Wave Chaos in Rotating Optical Cavities

Takahisa Harayama, Satoshi Sunada, and Tomohiro Miyasaka

Department of Nonlinear Science ATR Wave Engineering Laboratories
2-2-2 Hikaridai Seika-cho Soraku-gun Kyoto 619-0228 Japan

(Dated: September 19, 2006)

PACS numbers: 41.20.-q, 03.65.Pm, 42.55.Sa

In 1913 Sagnac pointed out that the path length of the clockwise (CW) propagating light in a rotating ring interferometer for one round trip is different from that of the counter-clockwise (CCW) propagating light and hence the Sagnac interferometer can be used as a rotation rate sensor [1,2].

The conventional theory of ring laser gyroscopes is based on the Sagnac's original idea, that is, the difference of the path lengths between counter-propagating lights derived from the special theory of relativity for ray-dynamical description [1,2]. More precise electromagnetic wave description of the Maxwell equations including the effect of rotation is given by the general theory of relativity [1,2,3]. These two different conventional ways describing the effect of rotation on the lasing frequency has given the same result because of the following two assumptions: (i) The ray-wave correspondence of the cavity-modes; the conventional theory assumes that there exist closed ring trajectories and the cavity-modes localize on these ring trajectories in a ring resonator. (ii) The equivalence of the descriptions for the cavity modes by standing waves and propagating waves; the conventional theory assumes that the size of the ring resonator is much larger than the wavelength of the laser, and in this short wavelength limit the cavity-modes are separated into two directions propagating along and transversal to the ring trajectory.

The assumption (ii) means that the cavity-modes can be always expressed as the unidirectionally rotating waves and the counter-propagating modes are degenerate. Precisely speaking, however, the cavity-modes of non-rotating cavities are the standing waves as far as the wavelength limit the cavity-modes are separated into two directions propagating along and transversal to the ring trajectory.

The assumption (ii) means that the cavity-modes can be always expressed as the unidirectionally rotating waves and the counter-propagating modes are degenerate. Precisely speaking, however, the cavity-modes of non-rotating cavities are the standing waves as far as the wavelength limit the cavity-modes are separated into two directions propagating along and transversal to the ring trajectory.

The researches on quantum chaos for these three decades have shown that there exist so-called wave-chaotic cavity-modes that do not localize on any ray-dynamical trajectories due to the chaotic property of the ray-dynamics in the cavity [4,5,6,7]. The ray-dynamical description for ring laser gyroscopes is not applicable for these wave-chaotic cavity-modes. On the other hand, the wave-dynamical description is still applicable. In this Letter, the recently established theory [7,8] of rotating resonant microcavities without the assumptions (i) and (ii) is applied to the cavity which shows the ray-dynamical properties of the mixed system. It is well-known that some eigenfunctions in the mixed system localize on the stable periodic trajectories while others are wave-chaotic and spread over the chaotic sea in the phase space. We show that the degenerate eigen-frequency corresponding to the wave-chaotic cavity-mode of the non-rotating cavity splits into two frequencies and their difference is proportional to the rotation rate although the splitting cavity-modes are still wave-chaotic and do not have any corresponding CW and CCW propagating modes as well as ray-dynamical counterparts, which cannot be explained by the conventional Sagnac effect.

First let us introduce the ray-dynamical properties and the wavefunctions of the eigenmodes when the cavity is not rotating. We will discuss the two-dimensional shape of the resonant microcavity defined by $R(\theta) = R_0 (1 + \epsilon \cos 4\theta)$ in the cylindrical coordinates. The parameters are set as follows: $R_0 = 6.2866 \mu m, \epsilon = 0.04$, and the refractive index $n = 1$.

Ray-dynamical properties can be easily seen by plotting the trajectories on the Poincaré surface of the section (SOS). It is convenient to use the Birkhoff coordinate $(\eta, s)$ for SOS, where $\eta$ is the normalized curvilinear distance measured along the edge of the cavity from a certain origin on the edge to the incident point and $s$ is the sine of the incident angle. The SOS of ray-dynamics in this cavity shown in Fig. 1 includes stable islands and a chaotic sea, which shows it is a typical mixed system. The stable islands around $s = \pm(1/\sqrt{2})$ correspond to CCW(CW) rotational ray-motions along a ring trajectory, as the ray-motion shown at the top (bottom) of the right hand side in Fig. 1. Then, typical example of a single trajectory in the chaotic sea are shown at the middle.

The wave properties are discussed only in the case of TM mode of electromagnetic field oscillating as $\mathbf{E}(r, t) = (0, 0, \psi(r)e^{-ikt} + c.c.)$ where $c$ is the velocity of light and the eigenfrequency $\omega$ equals $ck$, and the Dirichlet boundary condition are imposed for simplicity. By solving the Helmholtz equation $\nabla^2 \psi = 0$ where $n$ is the refractive index inside the cavity, we obtain the eigenmodes respectively corresponding to the stable islands and the chaotic sea as shown in Fig. 2(a) and 3(a). The
FIG. 1: The Poincaré surface of the section on the Birkhoff coordinate.

eigenmode in the Husimi representation \( \Psi \) of Fig. 2 (c) localizes on both the stable islands of the ring trajectories at \( s = \pm 1/\sqrt{2} \), which means that the eigenmode in the non-rotating cavity has both CW and CCW propagating wave components and is standing wave along the ring trajectory. On the other hand, Fig. 3 (c) shows that the eigenmode in Fig. 3 (a) does not localize on the stable islands but is distributed over the chaotic sea. As well known, no simple ray-wave correspondence exists for such an eigenmode.

These two modes belong to the same symmetry class that is respectively even and odd with respect to the \( y \)-axis and the \( x \)-axis because the cavity has \( C_4 \) symmetry. When these eigenmode are rotated by \( \pi/2 \), one can obtain the other eigenmodes that are respectively even and odd with respect to the \( x \)-axis and the \( y \)-axis. These modes created by \( \pi/2 \) rotation are orthogonal to the original ones. Thus, the pairs of degenerate eigenmodes in this cavity are obtained as shown in Fig. 2 (b) and 3 (b), and the Husimi representations of the eigenmodes shown in Fig. 2 (b) and 3 (b) are shown in Fig. 2 (d) and 3 (d), respectively.

Next we discuss the effect of rotation of the cavity on the above eigenmodes. The theory of the effect of the rotating resonant microcavity on the resonances has recently been obtained \( \Xi \) and is applicable to the wavechaotic eigenmodes where the electric fields spread over the whole phase space. Let us briefly review this theory for the case of the degenerate eigenmodes.

According to the general theory of relativity, the electromagnetic fields in a rotating resonant microcavity are subject to the Maxwell equations generalized to a non-inertial frame of reference in uniform rotation with angular velocity vector \( \Omega \). By neglecting \( O(|\Omega|^2) \) and assuming the 2D resonant microcavity is perpendicular to angular velocity vector \( \Omega \), we obtain the following equation for the eigenmodes of the rotating cavity,

\[
(\nabla_{xy}^2 + n^2k^2) \psi - 2ik (\mathbf{h} \cdot \nabla) \psi = 0, \tag{1}
\]

where \( \mathbf{h} = (\mathbf{r} \times \Omega)/c \) and the 2D resonant cavity is ro-

FIG. 2: (Color) (a-b) The eigenmodes corresponding to the stable islands of the ring trajectory in non-rotating cavity. Each eigenmodes labeled by (a) and (b) are odd(even) parity and even(odd) parity with respect to the \( x(y) \)- axis, and they are obtained at \( nkR_0 = 50.264118 \). (c-d) The Husimi representations corresponding to the eigenmodes (a) and (b).

FIG. 3: (Color) (a-b) The wave-chaotic eigenmodes in non-rotating cavity. Each eigenmodes labeled by (a) and (b) are odd(even) parity and even(odd) parity with respect to the \( x(y) \)-axis, and they are obtained at \( nkR_0 = 50.264118 \), which is the almost same value as that of the modes shown in Fig. 2 (c-d) The Husimi representations corresponding to the eigenmodes (a) and (b).
FIG. 4: (Color) (a-b) The eigenmodes corresponding to the stable islands of the ring trajectory in rotating cavity with $R_0\Omega/c = 6.28 \times 10^{-5}$. (c-d) the Husimi representations corresponding to the eigenmodes (a) and (b). (e-f) the angular momentum spectrum of the eigenmodes (a) and (b).

FIG. 5: (Color) (a-b) The wave-chaotic eigenmodes in rotating cavity with $R_0\Omega/c = 6.28 \times 10^{-5}$. (c-d) the Husimi representations corresponding to the eigenmodes (a) and (b). (e-f) the angular momentum spectrum of the eigenmodes (a) and (b).

By using the perturbation theory for degenerate states in quantum mechanics, the degenerate eigenmodes $\psi_0$ and $\psi_1$ of the non-rotating cavity are superposed to reproduce the solutions of Eq. (1) as

$$\psi_{\pm} = 1/\sqrt{2}\psi_0 \pm i/\sqrt{2}\psi_1, \quad (2)$$

where the degenerate wave number $k_0$ splits into two. Therefore, the frequency difference $\Delta \omega$ between the two eigenfunctions newly produced by rotation of the cavity is proportional to the angular velocity,

$$\Delta \omega = 2 \left| \int \int_D dr \psi_0 \left( y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \psi_1 \right| \frac{\Omega}{n^2}. \quad (3)$$

Applying this theory to the degenerate eigenmodes where the electric fields localize on the ring trajectories as shown in Fig. 2(a) and 2(b), we cannot expect the superposed solutions (2) are CW and CCW propagating modes. The cavity-mode $\psi_0$ in Fig. 3(a) can be expressed by the superposition of the Bessel functions as

$$\psi_0 = \sum_{m=0}^{\infty} a_{4m+1} J_{4m+1}(nk_0r) \sin(4m+1)\theta + \sum_{m=0}^{\infty} a_{4m+3} J_{4m+3}(nk_0r) \sin(4m+3)\theta, \quad (4)$$

because the wave function $\psi_0$ is odd (even) with respect to $x (y)$-axis. Then, the expression of the cavity-mode
ψ₁ in Fig. 3(b) can be obtained by π/2 rotation of ψ₀,

\[
ψ₁ = \sum_{m=0}^{∞} -a_{4m+1}J_{4m+1}(nk₀r) \cos(4m+1)θ \\
+ \sum_{m=0}^{∞} a_{4m+3}J_{4m+3}(nk₀r) \cos(4m+3)θ, \quad (5)
\]

which has the same eigen-frequency as that of ψ₀ and is even (odd) with respect to x (y)-axis. Accordingly, we have alternative expressions of the degenerate eigenmodes [2],

\[
ψ_± = \frac{i}{\sqrt{2}} \sum_{m=0}^{∞} a_{4m+1}J_{4m+1}(nk₀r) \exp\{±i(4m+1)θ\} \\
± \frac{i}{\sqrt{2}} \sum_{m=0}^{∞} a_{4m+3}J_{4m+3}(nk₀r) \exp\{±i(4m+3)θ\}. \quad (6)
\]

It is important that ψ± should be the eigen-mode of the rotating cavity, but it contains both CW and CCW propagating waves. Consequently it is impossible to discuss the difference of the path lengths between CW and CCW propagating lights. Nevertheless, the frequency splitting is still proportional to the angular velocity like the result from the conventional Sagnac effect. Therefore, it is concluded that the frequency splitting proportional to the rotation rate is the more general effect of rotation of the resonant cavity on its eigenmodes, and only in the special case that the eigenmodes localize on the ray-dynamical ring trajectories, this frequency splitting can be related to the Sagnac effect that is the difference of the path lengths between the CW and CCW propagating modes.

In order to confirm the above conclusion from the perturbation theory, we actually solved Eq. (1) numerically in the rotating cavity with angular velocity Ω, and obtained the eigenmodes, as shown in Fig. 4 and Fig. 5.

Fig. 4(a) and (b) show the wavefunctions of the eigenmodes corresponding to the stable island of the ring trajectory in rotating cavity. Then the Husimi representations of the eigenmodes of Fig. 4(a) and (b) are shown in Fig. 4(c) and (d), and the distribution of the angular momentum components |aₘ|² of the eigenmodes are shown in Fig. 4(e) and (f), respectively. As seen in these figures, the eigenmodes become almost unidirectionally propagating waves along the ring trajectory in the rotating cavity, while they are the standing waves along the ring trajectory in non-rotating cavity. Then, the frequency splitting is proportional to the angular velocity, as shown in Fig. 6, and the scale factor agrees with that calculated with the difference of the path lengths of the ring trajectory. That is, the conventional theory of the Sagnac effect can be reproduced in this case.

Fig. 5(a) and (b) show the wave-chaotic eigenmodes, and the Husimi representations corresponding to each eigenmodes are shown in Fig. 5(c) and (d). Unlike the case where the eigenmodes localize on the stable island, the distributions of the wave-chaotic modes in the Husimi representations clearly show that the eigenmodes do not become unidirectionally-propagating modes corresponding to the stable islands. Moreover, the angular momentum spectrum in Fig. 5(e) and (f) show the wave-chaotic eigenmodes have both the CW (m < 0) and CCW (m > 0) propagating wave components even when the cavity is rotated. However, the eigenfrequency actually splits into two, and the frequency splitting is proportional to the angular velocity as shown in Fig. 6.

Thus, one can confirm that the frequency splitting due to rotation can be observed even on the eigenmodes which do not split into CW and CCW propagating wave modes.

The work was supported by the National Institute of Information and Communication Technology of Japan.

---