP}}\) is obtained for \(\mu \gg m_{L,R,\tilde{\mu}}, M_2\) via diagram (N1). The values at the lower border are suppressed by the large \(m_{R,\tilde{\mu}} = 5M_2\) and are roughly
by a factor 3 smaller.

- **Panel (b), light yellow:** The situation is reversed in panel (b) for large $m_{L,\tilde{\mu}}$. In this case large $\mu$ does not lead to an enhancement of diagram (N1) or any other diagram, since then all diagrams are suppressed either by the large $\mu$ or the large $m_{L,\tilde{\mu}}$. Consequently the result for $\mu = m_{L,\tilde{\mu}} = 5M_2$ is dominated by diagram (C) and is almost independent of $m_{R,\tilde{\mu}}$. Hence the yellow region in panel (b) is narrow.

- **Panel (b), dark blue:** For large $m_{L,\tilde{\mu}}$ but small $\mu$, however, diagram (N2) with $\tilde{\mu}_R$-exchange becomes important. It is the unique diagram that is not suppressed by the large $m_{L,\tilde{\mu}}$, but it has the opposite sign and a smaller numerical prefactor than diagram (C). The lower border of the blue region in panel (b) corresponds to the case of small $m_{R,\tilde{\mu}}$, i.e. $m_{R,\tilde{\mu}} = \mu / 5$, where the contributions of (C) and (N2) almost cancel. For larger ratios than displayed in the plot, $m_{L,\tilde{\mu}} / m_{R,\tilde{\mu}} > 5$, $a^{\text{SUSY,IL}}_\mu$ can even change sign. The upper border of the same region corresponds to the case of large $m_{R,\tilde{\mu}}$, where (C) dominates and (N2) is suppressed.

Figure 9 shows how the results are modified for different choices of $M_2$ and $\tan\beta$. Generally, larger values of $M_2$ lead to a suppression of the results without dramatic change of the qualitative behaviour, and smaller values of $\tan\beta$ lead to a suppression of the results due to the almost linear $\tan\beta$-dependence. Figure 9 (a) shows the same as figure 8 (a) except that $M_2$ has been replaced by $5M_2$, i.e. $M_2$ is the largest of the four mass parameters and the other three are varied between $M_2/5 \ldots M_2$. We find that the qualitative features are essentially unchanged. The larger value of $M_2$ leads to a suppression by a factor of roughly 3, and the largest contributions are now obtained for small $\mu = M_2/5$, corresponding to the dark blue region. Panel (b) shows the same again but for $\tan\beta = 5$ (and small $M_2$). Here the results are reduced by almost a factor 10 compared to the case with $\tan\beta = 50$, as expected.

The two mass parameters we have not explicitly discussed yet are $A_\mu$ and $M_1$. $M_1$ has generally less influence than $M_2$ since the $M_1$-independent chargino diagrams dominate in most of the parameter space. However, as exemplified by the discussion of figure 8 (a) there are situations where the bino-exchange diagram (N1) becomes important. If $M_1$ is not tied to $M_2$ by the GUT relation, it is possible to choose $M_2, \mu \gg M_1$. In this case one can obtain large contributions to $a^{\text{SUSY,IL}}_\mu$ from diagram (N1) even if both charginos are very heavy [47].

The dependence of $a^{\text{SUSY,IL}}_\mu$ on $A_\mu$ arises only via the smuon mixing matrix, where $A_\mu$ is accompanied by the term $\mu \tan\beta$, which is typically much larger. Therefore, $A_\mu$ is quite insignificant for the SUSY contributions to $a_\mu$. The small influence of $A_\mu$ was verified in [65] by means of a scan of the SUSY parameter space, where $|A_\mu|$ was varied up to 100 TeV.

### 3.1.2. CP violation and flavour violation

So far, we have neglected the possibilities of complex phases and generation mixing in the SUSY parameters. The influence of complex phases has been studied in [47,66,67]. Clearly, the phase of the $\mu$-parameter has
the most significant impact, corresponding to the sign(\(\mu\))-dependence in the real case. At the same time, this phase is strongly restricted by negative results for electric dipole moment (EDM) measurements. Nevertheless, non-negligible effects of the complex phases can be obtained \([66, 67]\) even if only small phases that do not violate the EDM-bounds are considered. On the other hand, CP invariance is not a critical symmetry for the magnetic moment, in contrast to e.g. chiral invariance. Therefore, CP-violating phases do not lead to new enhancement factors, and the largest SUSY contributions are naturally obtained for real parameters \([47]\).

The situation is different for generation mixing in the slepton sector. If the smuons can mix with staus, it is possible to obtain contributions where the chirality is flipped at a stau-line instead of a smuon-line and thus contributes a factor \(m_\tau\) instead of \(m_\mu\) \([44]\).

An example is provided by diagram (N1), where the \(\tilde{\mu}_L-\tilde{\mu}_R\) insertion is replaced by a \(\tilde{\tau}_L-\tilde{\tau}_R\) insertion. Due to the large enhancement factor \(m_\tau/m_\mu\), even a small smuon–stau mixing can lead to sizable corrections to \(a_{\mu}^{\text{SUSY,1L}}\) \([44]\).

### 3.2. Two-loop contributions

#### 3.2.1. Chargino/neutralino-loop contributions

The two-loop contributions from closed chargino/neutralino loops have the same linear \(\tan \beta\)-dependence and \(1/M_{\text{SUSY}}^2\)-suppression as the one-loop contributions, see \((81)\). The detailed parameter-dependence, however, shows interesting differences, as shown in \([49]\).

In contrast to the one-loop contributions, the two-loop chargino/neutralino diagrams are independent of the smuon mass parameters. They only depend on \(\mu\),
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Figure 10. Panel (a): \( a_\mu^{(\chi^{\pm}H), (\chi^{00})} \) as a function of the mass of the lightest observable supersymmetric particle \( M_{\text{LOSP}} = \min(m_{\chi^\pm}, m_{\chi^0}) \), for \( \tan \beta = 50 \) and \( M_A = 150 \) GeV and for three different ratios \( \mu/M_2 = 1, 3, 5 \) (from top to bottom). Panel (b): Contours of \( a_\mu^{\text{SUSY,1L}} + a_\mu^{(\chi^{\pm}H), (\chi^{00})} \) (solid border) and \( a_\mu^{\text{SUSY,1L}} \) alone (dashed line) in the \( (\mu - M_2) \)-plane for \( M_A = 200 \) GeV, \( m_{L,R,\tilde{\mu}} = 1000 \) GeV and \( \tan \beta = 50 \). The contours are at \((24.5, 15.5, 6.5, -2.5, -11.5, -20.5) \times 10^{-10}\) . Likewise, the colours of the fully drawn areas correspond to the following values: white: \( > 24.5 \times 10^{-10} \), lightest grey: \( (15.5 \ldots 24.5) \times 10^{-10} \), .. , black: \( < -20.5 \times 10^{-10} \). The plot has been taken from [49], and the contours and regions correspond to \( \Delta a_\mu(\exp - \text{SM}) \) as taken in that reference.

\( M_1,2, \) and the Higgs boson mass \( M_A \). Moreover, the dependence on these parameters is quite straightforward, while the \( \mu \)-dependence at the one-loop level is very complicated due to the different behaviour of diagrams (C), (N1) in figure 2, for example.

Figure 10 (a) shows the result of the two-loop chargino/neutralino contributions \( a_\mu^{(\chi^{\pm}H), (\chi^{00})} \) for three different ratios \( \mu/M_2 = 1, 3, 5 \) as a function of \( M_{\text{LOSP}} \), defined here as \( \min(m_{\chi^\pm}, m_{\chi^0}) \). The other parameters are chosen as \( \tan \beta = 50 \) and \( M_A = 150 \) GeV, and the bino mass parameter \( M_1 \) is determined by \( M_2 \) via the GUT relation. For \( M_{\text{LOSP}} > 100 \) GeV, contributions of up to \( 5 \times 10^{-10} \) can be obtained. This is a lot smaller than the largest possible one-loop contributions, but it can be a significant correction if the one-loop contributions are suppressed by heavy smuons and sneutrinos.

This can be seen immediately by comparing the two-loop contributions in figure 10 (a) with the one-loop contributions in the blue region in figure 8 (b), where \( m_{L,R,\tilde{\mu}} = 5M_2 = 5\mu \). In this case the two-loop correction can amount to more than 20% of the one-loop result.

The result for a given \( M_{\text{LOSP}} \) is largest if \( \mu \) and \( M_2 \) are equal. There is no enhancement for larger \( \mu \) similar to the one-loop diagram (N1). Instead, increasing \( \mu \) to \( \mu = 5M_2 \) leads to a suppression of the result by a factor 2...3; for even larger
ratios than the ones displayed in the figure, the result becomes even smaller.

For larger values of $M_A$, the result is suppressed as well. We have checked that the results in figure 10 (a) would be smaller by a factor 1.5 (for large $M_{\text{LOSP}}$) to 2.5 (for small $M_{\text{LOSP}}$) for $M_A = 300$ GeV instead of 150 GeV. This is in agreement with the analysis of the $M_A$-dependence in [49] for a wider selection of values for $\mu$ and $M_2$.

Figure 10 (a) does not contain the case $\mu < M_2$. It is straightforward to show from the exact expression (63) that the dominant contribution $a_{\mu}^{(\chi \gamma H)}$ is symmetric under the exchange $\mu \leftrightarrow M_2$. Therefore the behaviour for $\mu < M_2$ is very similar to the one for $\mu > M_2$ and does not need to be analyzed separately.

The importance of the two-loop corrections is also visible in figure 10 (b), which has been taken from [49]. Here the one-loop contributions and the sum of one- and two-loop corrections are shown in a contour plot in the $(\mu-M_2)$-plane. The smuon mass parameters are heavy, $m_{L,R,\tilde{\mu}} = 1000$ GeV, and $\tan \beta = 50$, $M_A = 200$ GeV. The white and lightest grey regions correspond to the 1σ-region around the experimentally preferred value (the value $24.5(9.0) \times 10^{-10}$ used for the plot is very similar to the one quoted in (5)). For the chosen parameters, this region extends up to $\mu, M_2 \leq 600$ GeV. However, it is clearly visible that if the two-loop contributions were neglected, the contours would shift considerably, and the 1σ region would extend only up to $\mu, M_2 \leq 500$ GeV.

### 3.2.2. Sfermion-loop contributions

The two-loop contributions from sfermion-loop diagrams can be even larger than the ones from chargino/neutranino-loop diagrams. As discussed in [48, 59, 60], however, and as the discussion in section 2.1 and formulas (82), (83) show, these large contributions arise only in special corners of parameter space. The stop-loop diagrams become large if the stop mixing is large, one stop is light, $\mu$ is very large and $M_A$ is small. Similarly, sbottom-loop diagrams become large for large sbottom mixing and if $A_b/M_A$ is very large.

Such parameter regions are quite restricted for several reasons. Primarily, small stop or sbottom masses in conjunction with large $\tan \beta$ can have dramatic effects on the SUSY predictions for the lightest Higgs boson mass and $b$-decays and are therefore constrained by experimental limits on these observables. Furthermore, large $\mu$ is restricted by the requirement that the sbottom mass eigenvalues are positive, and by naturalness arguments due to its appearance in the Higgs potential.

Figure 11 (a) shows the results of the two-loop contributions from sfermion-loop diagrams as a function of the lightest sfermion mass in two particular parameter scenarios. The parameters have been chosen such that they are in agreement with experimental bounds and that the results are sizable. The Higgs boson mass $M_A = 300$ GeV, $\tan \beta = 50$, the sfermion mass parameters $m_{L,i}, m_{R,i}, m_{R,b}, m_{L,b}, m_{R,\tau}, m_{L,\tau}$ have been chosen equal to a common scale $M_{\text{SUSY}}$. The trilinear parameter $A_t = -2M_{\text{SUSY}}$, which leads to maximal stop mixing and values of $M_h$, evaluated using *FeynHiggs* [68–71] including higher-order corrections, which are in agreement with current bounds.

The light yellow region of figure 11 (a) corresponds to $\mu = -A_t = 2M_{\text{SUSY}}$ and thus
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Figure 11. Panel (a): $a_{\mu}^{\text{eff}}$ as a function of the mass of the lightest observable supersymmetric particle $M_{\text{LOSP}} = \min(m_{\tilde{t}}, m_{\tilde{b}})$, for the parameter values given in the plot. The light yellow (dark blue) region corresponds to $\mu = -A_t = 2M_{\text{SUSY}}$ ($\mu = -2A_t = 4M_{\text{SUSY}}$), where $M_{\text{SUSY}}$ is the common sfermion mass parameter, and $A_b = A_\tau = 300\ldots 3000$ GeV. Panel (b): Maximum values of $a_{\mu}^{\text{eff}}$ obtained in a parameter scan in the region (87) if the experimental constraints described in the text are incrementally applied. The plot has been taken from [48].

to a situation where the lightest stop and sbottom masses are approximately equal. In the dark blue region, $\mu = -2A_t = 4M_{\text{SUSY}}$, and thus the lightest sbottom is significantly lighter than the lightest stop. In both regions, $A_b = A_\tau = -300\ldots -3000$ GeV. The signs of the parameters are such that the sbottom and stop contributions add up constructively. The lower borders of the yellow and blue bands correspond to small $|A_b|$ and thus mainly to the pure stop-loop contributions. The width of the bands is essentially due to the sbottom-loop contributions, which are approximately linear in $A_b$.

The larger value of $\mu$ in the blue region has two effects. On the one hand, the lightest sbottom is lighter than the lightest stop, and thus the sbottom-loop contributions have more relative importance. Hence the blue band is wider than the yellow band. On the other hand, since the stop-loop contributions increase linearly with $\mu$, they are enhanced as well in spite of the heavier stops, and hence the blue band lies at higher values of $a_{\mu}^{\text{eff}}$.

The upper borders of the two bands almost exhaust the limits found in [48] for the largest possible values of $a_{\mu}^{\text{eff}}$. Figure 11 (a), taken from [48], shows these largest possible values depending on which experimental constraints are taken into account. The allowed parameter range is

$$\tan \beta = 50, \quad M_{\text{SUSY}} \leq 1 \text{ TeV}, \quad |\mu|, |A_{\text{L,b}}| \leq 3 \text{ TeV}, \quad 150 \text{ GeV} \leq M_A \leq 1 \text{ TeV} \quad (87)$$

with common sfermion mass parameters $m_{\tilde{L},\tilde{t}} = m_{\tilde{R},\tilde{t}} = m_{\tilde{R},\tilde{b}} = m_{\tilde{L},\tilde{\tau}} = m_{\tilde{R},\tilde{\tau}} = M_{\text{SUSY}}$.
and $A_\tau = A_b$. The experimental constraints are $M_h > 106.4$ GeV, $\Delta \rho^{\text{SUSY}} < 4 \times 10^{-3}$, $\text{BR}(B_s \to \mu^+\mu^-) < 1.2 \times 10^{-6}$, $|\text{BR}(B \to X_s\gamma) - 3.34 \times 10^{-4}| < 1.5 \times 10^{-4}$, corresponding to conservative bounds taking into account experimental and theoretical uncertainties (see [48] and references therein).

If all experimental constraints are ignored, $a_{\mu}^{\text{2L}} > 15 \times 10^{-10}$ is possible, in agreement with the results of [59, 60]. Taking into account the Higgs boson mass limit reduces the maximum contributions drastically, and if all constraints are taken into account $a_{\mu}^{\text{2L}}$ turns out to be smaller than $5 \times 10^{-10}$ in the parameter region (87).§

Significantly larger values of these contributions can be obtained if the sfermion mass parameters are non-universal. In particular, if the ratio of $m_{R,\tilde{b}}$ and $m_{R,\tilde{t}}$ is either very large or very small, values of $a_{\mu}^{\text{2L}} > 15 \times 10^{-10}$ can be in agreement with all experimental constraints [48]. We will come back to this possibility in section 4.6, where general scans of the MSSM parameter space are described.

### 3.3. Results for SPS benchmark points

After having discussed the general parameter dependence of the individual SUSY contributions to $a_{\mu}$, we present here the results obtained in the Snowmass Points and Slopes (SPS) benchmark points [64]. These results provide an overview and reference of the SUSY contributions that can be expected in various well-motivated and often considered parameter scenarios. Furthermore, we use these results to assess the relative importance of the individual one- and two-loop contributions.

Table 2 shows the results of the known contributions to $a_{\mu}^{\text{SUSY}}$, split up according to (79). The QED-improved one-loop and the two-loop contributions with photon exchange in the first line of (79) are listed explicitly. The remaining contributions are combined into the ones from diagrams with charginos, neutralinos or sfermions and the purely SM-like ones.

The SPS points 1a, 1b, 3, 6 correspond to various minimal supergravity (mSUGRA)-scenarios with $\tan \beta$ between 10 and 30 and SUSY masses in the range 100...1000 GeV. They lead to predictions of $a_{\mu}$ very close to the observed value (5). The same is true for points 7, 8, which correspond to gauge-mediated SUSY breaking. The point SPS 2 does not fit so well to (5) because it corresponds to the focus-point region in which sfermions are very heavy. SPS 4, 5 involve very large/small $\tan \beta$, respectively, and therefore yield too high/low values for $a_{\mu}^{\text{SUSY}}$, and SPS 9, which corresponds to anomaly-mediated SUSY breaking, involves negative ($\mu M_{1,2}$) and thus leads to negative $a_{\mu}^{\text{SUSY}}$.

The two-loop corrections are very small in all cases. In general, the chargino- or sfermion-loop contributions with photon exchange can be the largest two-loop contributions, but in all SPS points they are suppressed by moderate values of $\tan \beta$.

§ For very low sfermion masses the parameter scenarios used in figure 11 (a) violate the experimental bounds, which is why the results displayed in figure 11 (a) are larger than the limits found in figure 11 (b) at $M_{\text{LOSP}} \approx 100$ GeV.
Table 2. Results of the SUSY contributions to $a_\mu$ in units of $10^{-10}$ for the SPS benchmark parameter points. The one-loop contributions include the two-loop QED-logarithms (78). The SUSY two-loop corrections to SM one-loop diagrams have been split up as $a_\mu^{SUSY,2L(\alpha)} = a_\mu^{(\chi\gamma H)} + a_\mu^{(\tilde{f}\gamma H)} + a_\mu^{SUSY,\chi+\tilde{f},rest} + a_\mu^{SUSY,\text{ferm+}bos,2L}$ into the photon-loop contributions, the remaining chargino/neutralino and sfermion contributions, and the bosonic and fermionic contributions.

<table>
<thead>
<tr>
<th>SPS Point</th>
<th>$a_\mu^{SUSY,1L(\text{improved})}$</th>
<th>$a_\mu^{(\chi\gamma H)}$</th>
<th>$a_\mu^{(\tilde{f}\gamma H)}$</th>
<th>$a_\mu^{SUSY,\chi+\tilde{f},rest}$</th>
<th>$a_\mu^{SUSY,\text{ferm+}bos,2L}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPS 1a</td>
<td>29.29</td>
<td>0.168</td>
<td>0.029</td>
<td>0.056</td>
<td>0.267</td>
</tr>
<tr>
<td>SPS 1b</td>
<td>31.84</td>
<td>0.273</td>
<td>0.044</td>
<td>0.106</td>
<td>0.222</td>
</tr>
<tr>
<td>SPS 2</td>
<td>1.65</td>
<td>0.032</td>
<td>-0.002</td>
<td>0.027</td>
<td>0.068</td>
</tr>
<tr>
<td>SPS 3</td>
<td>13.55</td>
<td>0.078</td>
<td>0.009</td>
<td>0.029</td>
<td>0.187</td>
</tr>
<tr>
<td>SPS 4</td>
<td>49.04</td>
<td>0.786</td>
<td>0.085</td>
<td>0.288</td>
<td>0.349</td>
</tr>
<tr>
<td>SPS 5</td>
<td>8.59</td>
<td>0.029</td>
<td>0.135</td>
<td>-0.046</td>
<td>0.153</td>
</tr>
<tr>
<td>SPS 6</td>
<td>16.87</td>
<td>0.125</td>
<td>0.015</td>
<td>0.044</td>
<td>0.230</td>
</tr>
<tr>
<td>SPS 7</td>
<td>23.71</td>
<td>0.236</td>
<td>0</td>
<td>0.089</td>
<td>0.282</td>
</tr>
<tr>
<td>SPS 8</td>
<td>17.33</td>
<td>0.163</td>
<td>-0.001</td>
<td>0.062</td>
<td>0.211</td>
</tr>
<tr>
<td>SPS 9</td>
<td>-8.98</td>
<td>-0.046</td>
<td>-0.002</td>
<td>-0.018</td>
<td>0.115</td>
</tr>
</tbody>
</table>

and/or rather high values of $M_A$. Particularly the sfermion-loop contributions are suppressed in addition by the heavy stops and sbottoms in all points except SPS 5. The results show, however, that the photon-exchange contributions are larger than $a_\mu^{SUSY,\chi+\tilde{f},rest}$ by a factor $\approx 3$, in agreement with the discussion in section 2.3.1 and the error estimate (85). Likewise, the two-loop corrections from SM-like diagrams with fermionic or bosonic loops are in the expected range $\approx (0.1 \ldots 0.3) \times 10^{-10}$.

4. Impact on SUSY phenomenology

In this section we assume that the MSSM is the correct description of physics at the weak scale and above and interpret the muon $g - 2$ measurement within this framework. We discuss the impact of the result for $a_\mu$ on the SUSY parameter space and its relation to other relevant observables.

The deviation $\Delta a_\mu(\text{exp} - \text{SM})$ is positive and larger than the pure SM weak contribution, see (5), (9), (10). We have seen before that the MSSM can easily accommodate this deviation, preferably for a rather small SUSY mass scale and/or large $\tan \beta$ and a positive $\mu$-parameter. In the following, we first focus on the general MSSM and show that specific, quantitative upper and lower bounds on SUSY masses can be derived from $a_\mu$. Then we compare $a_\mu$ to $b$-decays, the Higgs boson mass and electroweak observables, and neutralino dark matter. Several of these observables exhibit correlations with $a_\mu$, while others lead to complementary information on SUSY parameters.

Parameter bounds as well as correlations between observables become much more
severe in more constrained models than the MSSM. We discuss here the cases of minimal supergravity, gauge-mediated SUSY breaking and anomaly-mediated SUSY breaking. We conclude the section with a full parameter scan in the general MSSM that summarizes the current status of $a_\mu$ in SUSY.

4.1. Constraints from $a_\mu$ on the general MSSM

Already before the Brookhaven $g - 2$ experiment, the muon magnetic moment was an important observable in SUSY phenomenology. Since the SM theory prediction agreed with the older CERN measurement of $a_\mu$ [19], it was possible to derive lower bounds on SUSY masses [33–43]. Even after many of these bounds were superseded by LEP bounds, taking into account $a_\mu$ remained complementary in corners of parameter space where light SUSY particles could escape LEP detection [72].

In 2001, the publication of the Brookhaven result [3] caused a lot of excitement since it showed a deviation of $43(16) \times 10^{-10}$ (2.6$\sigma$) from the SM theory prediction at the time (which involved a sign error in the light-by-light contributions). Many authors interpreted this result as a possible signal for supersymmetry, and $a_\mu$ was studied in general SUSY models [20,47,65,73–81], with emphasis on correlations and implications for particular other observables [82–88], and special scenarios or extended models were considered [89–96]. A general conclusion was that significant constraints, in particular upper mass limits could be derived even in the general MSSM.

The current deviation (5) between the corrected SM result and the final experimental value is $23.9 (9.9) \times 10^{-10}$. This is smaller but statistically almost as significant as the one in 2001, and it still allows to stringently constrain the MSSM parameter space. Without assuming any specific scenario of SUSY breaking it is possible to derive both lower and upper bounds on the masses of SUSY particles. It is particularly encouraging that not only one but several SUSY masses can be bounded from above.

Figure 12 from [97] shows the maximum values of the four lightest SUSY particle masses, depending on the value of $a_{\mu}^{\text{SUSY}}$ and $\tan \beta$. These values are obtained from a scan of the parameters $M_2, m_{L,\tilde{\ell}}, m_{R,\tilde{\ell}}, \mu$ up to 2 TeV, assuming the GUT relation for $M_1$. The bounds are independent of the identity of the SUSY particles but they mainly restrict the chargino/neutralino and smuon/sneutrino masses as these are most relevant for $a_{\mu}^{\text{SUSY}}$. The figure shows meaningful bounds even for the fourth lightest SUSY particle. For example, for $\tan \beta = 10$, requiring that $a_{\mu}^{\text{SUSY}} > 14 \times 10^{-10}$ ($a_{\mu}^{\text{SUSY}} > 4.1 \times 10^{-10}$), corresponding to a 1$\sigma$ (2$\sigma$) band in (5), implies that four SUSY particles are lighter than about 500 GeV (1 TeV) and two of them are even lighter than 350 GeV (600 GeV). The upper mass limits are tighter for smaller values of $\tan \beta$, but all values of $\tan \beta$ allow for contributions $a_{\mu}^{\text{SUSY}}$ larger than $20 \times 10^{-10}$, so $a_\mu$ alone does not place a lower bound on $\tan \beta$ [47,97].

The upper mass limits are very promising for the search for SUSY particles at the LHC and ILC. At a linear $e^+e^-$ collider, SUSY particles can be pair-produced if they are lighter than half of the center-of-mass energy. At the LHC, weakly interacting particles
like charginos, neutralinos and smuons can be best studied in cascade decays of squarks, provided they are lighter than the squarks. In view of figure 12, combined with lower squark mass limits from Tevatron, these criteria might be satisfied for a number of SUSY particles.

One might argue that it is too aggressive to require that $a_{\mu}^{\text{SUSY}}$ is within the $1\sigma$ or $2\sigma$ band of (5), especially given the difficulty in assessing the theoretical errors of the SM hadronic contributions. Clearly, if e.g. a $3\sigma$ band is admitted, zero SUSY contributions are allowed, and the upper limits on SUSY masses disappear. However, interestingly even in a “super-conservative” approach significant parameter constraints can be derived [98]. The only requirement made in [98] was that $a_{\mu}^{\text{SUSY}}$ falls into the interval

$$-36.8 \times 10^{-10} < a_{\mu}^{\text{SUSY}} < 89.9 \times 10^{-10}$$

[98],

which was supposed to be a region that nobody could seriously argue with. At the
time of publication, it corresponded to the $5\sigma$ allowed region, now it corresponds roughly to the $6\sigma$ region. The bounds derived in [98] from (88) can therefore be regarded as definite bounds, independent of any future theoretical or experimental developments. In addition to (88), only the following assumptions have been made: all SUSY parameters are real, $|\mu| > M_2$, $M_1 = M_2/2$, smuon masses are greater than 95 GeV, and $|A_\mu|/m_\tilde{\mu}_2 < 3$ in order to avoid electric charge-violating minima.

Figure 13 shows an example of the (lower) mass bounds that can be derived in this super-conservative approach in the plane of the lighter chargino and heavier smuon mass. For each $\tan \beta$, the region below the corresponding line is excluded. The excluded regions grow with $\tan \beta$, but even for small $\tan \beta$, $a_\mu$ excludes regions of parameter space that are not excluded by any other experiment.

4.2. General correlations with other observables

Many observables can be related to $a_\mu$ in a meaningful way. Here we discuss four particularly interesting examples: rare $b$-decays, the neutralino dark matter density and detection rate, the Higgs boson mass, and electroweak precision observables. Since the signs of the parameters play an important role, we fix the convention $M_2 > 0$ for simplicity in this and the following subsections.

We do not discuss in detail the possibility of lepton flavour violation. As mentioned in section 3.1.2, $\tilde{\mu}-\tilde{\tau}$ mixing can lead to substantial effects in $a_\mu^\text{SUSY}$, and conversely the measurement of $a_\mu$ implies bounds on lepton flavour-violating parameters. Furthermore, the diagrams contributing to $a_\mu$ and processes like $\tau \to \mu \gamma$ and $\mu \to e \gamma$ have a similar structure and are correlated. Corresponding detailed studies can be found in [91,94,95,99].
4.2.1. \textit{B}-decays The rare \textit{b}-decays $b \to s\gamma$ and $B_s \to \mu^+\mu^-$ are similar to $a_\mu$ in two respects. They are loop-induced, and they involve a chirality flip in the $b$–$s$ transition. Correspondingly, the SUSY contributions to both branching ratios are enhanced by $\tan \beta$, in the second case even $\propto \tan^6 \beta$. In the case of $b \to s\gamma$ also $\text{sign}(\mu)$ plays a crucial role as it determines whether diagrams with $H^\pm$ exchange (always positive) and diagrams with chargino or gluino exchange (the leading terms are $\propto \text{sign}(\mu A_t)$, $\text{sign}(\mu M_3)$, respectively) interfere constructively or destructively. The current experimental results for the two branching ratios are \cite{100, 101}

$$
\text{BR}(b \to s\gamma) = (3.39^{+0.30}_{-0.27}) \times 10^{-4}, \quad (89)
$$

$$
\text{BR}(B_s \to \mu^+\mu^-) < 1.0 \times 10^{-7} \quad (95\% \text{ C.L.}). \quad (90)
$$

Since both results are well in agreement with the SM theory predictions, the possible SUSY contributions are bounded from above. As long as minimal flavour violation is assumed, this implies for example that destructive interference between the $H^\pm$ and $\chi^\pm$ contributions to $b \to s\gamma$, and thus the specific sign $\mu A_t < 0$ is favoured (see \cite{102, 103} for reviews and references and \cite{104} for a recent thorough analysis of $b$-physics and SUSY).

The interplay between rare $b$-decays and $a_\mu$ is particularly strong in the framework of specific models such as minimal supergravity or gauge-mediated SUSY breaking. Such models relate squark and slepton masses and make specific predictions on the signs of $A_t$ and $M_3$ and thus also the sign of $\mu$ favoured by $b \to s\gamma$. Therefore an interesting tension can arise between $b \to s\gamma$ and $a_\mu$. Both observables favour a particular sign of $\mu$, but this might or might not be the same, depending on the values of $A_t$ and $M_3$. And both depend on $\tan \beta$ and SUSY masses, but $a_\mu$ prefers rather large and $b \to s\gamma$ rather small SUSY contributions.

4.2.2. \textit{Neutralino dark matter density and detection rate} If the lightest neutralino is stable it provides a promising candidate for cold dark matter. In this case, two observables are of particular interest: the relic density and the detection rate, governed by the neutralino-nucleon cross section (for reviews and references see \cite{105, 106}). The spin-independent contribution $\sigma_{\text{SI}}$ to this cross section is yet another chirality-flipping quantity, and it is also enhanced by large $\tan \beta$. Moreover, for positive $\mu$ this cross section is typically larger than $10^{-10}$ pb, which should be accessible to future detectors, while negative $\mu$ would allow for cancellations that could strongly suppress $\sigma_{\text{SI}}$, such that neutralino dark matter detection would be out of reach. Hence, the preference of $a_\mu$ for positive $\mu$ and not too small $\tan \beta$ has significant impact on the prospect of dark matter detection \cite{74, 78, 82, 107–110}. Figure 14 from \cite{108} shows the spin-independent neutralino–nucleon cross section versus the neutralino mass for a wide range of the MSSM parameters $\mu$, $M_2$, $\tan \beta$, $M_A$, $A_{b,t}$, $m_0$, where $m_0$ is a common sfermion mass parameter. The circles denote points that satisfy the $a_\mu$ constraint. Imposing this constraint increases the minimum value of $\sigma_{\text{SI}}$ from $10^{-16}$ pb to $10^{-10}$ pb, significantly improving the prospect for direct detection of galactic neutralinos.

Under the assumption that standard cosmology (involving a fixed cosmological
constant and a certain amount of cold dark matter) is valid and that neutralinos constitute the only component of cold dark matter, the neutralino relic density $\Omega_\chi$ is fixed by astrophysical observations, in particular from WMAP [111]. Fixing $\Omega_\chi$ effectively selects a one-dimensional hyper-surface from the MSSM parameter space. However, the dependence of $\Omega_\chi$ on the MSSM parameters is rather uncorrelated with the one of $a_\mu$. Hence, the measurement of $a_\mu$ and the increasingly precise determination of the cold dark matter density are complementary probes of the MSSM parameter space.

4.2.3. Lightest Higgs boson mass The lightest Higgs boson mass $M_h$ is an important observable since it is tightly constrained in the MSSM. At tree level, $M_h$ must be smaller than $M_Z$, but the current lower LEP-bound is $M_h > 114.4$ GeV (95% C.L.) [112]. Large quantum corrections can reconcile the MSSM prediction for $M_h$ with the lower bound, but this leads to severe constraints on the MSSM parameter space. The main quantum corrections are enhanced by $m_t^4 \log(m_t^2/m_{\tilde{t}}^2)$, and large $\tan \beta$ and large stop mixing can lead to further enhancements (see [27] for a review and references). Therefore, like

$\|$ This bound applies only if the MSSM Higgs boson $h$ is SM-like, which however is the case in the largest part of parameter space, in particular for $M_A \gtrsim 150$ GeV.
$a_\mu$, the $M_h$-constraint favours larger $\tan \beta$, but unlike $a_\mu$ it also favours larger SUSY masses, at least in the stop sector.

Hence there can be a certain tension between the $a_\mu$- and $M_h$-constraints, particularly in models that connect stop and smuon masses. This tension has become stronger recently owing to the latest determination $m_t = 171.4(2.1) \text{ GeV}$ [113,114]. This value is lower than previous ones, and it increases the tendency of the $M_h$-constraint to favour rather large stop masses. In [115], the tension is illustrated for a class of SU(5) GUT models by plots of the largest possible $a_\mu^{\text{SUSY}}$ as a function of $M_h$. The higher the Higgs boson mass, the lower the possible values for $a_\mu^{\text{SUSY}}$. However, the current bounds on $a_\mu$ and $M_h$ can still be simultaneously satisfied in considerable parameter regions.

### 4.2.4. Electroweak precision observables

Finally, we briefly comment on electroweak precision observables, particularly on $M_W$ and the effective weak mixing angle $\sin^2 \theta_{\text{eff}}$. The present experimental values are [114,116]

$$M_W^{\text{exp}} = 80.392(29) \text{ GeV}, \quad (91)$$

$$\sin^2 \theta_{\text{eff}}^{\text{exp}} = 0.23153(16). \quad (92)$$

The SM predictions for these observables sensitively depend on the values of the SM input parameters, in particular on $m_t$ and $M_h$. For the experimentally preferred values of the input parameters, the SM predictions agree with quite well $M_W^{\text{exp}}$ and $\sin^2 \theta_{\text{eff}}^{\text{exp}}$. It has been noted that the agreement with experiment can be even better in the MSSM (see e.g. [117] for an analysis that takes into account the most recent experimental data). The SUSY contributions to both observables are dominated by the quantity $\Delta \rho$, which is sensitive to the breaking of isospin invariance and thus e.g. to the mass splittings in the stop/sbottom sector. These SUSY contributions are enhanced for smaller stop/sbottom masses and also depend on the chargino and neutralino masses, but they are not particularly sensitive to $\tan \beta$. Therefore, $a_\mu$ as well as the electroweak precision observables have a tendency to favour not too heavy SUSY masses, and it is fruitful to combine analyses of both kinds of quantities [118–120].

### 4.3. MSUGRA

In the general MSSM, supersymmetry breaking is parametrized by a large set of free parameters. In specific scenarios of supersymmetry breaking these parameters find an explanation in terms of some underlying physical mechanism. Typically, then, the MSSM parameters can be related to a much smaller set of more fundamental quantities. Such supersymmetry breaking scenarios are very predictive, and parameter bounds implied by experimental constraints as well as correlations between observables are much more stringent than in the general MSSM.

The first scenario we discuss here is minimal supergravity (mSUGRA) and the very similar constrained MSSM (CMSSM) (see e.g. [121] and references therein). In this scenario supersymmetry breaking is assumed to take place in a hidden sector
and to be transmitted to the observable sector via gravitational interactions. The Kähler potential of the underlying supergravity theory is assumed to be minimal, i.e. in particular to involve no generation-dependent couplings. Furthermore, at the GUT scale $M_{\text{GUT}} \approx 2 \times 10^{16}$ GeV the SM gauge interactions unify. The free parameters of this model are

$$m_0, m_{1/2}, A_0, \tan \beta, \text{sign}(\mu),$$

where $m_0$, $m_{1/2}$ and $A_0$ are the universal values of all scalar mass, gaugino mass, and $A$ parameters of the MSSM at the GUT scale. The low-energy values of the MSSM soft-breaking parameters are determined by renormalization-group running, and the value of $|\mu|$ is determined by the requirement that electroweak symmetry breaking leads to the correct value of $M_Z$. (The mSUGRA scenario also leads to a prediction of the gravitino mass, but in the present review we will not make use of this prediction.)

The phenomenology of mSUGRA has been studied extensively, in particular in view of the $a_\mu$ constraint (see [74, 77, 78, 81, 88, 122–124] for mSUGRA studies focussing on $a_\mu$ and [119, 120, 125–131] for recent general analyses of mSUGRA).

As an example, figure 15 (a) from [132] shows the mSUGRA $m_0-m_{1/2}$ plane for

**Figure 15.** (a) The $m_0-m_{1/2}$ plane of the CMSSM for $\mu > 0$, $\tan \beta = 10$, $A_0 = 0$, taken from [132]. The medium pink band and the thin black dashed lines show the 2σ and 1σ contours for $a_\mu$. The near-vertical lines correspond to $M_h = 114$ GeV (red, dot-dashed), $m_{\tilde{\chi}_1^+} = 104$ GeV (black, dashed); the medium green area is excluded by $b \to s\gamma$; and the light turquoise band satisfies the relic density constraint. In the dark red area the LSP is charged.

(b) Likelihood function $\chi^2_{\text{tot}}$ for the observables $a_\mu$, $b \to s\gamma$, $M_h$, $M_W$, $\sin^2 \theta_W$ in the CMSSM for $\tan \beta = 10$ and various values of $A_0$. $m_0$ is chosen to yield the central value of the relic density constraint. This figure has been taken from [120].
$A_0 = 0$, $\tan \beta = 10$, $\mu > 0$ including contours corresponding to the most important observables. The region preferred by $a_\mu$ at the $2\sigma$ and $1\sigma$ level is shown as the medium pink band and thin black dashed lines. The Higgs boson mass bound $M_h > 114$ GeV is satisfied to the right of the near-vertical dot-dashed red line at $m_{1/2} \approx 400$ GeV, and to the right of the black dashed line at $m_{1/2} \approx 150$ GeV $m_{\chi^\pm} > 104$ GeV. The plot displays clearly the tension between $M_h$, which is increased by larger $m_{1/2}$, and $a_\mu$, which prefers smaller SUSY masses, but there is a non-vanishing region where both constraints are satisfied. For larger $\tan \beta$ this overlap region grows.

The small green region around $m_{1/2} \approx 100$ GeV is excluded by the decay $b \rightarrow s\gamma$. As mentioned before, the good agreement between SM theory and experiment favours destructive interference between the various SUSY contributions. In mSUGRA, the dominant contributions are the ones with charged Higgs or chargino exchange, and destructive interference requires $\mu A_t < 0$. Since mSUGRA predicts $A_t < 0$ almost independently of $A_0$, positive $\mu$ is preferred by $b \rightarrow s\gamma$. It is a non-trivial success of mSUGRA that both $a_\mu$ and $b \rightarrow s\gamma$ prefer the same sign of $\mu$, and this is the reason why the green region in figure 15 (a) is so small. However, for larger $\tan \beta$ the $b \rightarrow s\gamma$-constraint excludes a larger part of the mSUGRA parameter space, and this counterbalances the preference of $M_h$ and $a_\mu$ for larger $\tan \beta$. Contours for the decay $B_s \rightarrow \mu^+\mu^-$ are not shown in figure 15, but for large $\tan \beta \approx 50$ this decay becomes relevant, too. The current CDF limit (90) already begins to constrain the mSUGRA parameter space [88, 133].

The constraint imposed by the requirement that the lightest neutralino relic density coincides with the cold dark matter density preferred by WMAP [111] at the $2\sigma$ level is shown by the light turquoise band. This band is very narrow and thus allows to fix e.g. $m_0$ as a function of $m_{1/2}$. But since the relic density band is essentially orthogonal to the other regions it is possible to satisfy all constraints simultaneously.

All observables considered in figure 15 (a) have been combined with the electroweak precision observables $M_W$ and $\sin^2 \theta_{\text{eff}}$ in [119, 120]. In these references, $\chi^2$ fits within mSUGRA have been performed, and remarkably there are mSUGRA parameter points that are consistent with all constraints. Figure 15 (b) shows an example of the total $\chi^2$ as a function of $m_{1/2}$ for $\tan \beta = 10$ and various values of $A_0$ ($m_0$ is always fixed by the relic density constraint). The minimum $\chi^2_{\text{tot}} = 2.55$ is obtained for $A_0 = m_{1/2} = 320$ GeV.

In fact, a similarly good fit quality with a $\chi^2_{\text{tot}} < 3$ can be achieved for all values of $\tan \beta$ between 10 and 50, and the preferred mass range for $m_{1/2}$ is always between 300 and 600 GeV. For lower $\tan \beta$, the $M_h$- and $a_\mu$-constraint are difficult to reconcile. For higher $\tan \beta$, $b \rightarrow s\gamma$ is a serious constraint. It should be noted that the significance of these tensions has grown recently since the experimental value of the top quark mass has gone down [120, 131]. Nevertheless, the fact that mSUGRA fits well to all observables and that the preferred mass range $m_{1/2} = 300 \ldots 600$ GeV is rather low is very encouraging also in view of forthcoming SUSY searches at colliders.
Figure 16. The $M_1$–$\tan \beta$ plane in GMSB for $\mu > 0$ and for (a) $M_{\text{mess}} = 10^6$ GeV, $N_{\text{mess}} = 1$ and (b) $M_{\text{mess}} = 10^{15}$ GeV, $N_{\text{mess}} = 5$, taken from [133]. The shown contours correspond to $a_\mu \times 10^{10} = 10, 26, 42, 58$ (blue, dashed), to $\text{BR}(B_s \to \mu^+ \mu^-)$ (red, solid), and to $M_h$ (black, dot-dashed). The light green region visible only in panel (b) is excluded by $b \to s\gamma$. They grey and dark brown regions are excluded by mass bounds on SUSY particles and the lightest Higgs boson.

4.4. Gauge-mediated SUSY breaking

Gauge-mediated SUSY breaking (GMSB) assumes that supersymmetry breaking is mediated from a hidden sector to the observable sector by gauge fields. In the simplest case there is an integer number $N_{\text{mess}}$ of such messenger gauge fields, and these gauge fields have mass $M_{\text{mess}}$ and form vector like $5 + \bar{5}$ representations of SU(5). A major advantage of GMSB over the mSUGRA framework is that flavour universality naturally follows from the symmetries of the messenger interactions (see [134] for a review of GMSB).

The low-energy properties of GMSB are described by the following parameters:

$$M_{\text{mess}}, N_{\text{mess}}, \Lambda, \tan \beta, \text{sign}(\mu),$$

where the mass scale $\Lambda$ is related to the SUSY breaking scale $\sqrt{F}$ by $\Lambda = F/M_{\text{mess}}$. $\Lambda$ determines the overall scale of the soft breaking parameters and can be traded for one of them, e.g. $M_1$. At $M_{\text{mess}}$ boundary conditions are imposed on the MSSM soft parameters, and renormalization group running is used to determine the MSSM spectrum at the electroweak scale.

The GMSB predictions for $a_\mu$ and related observables have been studied in [72, 135–137] and compared to the predictions of mSUGRA and other models in [47, 80, 133]. Figure 16 from [133] shows two contour plots in the $M_1$–$\tan \beta$ plane for $\mu > 0$ and for $M_{\text{mess}} = 10^6$ GeV, $N_{\text{mess}} = 1$ and $M_{\text{mess}} = 10^{15}$ GeV, $N_{\text{mess}} = 5$. In both
cases, the experimental result for $a_\mu$ can be easily accommodated by GMSB. For any given $M_1$ and $\tan \beta$, $a_\mu$ is larger in panel (b) mainly due to the relative suppression of the sfermion masses by $1/\sqrt{N_{\text{mess}}}$. However, curiously due to the other constraints the highest values for $a_\mu^{\text{SUSY}}$ can be obtained for smaller $N_{\text{mess}}$ [47].

Like in mSUGRA $a_\mu$ and $b \to s\gamma$ both favour the same, positive sign of $\mu$. Moreover, in GMSB both stops and charged Higgs bosons are typically rather heavy, especially for small $M_{\text{mess}}$. As a result, GMSB is not very significantly constrained by the $b$-decays [133]. Conversely, observation of non-standard effects in $b$-decays would seriously constrain GMSB models.

4.5. Anomaly-mediated SUSY breaking

In anomaly-mediated SUSY breaking (AMSB) SUSY breaking takes place in a hidden sector and is transmitted to the observable sector via the anomalous breaking of superconformal (or super-Weyl) invariance [138, 139]. In this scenario the gaugino and scalar mass soft parameters are related to the breaking of scale invariance, expressed in terms of the gauge $\beta$ functions and the anomalous dimensions of the matter fields. Most notably, the AMSB contributions to the gaugino masses are $M_i \propto -b_i\alpha_i$ with the one-loop coefficients $(b_i) = (3, -1, -33/5)$ of the SU(3), SU(2) and U(1) $\beta$ functions. While these AMSB contributions to the soft SUSY breaking terms are always present in hidden sector models, they are usually subdominant. In the absence of any larger contributions, AMSB leads to a very predictive framework that is qualitatively very different from mSUGRA or GMSB.

The parameters of minimal AMSB (mAMSB) are

$$M_{\text{aux}}, m_0, \tan \beta, \text{sign}(\mu).$$

(95)

$M_{\text{aux}}$ is a common scale for all soft parameters, and $m_0$ is a universal additional scalar mass term that does not originate from the super-Weyl anomaly. This term is necessary in order to avoid tachyonic sleptons.

A crucial feature of AMSB is that the signs of $M_3$ and $A_t$ (in the convention that $M_2 > 0$) are reversed compared to the mSUGRA and GMSB cases. As a result, the $b \to s\gamma$ constraint favours negative $\mu$ and thus negative $a_\mu^{\text{SUSY}}$. If the observed deviation of the experimental from the SM theory value of $a_\mu$ would have turned out to have a negative sign, AMSB would be favoured over virtually all other models of SUSY breaking [140, 141]. However, since the deviation is positive AMSB is disfavoured compared to other scenarios such as mSUGRA or GMSB [65]. This clearly shows how combining information on low-energy physics can discriminate between different well-motivated SUSY breaking mechanisms.

Figure 17 shows an example from [133] of the $m_0-\tan \beta$ plane of mAMSB for $M_{\text{aux}} = 50$ TeV and $\mu > 0$. The region for $\tan \beta > 30$ is almost entirely excluded by the $b \to s\gamma$ constraint (light green). For low $\tan \beta$, the Higgs boson mass bound and $a_\mu$ restrict the parameter space. Nevertheless, for moderate values of $\tan \beta$ and $m_0$ it is possible to consistently accommodate $b$-decays, $a_\mu$ and $M_h$ within mAMSB.
4.6. MSSM parameter scan

In this subsection we present a general, model-independent MSSM parameter scan that summarizes the current status of $a_\mu$ in SUSY. This scan shows the maximum results for $a_\mu$ in the MSSM if all parameters are independently varied and the range of SUSY masses for which the MSSM can accommodate the experimental result. The full set of one- and two-loop contributions in (79) are taken into account, and the results are compared with the current deviation between the experimental and SM result (5). The scan can be viewed as an update of the one presented in [65].

The MSSM parameters have been varied in the ranges

$$|\mu|, M_2, m_{L,j}, m_{R,j}, |A_f| \leq 3000 \text{ GeV}, \quad M_A = 90 \ldots 3000 \text{ GeV}, \quad \tan \beta = 50,$$

where the upper limit is motivated by naturalness arguments, and the lower limit on $M_A$ corresponds to the experimental limit. The parameter $\tan \beta$ has not been varied because the essentially linear $\tan \beta$ dependence of $a_\mu^{\text{SUSY}}$ has been sufficiently established before. All other parameters have been varied independently, except that $M_1$ has been fixed via the GUT relation, $A_\mu = 0$, $m_{L,\tilde{b}} = m_{L,\tilde{t}}$, $m_{R,\tilde{b}} = m_{R,\tilde{t}}$, and $A_b = A_\tau$. These constraints do not have much impact on the maximum contributions [65]. The 3rd generation sfermion parameters are significantly restricted by experimental constraints on $M_h$, $\Delta \rho$ and $b$-decays. Only parameter points have been considered that are in agreement with these constraints, according to the same criteria as in the “weak bounds” of [48]. Note that the precise values used in these constraints have not much influence on the maximum
results in figure 18 as they affect essentially only the two-loop sfermion contributions.

Figure 18 shows the results of the scan. The light yellow region corresponds to the possible results for \( a_\mu^{\text{SUSY}} \), given by (79), as a function of \( M_{\text{LOSP}} \), if all parameters are varied in the range (96). The red region corresponds to the situation that the smuons and sneutrinos are heavy,

\[
m_{\tilde{\mu}_{1,2}} \, m_{\tilde{\nu}_\mu} > 1000 \text{ GeV},
\]

while charginos, neutralinos and stops and sbottoms can still be light. A dedicated analysis of this situation is of interest since heavy 1st and 2nd generation sfermions are sometimes considered as a possible explanation of the absence of observable SUSY contributions to flavour-changing neutral currents and CP-violating observables.

The dashed lines in figure 18 correspond to the results if the genuine two-loop diagrams are neglected and only the one-loop contributions and the logarithmic QED-corrections (78) are taken into account.

The yellow region corresponds to all data points and thus to the maximum possible values of \( a_\mu^{\text{SUSY}} \) compatible with the parameter range (96) and the experimental constraints from \( b \)-decays, \( M_h \) and \( \Delta \rho \). The SUSY contributions can accommodate the observed result (5) within \( 1\sigma \) for LOSP masses below about 600 GeV. For LOSP masses below about 440 GeV, the SUSY contributions can be even too large, and thus the \( a_\mu \)-
The red region of figure 18 shows that heavy smuons and sneutrinos significantly suppress the maximum SUSY contributions to $a_\mu$. Nevertheless, the contributions can be in the 1σ region (5) for $M_{\text{LOS}}$ between about 150 and 470 GeV.

We close the section with a couple of instructive but more technical remarks. On the one hand we analyze which parameter regions give rise to the maximum contributions found in figure 18. On the other hand we explain the reasons for the different behaviour of the two-loop contributions in the red and yellow regions.

In the yellow region, corresponding to all data points, the maximum values are obtained for $\mu = 3000$ GeV, the upper border of the allowed range, and for $M_2 \approx m_{\tilde{L},R,\tilde{\mu}}$. The reason is that the one-loop bino-exchange diagram (N1) of figure 2 is linear in $\mu$ and therefore dominant for large $\mu$. In comparing our results with the ones of [65], one should note that since the maximum contributions are obtained at the border of the allowed multi-dimensional parameter region, an unbiased random generation of parameter points as the one used in [65] has difficulties in finding the maximum contributions. Therefore the results displayed in figure 1 and equation (3) of [65] slightly underestimate the maximum contributions allowed by the employed parameter ranges. Since the maximum contributions are obtained for large $\mu$, the two-loop chargino/neutralino contributions are negligible, but the two-loop sfermion contributions shift the contour by about 5%. Another consequence of the dominance of diagram (N1) is that the border of the yellow contour scales linearly with the maximum value for $\mu$ used in scanning over the SUSY parameters.

The one-loop contributions in the red region are maximized for smuon masses of 1 TeV and $\mu = M_2$. Since therefore $\mu$, $M_2$ and $M_A$ can be simultaneously small, substantial two-loop chargino/neutralino contributions are possible. Furthermore, the two-loop sfermion contributions can be large as well since the stop and sbottom mass parameters are not required to be universal. The largest results in the red region are obtained if both $m_{\tilde{L},\tilde{b}}$ and $m_{\tilde{R},\tilde{b}}$ are small but $m_{\tilde{R},\tilde{t}}$ is large. In this situation, the sbottom-loop contribution can be larger than the results displayed in figure 11 without any violation of the experimental constraints on $M_h$, $\Delta \rho$ or $b$-decays.

The significance of the two-loop corrections in the red region is reflected in the fact that the contour shifts by more than 30% in the low-mass region if the two-loop corrections are neglected.

5. Concluding remarks and outlook

The final result of the Brookhaven $a_\mu = (g_\mu - 2)/2$ measurement shows a deviation of $23.9 \times 10^{-10}$, corresponding to 2 ppm and more than 2σ, from the corresponding SM prediction based on $e^+e^-$ data. The Brookhaven experiment is the first magnetic dipole moment measurement that is sensitive to physics at the electroweak scale, just like the previous CERN experiment was the first to be sensitive to hadronic effects. But unlike the CERN experiment, which confirmed the SM prediction of the hadronic
effects, the Brookhaven experiment would prefer electroweak effects that are about 2.5 times larger than predicted by the SM.

The SUSY contributions to $a_\mu$ are essentially proportional to $\tan \beta \frac{\text{sign}(\mu)}{M_{\text{SUSY}}^2}$. Hence for moderate or large $\tan \beta$ these contributions can easily be larger than the electroweak SM contributions and thus constitute the origin of the discrepancy between the experiment and the SM prediction. Furthermore, in this case $a_\mu$ strongly favours positive $\mu$, which is a very important piece of information for SUSY phenomenology. The qualitative behaviour of $a_\mu^{\text{SUSY}}$ can be well understood using the mass insertion technique. The $\tan \beta$-enhancement arises in diagrams where the necessary chirality flip occurs at a muon Yukawa coupling, either to a Higgsino or Higgs boson, because this coupling is enhanced by $1/\cos \beta \approx \tan \beta$ compared to its SM value. The $\mu$-parameter mediates the transition between the two Higgs/Higgsino doublets $H_{1,2}$, and this transition enhances diagrams because only $H_1$ couples to muons while $H_2$ has the larger vacuum expectation value.

For a quantitative analysis, the MSSM prediction of $a_\mu$ has to be known with an accuracy that matches the one of the SM prediction. This goal has been achieved with the calculation of the full one-loop and leading two-loop contributions, as given in (79). Since the remaining theory uncertainty is dominated by one particular class of unknown two-loop diagrams, an even more satisfactory MSSM prediction could be obtained by evaluating this missing class of diagrams.

If supersymmetry is assumed to exist, many non-trivial details about the SUSY spectrum can be derived from $a_\mu$. For example, for $\tan \beta = 10$ four SUSY particles must be lighter than 500(1000) GeV at the 1$\sigma$(2$\sigma$) level. And even in a very conservative interpretation, where a 5$\sigma$ deviation in $a_\mu$ is tolerated, significant lower bounds on the SUSY masses can be derived in a model-independent way. These bounds cannot be obtained from any other observable and establish $a_\mu$ as one of the most important indirect probes of SUSY.

There are several other observables that are relevant for SUSY phenomenology and provide constraints on SUSY parameters, although $a_\mu$ is unique in its largest discrepancy between experimental and SM values. On a qualitative level, the relation between $a_\mu$ and these other observables is the following. $a_\mu$ favours rather large $\tan \beta$ and/or small SUSY masses and positive $\mu$. If neutralinos constitute a component of dark matter, the dark matter detection rate is enhanced by large $\tan \beta$ and positive $\mu$. Therefore, the $a_\mu$ result is encouraging for future dark matter detection experiments.

The preference of $a_\mu$ for rather light SUSY masses is supported by the experimental results for the electroweak precision observables $M_W$ and $\sin^2 \theta_{\text{eff}}$, which favour small but non-zero SUSY contributions. The preference of $a_\mu$ for not too small $\tan \beta$ is supported by the constraint derived from the lower bound on $M_h$. However, the bound on $M_h$ also favours large SUSY masses (particularly stop masses) over light ones. Furthermore, the MSSM parameter space for large $\tan \beta$ is significantly constrained by the rare $b$-decays $b \rightarrow s\gamma$ and $B_s \rightarrow \mu^+\mu^-$. The decay $b \rightarrow s\gamma$ shows a similar dependence on $\text{sign}(\mu)$ as $a_\mu$. This already constitutes a crucial test of minimal models. For instance, in mSUGRA
or GMSB, but not in AMSB, $a_\mu$ and $b \rightarrow s\gamma$ favour the same sign of $\mu$.

Given the tension between all these observables it is non-trivial that all constraints can be simultaneously satisfied, even in the simplest but well-motivated models such as mSUGRA or GMSB. Mainly driven by $a_\mu$, the parameter points satisfying all constraints prefer a number of (though not necessarily all) SUSY particles to be light. This result is clearly encouraging for the SUSY search at the LHC and the ILC.

We have summarized the current status of $a_\mu$ and SUSY in figure 18, which shows the possible values of $a_\mu^{\text{SUSY}}$, compared with the observed deviation between experiment and the SM prediction. The MSSM parameters have been scanned over in the range allowed by all relevant constraints from other collider experiments, and in the evaluation of $a_\mu^{\text{SUSY}}$ all known one- and two-loop contributions have been taken into account. The result confirms again that SUSY can accommodate the $a_\mu$ result consistently with all other constraints and that the preferred mass range is rather low. On a more technical level, the figure also shows that two-loop SUSY effects can be important.

In spite of the impressive current status, progress on $a_\mu$ is important and will come both from the experimental and the theoretical side. Already a small improvement of the precision could be sufficient to increase the discrepancy between the experimental and SM values of $a_\mu$ to 4–5$\sigma$ and thus to establish the existence of non-SM contributions. Currently the SM theory prediction has a larger uncertainty than the experimental value, and within the SM prediction the main uncertainty is related to the hadronic vacuum polarization. However, this uncertainty can be expected to be significantly reduced as a result of ongoing measurements of $e^+e^- \rightarrow$ hadrons by CMD2 and SND in Novosibirsk and, using radiative return, by $B$ factories and KLOE. Another important source of theoretical uncertainty is related to the hadronic light-by-light scattering contribution. It is a very challenging theoretical task to evaluate this contribution, and current error estimates vary between $2.5\ldots4 \times 10^{-10}$. However, further progress in the understanding of this contribution can be expected, and one can hope that the full SM theory uncertainty can be reduced to below $4 \times 10^{-10}$ in the foreseeable future [17]. Finally, progress can be expected from the experimental determination of $a_\mu$. Since the current measurement precision is statistics limited, a new experiment using similar methods with only straightforward improvements could reduce the uncertainty by a factor of 2.5 or more, and an according plan has already been outlined [6, 142].

After more than 40 years of experimental and theoretical progress, the observable $a_\mu$ has become a sensitive probe of physics at the electroweak scale. Already today it is one of the most important constraints of physics beyond the SM. In the near future, particle physics will enter a new era where the detailed structure of physics at the electroweak scale and above will be unravelled by experiments at the LHC and possibly later the ILC. Not only new particles like smuons or charginos might be discovered, but also the other indirect constraints from the observables mentioned above will become more stringent. The magnetic moment of the muon will provide a crucial cross-check and complement of the forthcoming experiments.
The Muon Magnetic Moment and Supersymmetry

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