Superscaling analysis of inclusive electron scattering and its extension to charge-changing neutrino-nucleus cross sections beyond the relativistic Fermi gas approach


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Abstract

Superscaling analyses of inclusive electron scattering from nuclei are extended from the quasielastic processes to the delta excitation region. The calculations of \((e,e')\) cross sections for the target nucleus \(^{12}\text{C}\) at various incident electron energies are performed using scaling functions \(f(\psi')\) obtained in approaches going beyond the mean-field approximation, such as the coherent density fluctuation model (CDFM) and the one based on the light-front dynamics (LFD) method. The results are compared with those obtained using the relativistic Fermi gas (RFG) model and the extended RFG model (ERFG). Our method utilizes in an equivalent way both basic nuclear quantities, density and momentum distributions, showing their role for the scaling and superscaling phenomena. The approach is extended to consider scaling function for medium and heavy nuclei with \(Z \neq N\) for which the proton and neutron densities are not similar. The asymmetry of the CDFM quasielastic scaling function is introduced, simulating in a phenomenological way the effects which violate the symmetry for \(\psi' \geq 0\) including the role of the final-state interaction (FSI). The superscaling properties of the electron scattering are used to predict charge-changing neutrino-nucleus cross sections at energies from 1 to 2 GeV. A comparison with the results of the ERFG model is made. The analyses make it possible to gain information about the nucleon correlation effects on both local density and nucleon momentum distributions.

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I. INTRODUCTION

Over the past four decades electron scattering has provided a wealth of information on nuclear structure and dynamics. Form factors and charge distributions have been extracted from elastic scattering data, while inelastic measurements have allowed for a systematic study of the dynamic response over a broad range of momentum and energy transfer. The nuclear $y$-scaling analysis of inclusive electron scattering from a large variety of nuclei (e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]) showed the existence of high-momentum components in the nucleon momentum distributions $n(k)$ at momenta $k > 2 \text{ fm}^{-1}$ due to the presence of nucleon-nucleon (NN) correlations. It was shown (see, e.g. [11, 12, 13, 14, 15]) that this specific feature of $n(k)$, which is similar for all nuclei, is a physical reason for the scaling and superscaling phenomena in nuclei.

The concepts of scaling [1, 2, 3, 4, 5, 6, 7, 8] and superscaling [10, 11, 12, 13, 14, 15, 16] have been explored in [12, 17] for extensive analyses of the $(e,e')$ world data (see also [18]). Scaling of the first kind (no dependence on the momentum transfer) is reasonably good as expected, at excitation energies below the quasielastic (QE) peak, whereas scaling of second kind (no dependence on the mass number) is excellent in the same region. When both types of scaling behavior occur one says that superscaling takes place. At energies above the QE peak both scaling of the first and, to a lesser extent, of the second kind are shown to be violated because of important contributions introduced by effects beyond the impulse approximation, namely, inelastic scattering [19, 20] together with correlation contributions and meson exchange currents [21, 22].

The superscaling analyses of inclusive electron scattering from nuclei for relatively high energies (several hundred MeV to a few GeV) have recently been extended to include not only quasielastic processes, but also the region where $\Delta$-excitation dominates [23]. A good representation of the electromagnetic response in both quasielastic and $\Delta$ regions has been obtained using the scaling ideas, importantly, with an asymmetric QE scaling function $f^{QE}(\psi') \ (\psi' \text{ is the scaling variable in the QE region})$ and a scaling function $f^\Delta(\psi'_\Delta)$ in the region up to inelasticities where the $\Delta$ contribution reaches its maximum. Both functions were deduced from phenomenological fits to electron scattering data. Particularly, for the scaling function in the quasielastic region it has been shown in Ref. [23] that, in contrast to the relativistic Fermi gas model scaling function, which is symmetric, limited strictly to the
region $-1 \leq \psi' \leq +1$, and with a maximum value $3/4$, the empirically determined $f^{QE}(\psi')$ has a somewhat asymmetric shape with a tail that extends towards positive values of $\psi'$ and its maximum is only about 0.6. Of course, the specific features of the scaling function should be accounted for by reliable microscopic calculations that take FSI into account. In particular, the asymmetric shape of $f^{QE}$ tested in Refs. [24, 25] by using a relativistic mean field (RMF) for the final states shows a very good agreement with the behavior presented by the experimental scaling function.

The superscaling analyses and the present knowledge of inclusive electron scattering allowed one to start studies of neutrino scattering off nuclei on the same basis. The reactions of incident neutrino beams interacting with a complex nucleus have offered unique opportunities for exploring fundamental questions in different domains in physics. Recently, positive signals of neutrino oscillations confirmed the hypothesis of non-zero neutrino masses and triggered much interest on this issue [26]. To better analyze the next generation of high-precision neutrino oscillation experiments and to reduce their systematic uncertainty both neutral- (e.g. [27, 28, 29, 30, 31]) and charged-current (e.g. [23, 24, 29, 30, 32, 33, 34, 35, 36, 37]) neutrino-nucleus scattering have stimulated detailed investigations.

The neutrino-nucleus interactions have been studied within several approaches investigating a variety of effects. Using the superscaling analysis of few-GeV inclusive electron-scattering data, a method was proposed in Ref. [23] to predict the inclusive $\nu A$ and $\bar{\nu} A$ cross sections for the case of $^{12}$C in the nuclear resonance region, thereby effectively including delta isobar degrees of freedom. It was shown in Refs. [27, 34] that the important final-state interaction effects arising from the use of relativistic optical potentials within a relativistic Green’s function approach lower the cross section by at least a 14% factor at incoming neutrino energies of 1 GeV. A similar result has been obtained in Refs. [38, 39] where the use of Random Phase Approximation (RPA) to predict the neutrino-nucleus cross section was discussed. Apart from relativistic dynamics and FSI, other effects may influence the neutrino-nucleus reactions. The role of Pauli blocking and FSI in charged-current neutrino induced reactions is analyzed in Refs. [35, 36, 37].

In this article we follow our method presented in Refs. [13, 14, 15] to calculate the scaling function in finite nuclei firstly within the coherent density fluctuation model (e.g., Refs. [40, 41, 42, 43]). This approach, which is a natural extension of the RFG model, has shown how both basic quantities, density and momentum distributions, are responsible for the scaling
and superscaling phenomena in various nuclei. Although the scaling function obtained in [13] is symmetrical around \( \psi' = 0 \), the results agree with the available experimental data at different transferred momenta and energies below the quasielastic peak position, showing superscaling for \( \psi' < 0 \) including \( \psi' < -1 \), whereas in the RFG model \( f(\psi') = 0 \) for \( \psi' \leq -1 \). It was shown in [14] that the QE scaling function can be obtained within the CDFM in two equivalent ways, on the basis of the local density distribution, as well as of the nucleon momentum distribution. As pointed out in [14], the nucleon momentum distributions \( n(k) \) for various nuclei obtained in [44] within a parameter-free theoretical approach based on the light-front dynamics method (e.g., [45, 46] and references therein) can also be used to describe both \( y \)- and \( \psi' \)-scaling data. So, in our present work we explore both methods, CDFM and LFD, to investigate further the scaling functions and their applications to analyses of electron- and neutrino scattering off nuclei.

Our work is motivated by the fact that different models of the nuclear dynamics (including those with RMF dynamics and with RPA-type correlations accounted for) describe with different success the basic size and shape of the cross sections in studies of high-energy inclusive lepton scattering used so far. For this reason we extend further our consideration and calculate within the CDFM and LFD the scaling functions in the kinematical regions of the QE and \( \Delta \) peak on the basis of momentum and density distributions of finite nuclear systems in which nucleon correlations are included. This can be done either by using available empirical data for these quantities or theoretical calculations in which correlations are included to some extent. Then, the obtained scaling functions are applied to calculate electron-nucleus cross sections in QE and \( \Delta \) regions in the energy range from 500 MeV to 2 GeV for the target nucleus \(^{12}\text{C}\) and to predict charge-changing neutrino and antineutrino reaction cross sections from the scaling region to the QE peak at energies of few GeV. We also make comparisons of the results obtained using our methods with those obtained using the RFG model and other theoretical schemes.

The paper is organized in the following way. In Sec. II we present the formalism needed in studies of scaling functions in the quasielastic region and validate the superscaling within the CDFM and LFD for a variety of nuclei with \( Z = N \) and \( Z \neq N \). Then, we consider the nucleon momentum distributions and their applications in both approaches showing the sensitivity of the calculated scaling functions to the peculiarities of \( n(k) \) in different regions of momenta. Section III contains the CDFM and LFD methods to build up the scaling
function in the $\Delta$ region. The formalism involved in obtaining the electron-nucleus cross sections in QE and $\Delta$ kinematical regions and the results of the numerical calculations are presented in Sec. IV A. In Sec. IV B we give our theoretical predictions for cross sections of quasielastic charge-changing neutrino reactions. Finally, in Sec. V we summarize the results of our work.

II. SCALING FUNCTION IN THE QUASIELASTIC REGION

A. QE scaling function in the CDFM

As already mentioned in the Introduction, the superscaling behavior was firstly considered within the framework of the RFG model \[10, 11, 12, 16, 17, 19\], where a properly defined function of the $\psi'$-variable was introduced. As pointed out in \[12\], however, the actual nuclear dynamical content of the superscaling is more complex than that provided by the RFG model. It was observed that the experimental data have a superscaling behavior in the low-$\omega$ side ($\omega$ being the transfer energy) of the quasielastic peak for large negative values of $\psi'$ (up to $\psi' \approx -2$), while the predictions of the RFG model are $f(\psi') = 0$ for $\psi' \leq -1$. This imposes the consideration of the superscaling in realistic finite systems. One of the approaches to do this was developed \[13, 14\] in the CDFM \[40, 41, 42, 43\] which is related to the $\delta$-function limit of the generator coordinate method \[13, 47\]. It was shown in \[13, 14, 15\] that the superscaling in nuclei can be explained quantitatively on the basis of the similar behavior of the high-momentum components of the nucleon momentum distribution in light, medium and heavy nuclei. As already mentioned, the latter is related to the effects of the NN correlations in nuclei (see, e.g. \[40, 41\]).

The scaling function in the CDFM was obtained starting from that in the RFG model \[10, 11, 12, 16\] in two equivalent ways, on the basis of the local density distribution $\rho(r)$ and of the nucleon momentum distribution $n(k)$. This allows one to study simultaneously the role of the NN correlations included in $\rho(r)$ and $n(k)$ in the case of the superscaling phenomenon. To explore these properties the scaling function $f(\psi')$ has been derived in two ways in CDFM in \[14\]. Firstly, by means of the density distribution $\rho(r)$, it leads to

$$f^{QE}(\psi') = \int_{0}^{\alpha/(k_F|\psi'|)} dR |F(R)|^2 f_{RFG}^{QE}(\psi'(R)), \quad (1)$$
with a weight function of the form

\[
|F(R)|^2 = -\frac{1}{\rho_0(R)} \left. \frac{d\rho(r)}{dr} \right|_{r=R},
\]

(2)

where

\[
\rho_0(R) = \frac{3A}{4\pi R^3}.
\]

(3)

\(f_{RFG}^Q(\psi'(R))\) with \(\psi'(R) = k_F R \psi'/\alpha\) is the scaling function related to the RFG model

\[
f_{RFG}^Q(\psi'(R)) = \frac{3}{4} \left[ 1 - \left( \frac{k_F |\psi'|}{\alpha} \right)^2 \right] \left\{ 1 + \left( \frac{R m_N}{\alpha} \right)^2 \left( \frac{k_F |\psi'|}{\alpha} \right)^2 \right\}^{-1/2}
\times \left[ 2 + \left( \frac{\alpha}{R m_N} \right)^2 - 2 \sqrt{1 + \left( \frac{\alpha}{R m_N} \right)^2} \right]
\]

(4)

\(m_N\) being the nucleon mass and \(\alpha = (9\pi A/8)^{1/3} \simeq 1.52A^{1/3}\). Secondly, by means of the momentum distribution \(n(k)\), the scaling function is expressed by

\[
f_{FG}^Q(\psi') = \int_{k_F|\psi'|}^{\infty} d\bar{k}_F |G(\bar{k}_F)|^2 f_{RFG}^Q(\psi'(\bar{k}_F)),
\]

(5)

where \(\psi'(\bar{k}_F) = k_F \psi'/\bar{k}_F\) and the weight function is

\[
|G(\bar{k}_F)|^2 = -\frac{1}{n_0(\bar{k}_F)} \left. \frac{dn(k)}{dk} \right|_{k=\bar{k}_F}
\]

(6)

with

\[
n_0(\bar{k}_F) = \frac{3A}{4\pi \bar{k}_F^3}.
\]

(7)

In Eq. (5), the RFG scaling function \(f_{RFG}^Q(\psi'(\bar{k}_F))\) can be obtained from \(f_{RFG}^Q(\psi'(R))\) [Eq. (4)] by changing there \(\alpha/R\) by \(\bar{k}_F\). In Eqs. (4), (4) and (5), the Fermi momentum \(k_F\) is not a free parameter for different nuclei as it is in the RFG model, but \(k_F\) is calculated within the CDFM for each nucleus using the corresponding expressions:

\[
k_F = \int_0^\infty dR k_F(R) |F(R)|^2 = \alpha \int_0^\infty dR \frac{1}{R} |F(R)|^2 = \frac{4\pi(9\pi)^{1/3}}{3A^{2/3}} \int_0^\infty dR \rho(R) R
\]

(8)

when the condition

\[
\lim_{R \to \infty} \left[ \rho(R) R^2 \right] = 0
\]

(9)

is fulfilled and

\[
k_F = \frac{16\pi}{3A} \int_0^\infty d\bar{k}_F n(\bar{k}_F) \bar{k}_F^3
\]

(10)
when the condition
\[ \lim_{k_F \to \infty} \left[ n(k_F)k_F^4 \right] = 0 \]  
(11)
is fulfilled.

As shown in 14, the integration in Eqs. (1) and (5), using Eqs. (2) and (6), leads to the explicit relationships of the scaling functions with the density and momentum distributions:

\[ f^{QE}(\psi') = \frac{4\pi}{A} \int_0^{\alpha/(k_F|\psi'|)} dR \rho(R) \left[ R^2 f^{QE}_{RFG}(\psi'(R)) + \frac{R^3}{3} \frac{\partial f^{QE}_{RFG}(\psi'(R))}{\partial R} \right] \]  
(12)
and

\[ f^{QE}(\psi') = \frac{4\pi}{A} \int_{k_F|\psi'|}^{\infty} d\bar{k}_F n(\bar{k}_F) \left[ \bar{k}_F^2 f^{QE}_{RFG}(\psi'(\bar{k}_F)) + \frac{\bar{k}_F^3}{3} \frac{\partial f^{QE}_{RFG}(\psi'(\bar{k}_F))}{\partial \bar{k}_F} \right] , \]  
(13)
the latter at
\[ \lim_{k_F \to \infty} \left[ n(k_F)k_F^3 \right] = 0. \]  
(14)

One can see the symmetry in both Eqs. (12) and (13) written in r- and k-space. We also note that in the consideration up to here the CDFM scaling function \( f^{QE}(\psi') \) is symmetric under the change of \( \psi' \) by \(-\psi'\).

In Refs. [13, 14] we used the charge density distributions to determine the weight function \( |F(R)|^2 \) and \( f^{QE}(\psi') \) in Eqs. (1), (2) and (8) for the cases of \(^4\)He, \(^{12}\)C, \(^{27}\)Al, \(^{56}\)Fe and \(^{197}\)Au. The results for the scaling function \( f^{QE}(\psi') \) agree well with the available data from the inclusive quasieelastic electron scattering for \(^4\)He, \(^{12}\)C, \(^{27}\)Al, \(^{56}\)Fe and only approximately for \(^{197}\)Au for various values of the transfer momentum \( q = 500, 1000, 1650 \) MeV/c [13] and \( q = 1560 \) MeV/c [14], showing superscaling for negative values of \( \psi' \) including also those smaller than -1, whereas in the RFG model \( f(\psi') = 0 \) for \( \psi' \leq -1 \). One can see this in Fig. 1 for \(^4\)He, \(^{12}\)C and \(^{27}\)Al at \( q = 1000 \) MeV/c. At the same time, however, in [13, 14] we encountered some difficulties to describe the superscaling in \(^{197}\)Au which was the heaviest nucleus considered. We related this in [13, 14] to the particular A-dependence of \( n(k) \) in the model that does not lead to realistic high-momentum components of \( n(k) \) in the heaviest nuclei. We followed in Refs. [13, 14] an artificial way to "improve" the high-momentum tail of \( n(k) \) in \(^{197}\)Au by taking the value of the diffuseness parameter \( b \) in the Fermi-type charge density distribution of this nucleus to be \( b = 1 \) fm instead of the value \( b = 0.449 \) fm (as obtained from electron elastic scattering experiments, see e.g. [48]). In this way the high-momentum tail of \( n(k) \) for \(^{197}\)Au in CDFM becomes similar to those of \(^4\)He, \(^{12}\)C, \(^{27}\)Al, and \(^{56}\)Fe and this
leads to a good agreement of the scaling function $f^{QE}(\psi')$ with the data also for $^{197}$Au. Still in [13] we pointed out, however, that all the nucleons (not just the protons) may contribute to $f^{QE}(\psi')$ for the transverse electron scattering and this could be simulated by increasing of the diffuseness of the matter density with respect to that of the charge density for a nucleus like $^{197}$Au that has much larger number of neutron than of protons.

FIG. 1: The quasielastic scaling function $f^{QE}(\psi')$ at $q = 1000$ MeV/c for $^{4}$He, $^{12}$C, $^{27}$Al, $^{82}$Kr and $^{197}$Au calculated in CDFM. Dotted line: RFG model result. The curves for $^{4}$He, $^{12}$C and $^{27}$Al nuclei almost coincide. Grey area: experimental data [11, 12].

In [15] we assumed that the reason why the CDFM does not work properly in the case of $^{197}$Au is that we had used in [13, 14] only the phenomenological charge density, while this nucleus has many more neutrons than protons ($N=118$ and $Z=79$) and therefore, proton and neutron densities may differ considerably. In the case when $Z \neq N$ and the proton and neutron densities are not similar, the total scaling function will be expressed by the sum of the proton $f^{QE}_p(\psi')$ and neutron $f^{QE}_n(\psi')$ scaling functions which are determined by the proton and neutron densities $\rho_p(r)$ and $\rho_n(r)$, respectively:

$$f^{QE}_p(\psi') = \int_0^{\alpha_p(r)/|F_F^{p(n)}(\psi')|} dR |F_{p(n)}(R)|^2 f^{p(n)}_{RFG}(\psi'(R)). \tag{15}$$

In Eq. (15) the proton and neutron weight functions are obtained from the corresponding
The Fermi momentum for the protons and neutrons is given by
\[ k_F^{(n)} = \alpha_p^{(n)} \int_0^\infty dR \frac{1}{R} |F_p^{(n)}(R)|^2. \] (19)

The RFG proton and neutron scaling functions \( f_{\text{RFG}}^{p(n)}(\psi'(R)) \) have the form of Eq. (4), where \( \alpha \) and \( k_F \) stand for \( \alpha_p^{(n)} \) from Eq. (17) and \( k_F^{(n)} \) from Eq. (19), respectively. The functions are normalized as follows:
\[ \int_0^\infty |F_p^{(n)}(R)|^2 dR = 1, \] (20)
\[ \int_{-\infty}^\infty f_{\text{QE}}^{p(n)}(\psi') d\psi' = 1. \] (21)

Then the total scaling function can be expressed by means of both proton and neutron scaling functions:
\[ f^{\text{QE}}(\psi') = \frac{1}{A} [Z f_p^{\text{QE}}(\psi') + N f_n^{\text{QE}}(\psi')] \] (22)
and is normalized to unity.

The same consideration can be performed equivalently on the basis of the nucleon momentum distributions for protons \( n_p^p(k) \) and neutrons \( n_n^p(k) \) presenting \( f^{\text{QE}}(\psi') \) by the sum of proton and neutron scaling functions (22) calculated similarly to Eqs. (15)-(22) (and to Eqs. (5), (6), (10) and (11)):
\[ f_p^{\text{QE}}(\psi') = \int_{k_F^{(n)}(\psi')}^\infty d\bar{k}_F |G_p^{(n)}(\bar{k}_F)|^2 f_{\text{RFG}}^{p(n)}(\psi'(\bar{k}_F)), \] (23)
where
\[ |G_p^{(n)}(\bar{k}_F)|^2 = -\frac{4\pi \bar{k}_F^3}{3Z(N)} \left. \frac{dn_p^{(n)}(k)}{dk} \right|_{k=\bar{k}_F} \] (24)
with \( f_{RFG}^{p(n)}(\psi'(|k_F|)) \) containing \( \alpha_{p(n)} \) from Eq. (17) and \( k_F^{p(n)} \) calculated as

\[
k_F^{p(n)} = \int_0^\infty d|k_F| |G_{p(n)}(|k_F|)|^2.
\] (25)

The scaling functions for several examples, such as the medium stable nuclei \(^{62}\text{Ni}\) and \(^{82}\text{Kr}\) and the heavy nuclei \(^{118}\text{Sn}\) and \(^{197}\text{Au}\) are calculated following Eqs. (15)-(22) using the corresponding proton and neutron densities obtained in deformed self-consistent mean-field (HF+BCS) calculations with density-dependent Skyrme effective interaction (SG2) and a large harmonic-oscillator basis with 11 major shells \([49, 50]\). In Fig. 1 we give the results for the \(^{82}\text{Kr}\) and \(^{197}\text{Au}\) nuclei in which \( Z \neq N \) and compare them with the results for \(^{4}\text{He}\), \(^{12}\text{C}\) (\( Z = N \)) and \(^{27}\text{Al}\) (\( Z \approx N \)), as well as with the experimental data (presented by a grey area and taken from \([12]\)) obtained for \(^{4}\text{He}\), \(^{12}\text{C}\), \(^{27}\text{Al}\), \(^{56}\text{Fe}\), and \(^{197}\text{Au}\). The scaling functions are in a reasonable agreement with the data, which was not the case for \(^{197}\text{Au}\) calculated in \([13]\) by using only the Fermi-type charge density with phenomenological parameter values \( b = 0.449 \text{ fm} \) and \( R = 6.419 \text{ fm} \) from \([48]\). At the same time we note also the improvement in comparison with the RFG model result in which \( f^{QE}(\psi') = 0 \) for \( \psi' \leq -1 \). Thus, it can be concluded that the scaling function \( f^{QE}(\psi') \) for nuclei with \( Z \neq N \) for which the proton and neutron densities are not similar has to be expressed by the sum of the proton and neutron scaling functions. The latter can be calculated by means of theoretically and/or experimentally obtained proton and neutron local density distributions or momentum distributions.

As known (e.g. \([12, 23]\)), the total inclusive electron scattering response is assumed to be composed of several contributions: i) the entire longitudinal contribution which superscales and is represented by the QE scaling function \( f^{QE}(\psi') \); ii) a part of the transverse response, which arises from the quasielastic scaling function \( f^{QE}(\psi') \), and iii) the additional contribution of the transverse response from MEC effects and from inelastic single-nucleon processes including the excitation of the \( \Delta \) isobar. The effects of point iii) break the scaling. In \([23]\) an universal scaling function \( f^{QE}(\psi') \) has been determined by reliable separations of the empirical data into their longitudinal and transverse contributions for \( A > 4 \). Such separations are available only for a few nuclei \([51]\). All of these response functions have been used to extract the "universal" QE response function \( f^{QE}(\psi') \) (see Fig. 1 of \([23]\)) which is parametrized by a simple function. This
function has a somewhat asymmetric shape. Its left tail ($\psi' < 0$) passes through the grey area of Fig. The right tail ($\psi' > 0$) extends larger towards positive values of $\psi'$. In contrast, the RFG scaling function is symmetric. The CDFM scaling function discussed so far, which is based on the RFG one, is also symmetric. As mentioned, the maximum value of $f^{QE}(\psi')$ in RFG (and in CDFM) is $3/4$, while the empirical scaling function extracted in Ref. [23] reaches about 0.6.

As mentioned in [23], if FSI are neglected, the RMF theory [33, 52, 53] and relativized shell-model studies [54] provide rather modest differences from the RFG predictions. Another possible reason for the differences between the RFG (or mean-field results) and the empirically determined scaling function arises from high-momentum components in realistic wave functions which may be large enough. In [23] the scaling function was taken from the experiment. In the present work we also limit our approach to phenomenology when considering the asymmetric shape and the maximum value of the quasielastic scaling function.

In order to simulate the role of all the effects which lead to asymmetry, we impose the latter on the RFG scaling function (and, correspondingly, on the CDFM one) by introducing a parameter which gives the correct maximum value of the scaling function ($c_1$ in our expressions given below) and also an asymmetry in $f^{QE}(\psi')$ for $\psi' \geq 0$. We consider the main parts of the RFG scaling function for $\psi' \leq 0$ and $\psi' \geq 0$ in the following forms, keeping the parabolic dependence on $\psi'$ as required in Ref. [10]:

$$f^{QE}_{RFG,1}(\psi') = c_1 (1 - \psi'^2) \Theta(1 - \psi'^2), \quad \psi' \leq 0,$$

$$f^{QE}_{RFG,2}(\psi') = c_1 \left[ 1 - \left( \frac{\psi'}{c_2} \right)^2 \right] \Theta \left[ 1 - \left( \frac{\psi'}{c_2} \right)^2 \right], \quad \psi' \geq 0. \quad (27)$$

The total RFG scaling function is normalized to unity:

$$\int_{-\infty}^{\infty} f^{QE}_{RFG}(\psi') d\psi' = \int_{-\infty}^{\infty} [f^{QE}_{RFG,1}(\psi') + f^{QE}_{RFG,2}(\psi')] d\psi' = 1. \quad (28)$$

If the normalization of the scaling function for negative values of $\psi'$ is equal to

$$a = \int_{-\infty}^{0} d\psi' f^{QE}_{RFG,1}(\psi') = \frac{2}{3} c_1, \quad (29)$$

then, in order to keep the total normalization [Eq. (28)], the normalization for positive $\psi'$ has to be:

$$1 - a = \int_{0}^{\infty} d\psi' f^{QE}_{RFG,2}(\psi') = \frac{2}{3} c_1 c_2. \quad (30)$$
From Eqs. (29) and (30) we get the relationship between $c_2$ and $c_1$:

$$c_2 = \frac{3}{2c_1} - 1.$$  \hfill (31)

In the RFG $c_1 = 3/4$ and, correspondingly, $c_2 = 1$. In the CDFM the QE scaling function will be:

$$f_{QE}^{OE}(\psi^I) = f_1^{OE}(\psi^I) + f_2^{OE}(\psi^I),$$  \hfill (32)

where

$$f_1^{OE}(\psi^I) \equiv \int_0^{\alpha/k_F|\psi^I|} dR |F(R)|^2 c_1 \left[ 1 - \left( \frac{k_F R |\psi^I|}{\alpha} \right)^2 \right], \psi^I \leq 0,$$  \hfill (33)

$$f_2^{OE}(\psi^I) \equiv \int_0^{c_2\alpha/k_F|\psi^I|} dR |F(R)|^2 c_1 \left[ 1 - \left( \frac{k_F R |\psi^I|}{c_2\alpha} \right)^2 \right], \psi^I \geq 0.$$  \hfill (34)

In this approach, parametrizing the RFG scaling function by the coefficient $c_1$ we account for the experimental fact that $c_1 \neq 3/4$ and take this value in accordance with the empirical data. Then from the normalization [Eqs. (28)-(30)] we determine the corresponding value of $c_2$ using Eq. (31). As in [13, 14], the CDFM scaling function is obtained [Eqs. (32)-(34)] by averaging the RFG scaling function. As an example, we give in Fig. 2 the CDFM QE scaling function for different values of $c_1$ (0.75, 0.72, 0.60 and 0.50) in comparison with the empirical data and the phenomenological fit. We also include for reference the scaling function obtained from calculations for $(e,e')$ reaction based on the relativistic impulse approximation (RIA) with FSI described using the RMF potential (see [24, 25] for details). In this way we simulate in a phenomenological way the role of the effects which violate the symmetry for positive values of $\psi^I$ of the QE scaling function, which in the RMF approximation are seen to be due to the FSI.

**B. QE scaling function in the LFD method**

In this Subsection we will obtain the QE scaling function on the basis of calculations of nucleon momentum distribution (using Eqs. (5)-(7) or Eq. (13)) obtained within a modification of the approach from [44]. The latter is based on the momentum distribution in the deuteron from the light-front dynamics (LFD) method (e.g., [45, 46] and references therein). Using the natural-orbital representation of the one-body density matrix [55], $n(k)$
FIG. 2: The quasielastic scaling function $f^{QE}(\psi')$ for $^{12}$C calculated in CDFM in comparison with the experimental data (black squares) [23]. The CDFM results for different values of $c_1$ are presented by solid lines. Also shown for comparison is the phenomenological curve which fits the data (dash-two dots), as well as the curve (dash-dot line) corresponding to the result for $(e,e')$ obtained within the relativistic impulse approximation and FSI using the relativistic mean field (see Refs. [24, 25]).

was written as a sum of contributions from hole-states $[n^h(k)]$ and particle-states $[n^p(k)]$ (see also [14])

$$n_A(k) = N_A \left[ n^h(k) + n^p(k) \right]. \quad (35)$$

In [35]

$$n^h(k) = C(k) \sum_{nlj} F.L. 2(2j+1)\lambda_{nlj} |R_{nlj}(k)|^2, \quad (36)$$

where F.L. denotes the Fermi level, and

$$C(k) = \frac{m_N}{(2\pi)^3 \sqrt{k^2 + m_N^2}}, \quad (37)$$

$m_N$ being the nucleon mass. To a good approximation for the hole states, the natural occupation numbers $\lambda_{nlj}$ are close to unity in [44] and the natural orbitals $R_{nlj}(k)$ are
replaced by single-particle wave functions from the self-consistent mean-field approximation. In [44] Woods-Saxon single-particle wave functions were used for protons and neutrons. \( N_A \) is the normalization factor. Concerning the particle-state \([n^p(k)]\) contribution in [35], we used in [44] and [14] the well-known facts that: (i) the high-momentum components of \( n(k) \) caused by short-range and tensor correlations are almost completely determined by the contributions of the particle-state natural orbitals (e.g. [56]), and (ii) the high-momentum tails of \( n_A(k)/A \) are approximately equal for all nuclei and are a rescaled version of the nucleon momentum distribution in the deuteron \( n_d(k) \),

\[
n_A(k) \simeq \alpha_A n_d(k),
\]

where \( \alpha_A \) is a constant. These facts made it possible to assume in [44] and [14] that \( n^p(k) \) is related to the high-momentum components \( n_5(k) \) of the deuteron, that is,

\[
n^p(k) = \frac{A}{2} n_5(k).
\]

In [39] \( n_5(k) \) is expressed by an angle-averaged function [44] as

\[
n_5(k) = C(k)(1 - z^2)f_5^z(k).
\]

In [40] \( z = \cos(\vec{k}, \vec{n}) \), \( \vec{n} \) being a unit vector along the three vector (\( \vec{\omega} \)) component of the four-vector \( \omega \) which determines the position of the light-front surface [45, 46]. The function \( f_5(k) \) is one of the six scalar functions \( f_{1-6}(k^2, \vec{n}, \vec{k}) \) which are the components of the deuteron total wave function \( \Psi(\vec{k}, \vec{n}) \). It was shown [45] that \( f_5 \) largely exceeds other \( f \)-components for \( k \geq 2.0-2.5 \text{ fm}^{-1} \) and is the main contribution to the high-momentum component of \( n_d(k) \), incorporating the main part of the short-range properties of the nucleon-nucleon interaction.

It was shown in Fig. 2 of [14] that the calculated LFD \( n(k) \)'s are in good agreement with the “\( y \)-scaling data” for \( ^4\text{He}, ^{12}\text{C} \) and \( ^{56}\text{Fe} \) from [5] and also with the \( y_{\text{CW}} \) analysis [7, 8] up to \( k \lesssim 2.8 \text{ fm}^{-1} \). For larger \( k \) the momentum distributions from LFD exceeds that obtained from \( y_{\text{CW}} \) analysis. We should note also that the calculated scaling function \( f^{QE}(\psi') \) using the approximate relationship (see Eq. (75) and Fig. 4 of [14])

\[
f^{QE}(\psi') \simeq 3\pi \int_{|y|}^\infty d\bar{k}_F n(\bar{k}_F) \bar{k}_F^2, \quad |y| = \frac{1 - \sqrt{1 - 4ck_F|\psi'|}}{2c}, \quad c \equiv \frac{\sqrt{1 + m_N^2/q^2}}{2m_N},
\]

for \( ^{56}\text{Fe} \) at \( q = 1000 \text{ MeV}/c \) is in agreement with the data for \(-0.5 \leq \psi' \leq 0 \), while in the region \(-1.1 \leq \psi' \leq -0.5 \) it shows a dip in the interval \(-0.9 \leq \psi' \leq -0.6 \). This difference
is due to the particular form of \( n(k) \) from LFD shown in Fig. 2 of Ref. \[14\] (a dip around \( k \approx 1.7 \text{ fm}^{-1} \) and a very high-momentum tail at \( k \gtrsim 2.8 \text{ fm}^{-1} \)). This result showed that the assumption (39) for the particle-state contribution is a rather rough one. In this paper we consider a modification of the approach in which we include partially in the particle-state part \( n^p(k) \) not only \( n_5(k) \) but also \( n_2(k) \) which is related to the angle-averaged function \( f_2(k) \):

\[
n_2(k) = C(k) f_2^2(k).
\]

(42)

Then the particle-state part can be written in the form

\[
n^p(k) = \beta [n_2(k) + n_5(k)],
\]

(43)

where \( \beta \) is a parameter. Then the LFD nucleon momentum distribution for the nucleus with \( A \) nucleons will be:

\[
n_{\text{LFD}}(k) = N_A [n^h(k) + \beta (n_2(k) + n_5(k))],
\]

(44)

with \( n^h(k) \) from Eq. (36) and

\[
N_A = \left\{ 4\pi \int_0^\infty dq q^2 \left[ \sum_{nlj} (2j+1)\lambda_{nlj} C(q) |R_{nlj}(q)|^2 + \beta (n_2(q) + n_5(q)) \right] \right\}^{-1}.
\]

(45)

In Fig. 3 we present the nucleon momentum distribution for \(^{12}\text{C}\) calculated within the LFD method using Eqs. (35)–(37), (40), (42)–(45) with the parameter value \( \beta = 0.80 \). It is compared with the band of CDFM momentum distributions for \(^4\text{He}, ^{12}\text{C}, ^{27}\text{Al}, ^{56}\text{Fe}, ^{197}\text{Au} \) (grey area), with \( n_{\text{CW}}(k) \) from the \( y_{\text{CW}} \) analysis \[7, 8\] and with the \( y \)-scaling data \[5\] for \(^4\text{He}, ^{12}\text{C}, \) and \(^{56}\text{Fe}. \) It can be seen that up to \( k \approx 2.8 \text{ fm}^{-1} \) \( n_{\text{LFD}} \) curve is close to the results of \[3, 7, 8\]. For the region \( 1 \leq k \leq 2.5 \text{ fm}^{-1} \) it is between them and for \( k \geq 2.8 \text{ fm}^{-1} \) it is close to \( n_{\text{CW}}(k) \), in contrast to our previous results in Fig. 2 of Ref. \[14\] (see also \[44\]) which were based on Eq. (39) and which are also shown for comparison in Fig. 3. This behavior of \( n_{\text{LFD}}(k) \) reflects in the result of the calculation of the QE scaling function using Eq. (41) given in Fig. 4. It can be seen that both momentum distributions \( n_{\text{CW}} \) \[7\] and \( n_{\text{LFD}}(k) \) [Eq. (44)] give a good agreement with the experimental data for the QE scaling function at least up to \( \psi' \approx -1.2 \). This result is an improvement of that for LFD shown in Fig. 4 of Ref. \[14\], where only the contribution \( n_5 \) was used in the calculation of \( n^p(k) \) (39) and \( n_{\text{LFD}}(k) \).
FIG. 3: The nucleon momentum distribution $n(k)$. Grey area: CDFM combined results for $^4$He, $^{12}$C, $^{27}$Al, $^{56}$Fe and $^{197}$Au. Solid line: result of the present work for $^{12}$C using the modified LFD approach with $\beta = 0.80$. Dashed line: $y_{CW}$-scaling result [7, 8]. Dash-dotted line: result of LFD for $^{12}$C from [44]. Dotted line: the mean-field result using Wood-Saxon single-particle wave functions for $^{56}$Fe. Open squares, circles and triangles are $y$-scaling data [5] for $^4$He, $^{12}$C and $^{56}$Fe, respectively. The normalization is: $\int n(k) d^3k = 1$.

III. SCALING FUNCTION IN THE QUASIELASTIC DELTA REGION

In this Section we will extend our analysis within both CDFM and LFD to the $\Delta$-peak region, which is not too far above the QE peak region and is the main contribution to the inelastic scattering. Dividing the cross section by the appropriate single-nucleon cross section, now for $N \rightarrow \Delta$ transition, and displaying the results versus a new scaling variable ($\psi_\Delta$) (in which the kinematics of resonance electro-production is accounted for) it is obtained in [23] that the results scale quite well. This is considered as an indication that the procedure has identified the dominant contributions not only in the QE region, but also in the $\Delta$-region.
FIG. 4: The quasielastic scaling function $f_{QE}(\psi')$ calculated using Eq. (41) at $q = 1000$ MeV/c with $n_{CW}(k)$ from the $y_{CW}$-scaling analysis [7, 8] for $^{56}\text{Fe}$ (solid line) and $n_{LFD}(k)$ from modified LFD approach [Eq. (44)] for $^{12}\text{C}$ (dashed line).

The shifted dimensionless scaling variable in the $\Delta$-region $\psi'_\Delta$ is introduced (see, e.g. [23]) by the expression:

$$
\psi'_\Delta = \left[ \frac{1}{\xi_F} \left( \kappa \sqrt{\rho'^2_\Delta + 1/\tau'} - \lambda' \rho'^2_\Delta - 1 \right) \right]^{1/2} \times \begin{cases} 
+1, & \lambda' \geq \lambda'^0_\Delta \\
-1, & \lambda' \leq \lambda'^0_\Delta
\end{cases}, 
$$

where

$$
\xi_F \equiv \sqrt{1 + \eta^2_F} - 1, \quad \eta_F \equiv \frac{k_F}{m_N},
$$

$$
\lambda' = \lambda - \frac{E_{\text{shift}}}{2m_N}, \quad \tau' = \kappa^2 - \lambda'^2,
$$

$$
\lambda = \frac{\omega}{2m_N}, \quad \kappa = \frac{q}{2m_N}, \quad \tau = \kappa^2 - \lambda^2,
$$

$$
\lambda'^0_\Delta = \lambda'^0_\Delta - \frac{E_{\text{shift}}}{2m_N}, \quad \lambda'^0_\Delta = \frac{1}{2} \left[ \sqrt{\mu^2_\Delta + 4\kappa^2} - 1 \right],
$$

$$
\mu_\Delta = m_\Delta/m_N, 
$$

$$
\rho_\Delta = 1 + \frac{\beta_\Delta}{\tau}, \quad \rho'^0_\Delta = 1 + \frac{\beta_\Delta}{\tau'},
$$

(51)
\[ \beta_{\Delta} = \frac{1}{4} (\mu_{\Delta}^2 - 1). \quad (53) \]

The relativistic Fermi gas superscaling function in the \( \Delta \) domain is given by \[23]\:

\[ f_{RFG}^{\Delta}(\psi_{\Delta}') = \frac{3}{4}(1 - \psi_{\Delta}'^2)\Theta(1 - \psi_{\Delta}'^2). \quad (54) \]

Following the CDFM application to the scaling phenomenon, the \( \Delta \)-scaling function in the model will be:

\[ f^{\Delta}(\psi_{\Delta}') = \int_0^\infty dR |F_{\Delta}(R)|^2 f_{RFG}^{\Delta}(\psi_{\Delta}'(R)). \quad (55) \]

In Eq. (55):

\[ \psi_{\Delta}'^2(R) = \frac{1}{\sqrt{1 + \frac{k_F^2(R)}{m_N^2} - 1}} \left[ \kappa \sqrt{\rho_{\Delta}'^2 + \frac{1}{\tau}} - \lambda' \rho_{\Delta}' - 1 \right] \equiv t(R).\psi_{\Delta}'^2, \quad (56) \]

where

\[ t(R) \equiv \frac{\left[ \sqrt{1 + \frac{k_F^2}{m_N^2} - 1} \right]}{\left[ \sqrt{1 + \frac{k_F^2}{m_N^2} - 1} \right]}, \quad \text{and} \quad k_F(R) = \frac{\alpha}{R}. \quad (57) \]

In the CDFM \( k_F \) can be calculated using the density distribution (Eqs. (8), (9) or (19) and (16)) or the momentum distribution (Eqs. (10), (11) or (25) and (24)). The weight function \( |F_{\Delta}(R)|^2 \) is related to the density distributions (Eqs. (2) or (16)). In the equivalent form of the CDFM, the scaling function can be written in the form:

\[ f^\Delta(\psi_{\Delta}') = \int_0^\infty d\bar{k}_F |G_{\Delta}(\bar{k}_F)|^2 f_{RFG}^{\Delta}(\psi_{\Delta}'(\bar{k}_F)), \quad (58) \]

where \( G_{\Delta}(\bar{k}_F) \) is determined by means of the momentum distribution (Eqs. (6) or (24)) and

\[ \psi_{\Delta}'^2(\bar{k}_F) \equiv \tilde{t}(\bar{k}_F).\psi_{\Delta}'^2 \]

with

\[ \tilde{t}(\bar{k}_F) \equiv \frac{\left[ \sqrt{1 + \frac{k_F^2}{m_N^2} - 1} \right]}{\left[ \sqrt{1 + \frac{k_F^2}{m_N^2} - 1} \right]}. \quad (60) \]
Here we would like to note that though the functional forms of $f^\Delta(\psi'_\Delta)$ [Eq. (55)] and the weight function $|F_\Delta(R)|^2$ (Eqs. (2) or (10)) are like before, i.e. as in the case of the QE region, the parameters of the densities (e.g. the half-radius $R_\Delta$ and the diffuseness $b_\Delta$ when Fermi-type forms have been used) may be different from those ($R$ and $b$) in the QE case. Along this line, we calculated firstly the scaling function $f^\Delta(\psi'_\Delta)$ by means of Eqs. (55)–(57) using the Fermi-type density for $^{12}\text{C}$. We found the values of $R_\Delta$ and $b_\Delta$ fitting the scaling data at the $\Delta$ peak extracted from the high-quality world data for inclusive electron scattering (given in Fig. 2 of [23]). Our results are presented in Fig. 5. As mentioned already in the QE case, the empirical data require to use a value of the coefficient in the right-hand side of Eq. (54) for the RFG scaling functions $f^\Delta_{\text{RFG}}(\psi'_\Delta)$ different from $3/4$. In our calculations in the $\Delta$-region we use the value $0.54$. We found that reasonable agreement with the data can be achieved using the parameter values $R_\Delta = 1.565 \text{ fm}$ and $b_\Delta = 0.420 \text{ fm}$ (the Fermi-momentum value is taken to be $k_F = 1.20 \text{ fm}^{-1}$ and this choice leads to normalization to unity of $f^\Delta_{\text{RFG}}(\psi'_\Delta)$). The value of $R_\Delta$ is smaller than that used in the description of the QE superscaling function for $^{12}\text{C}$ [13, 14] ($R = 2.470 \text{ fm}$) while the value of $b_\Delta$ is the same as $b$ in the QE case. Secondly, we calculated $f^\Delta(\psi'_\Delta)$ using Eqs. (58), (59) and (60). In Eq. (58) the weight function $|G_\Delta(k_F)|^2$ was determined by means of Eq. (6) and the nucleon momentum distribution $n_{\text{LFD}}$ (Eqs. (44) and (45)) calculated with the parameter value $\beta = 0.80$ (shown in Fig. 3). We note that the use of $n_{\text{LFD}}(k)$ with this value of $\beta$ gives simultaneously a reasonable agreement both with the results for the momentum distribution from the $y$-scaling data shown in Fig. 3 as well as with the QE scaling function shown in Fig. 3.

IV. SCALING FUNCTIONS AND INCLUSIVE LEPTON SCATTERING

A. Scaling functions and $(e,e')$ reaction cross sections

In the beginning of this Subsection we will give some basic relationships concerning inclusive electron scattering from nuclei. An electron with four-momentum $k^\mu = (\epsilon, \mathbf{k})$ is scattered through an angle $\theta$ to four-momentum $k'^\mu = (\epsilon', \mathbf{k'})$. The four-momentum transfer is then

$$Q^\mu = (k - k')^\mu = (\omega, \mathbf{q}),$$

(61)
FIG. 5: The $f^\Delta(\psi_\Delta')$ scaling function for $^{12}$C in the $\Delta$-region. Dashed line: CDFM result (with $R_\Delta = 1.565$ fm, $b_\Delta = 0.420$ fm, $k_F = 1.20$ fm$^{-1}$). Solid line: result of modified LFD approach ($\beta = 0.80$, $k_F = 1.20$ fm$^{-1}$). The coefficient $c_1 = 0.54$ in both CDFM and LFD cases. Averaged experimental values of $f^\Delta(\psi_\Delta')$ are taken from [23].

where $\omega = \epsilon - \epsilon'$, $q = |q| = |k - k'|$ and

$$Q^2 = \omega^2 - q^2 \leq 0. \quad (62)$$

In the one-photon-exchange approximation, the double-differential cross section in the laboratory system can be written in the form (e.g. [10]):

$$\frac{d^2\sigma}{d\Omega_{\epsilon'}^{}d\epsilon'} = \sigma_M \left[ \left( \frac{Q^2}{q^2} \right)^2 R_L(q, \omega) + \left( \frac{1}{2} \left| \frac{Q^2}{q^2} \right| + \tan^2 \frac{\theta}{2} \right) R_T(q, \omega) \right], \quad (63)$$

where

$$\sigma_M = \left[ \frac{\alpha \cos(\theta/2)}{2\epsilon \sin^2(\theta/2)} \right]^2 \quad (64)$$

is the Mott cross section and $\alpha$ is the fine structure constant. In Eq. (63) $R_L$ and $R_T$ are the longitudinal and transverse response functions which contain all the information on the distribution of the nuclear electromagnetic charge and current densities, being projections (with respect to the momentum transfer direction) of the nuclear currents. They can be separated experimentally by plotting the cross section against $\tan^2(\theta/2)$ at fixed $(q, \omega)$ (the so-called "Rosenbluth plot"). These functions can be evaluated as components of the nuclear
tensor $W_{\mu\nu}$. In [10] this tensor is computed in the framework of the RFG model and $R_{L(T)}$ for the QE electron scattering are expressed by means of the RFG scaling function (Eq. 9 of Ref. 10).

At leading-order in the parameter $k_F/m_N$ the QE responses have the form [23]:

$R_{QE}^{L}(\kappa, \lambda) = \Lambda_0 \kappa^2 \tau [(1 + \tau)W_2(\tau) - W_1(\tau)] \times f_{RFG}^{QE}(\psi'), \quad (65)$

$R_{QE}^{T}(\kappa, \lambda) = \Lambda_0 [2W_1(\tau)] \times f_{RFG}^{QE}(\psi'), \quad (66)$

with

$\Lambda_0 \equiv \frac{N\xi_F}{m_N^3\eta_F^3}, \quad (67)$

where $N = Z$ or $N$ and $W_1, W_2$ are the structure functions for elastic scattering which are linked to the Sachs form factors

$(1 + \tau)W_2(\tau) - W_1(\tau) = G_E^2(\tau), \quad (68)$

$2W_1(\tau) = 2\tau G_M^2(\tau). \quad (69)$

In [23, 57] the electro-production of the $\Delta$-resonance is considered computing the nuclear tensor also within the RFG model and analytical expressions for the response functions are obtained. The latter contain the RFG $\Delta$-peak scaling function (54) and read [57]:

$R_{L}(\kappa, \lambda) = \frac{3N\xi_F}{2m_N^3\eta_F^3}\kappa^2 \tau [(1 + \tau\rho^2)w_2(\tau) - w_1(\tau) + w_2(\tau)D(\kappa, \lambda)] \times f_{RFG}^{\Delta}(\psi'), \quad (70)$

$R_{T}(\kappa, \lambda) = \frac{3N\xi_F}{2m_N^3\eta_F^3}[2w_1(\tau) + w_2(\tau)D(\kappa, \lambda)] \times f_{RFG}^{\Delta}(\psi'), \quad (71)$

where $N = Z$ or $N,$

$D(\kappa, \lambda) \equiv \frac{\tau}{\kappa^2}[(\lambda\rho + 1)^2 + (\lambda\rho + 1)(1 + \psi_\Delta^2)\xi_F$

$\times \frac{1}{3}(1 + \psi_\Delta^2 + \psi_\Delta^4)\xi_F^2] - (1 + \tau\rho^2). \quad (72)$

The single-baryon structure functions can be expressed by means of the electric ($G_E$), magnetic ($G_M$) and Coulomb ($G_C$) delta form factors [50]:

$w_1(\tau) = \frac{1}{2}(\mu_\Delta + 1)^2(2\tau\rho + 1 - \mu_\Delta)(G_M^2 + 3G_E^2), \quad (73)$
\[ w_2(\tau) = \frac{1}{2} \left( \mu_\Delta + 1 \right)^2 \left( 1 + \frac{1 - \mu_\Delta}{1 + \tau_\rho^2} \right) \left( G_M^2 + 3G_E^2 + 4 \frac{\tau}{\mu_\Delta} G_C^2 \right). \] (74)

These form factors are parametrized as follows [57]:

\[ G_M(Q^2) = 2.97 f(Q^2), \] (75)

\[ G_E(Q^2) = -0.03 f(Q^2), \] (76)

\[ G_C(Q^2) = -0.15 G_M(Q^2), \] (77)

where

\[ f(Q^2) = G_E^P(Q^2) \frac{1}{\left[ 1 - \frac{Q^2}{3.5(\text{GeV/c})^2} \right]^{1/2}} \] (78)

with

\[ G_E^P = \frac{1}{(1 + 4.97\tau)^2} \] (79)

being the Galster parametrization [58] of the electric form factor.

In the CDFM the longitudinal and transverse response functions can be obtained by averaging the RFG response functions in the QE region [Eqs. (65) and (66)] and \( \Delta \)-region [Eqs. (70) and (71)] by means of the weight functions in \( r \)-space \( |F(R)|^2 \) and \( k \)-space \( |G(k_F)|^2 \), similarly as in the case of the QE- and \( \Delta \)-scaling functions (Eqs. (1), (5), (15), (23) and (55), (58), respectively). As a result, accounting for the different behavior of the RFG scaling functions and terms containing \( \eta_F(R) = k_F(R)/m_N \) as functions of \( R \) or \( k_F = \alpha/R \) in (69), (68), (70) and (71), the CDFM response functions \( R_{L(T)} \) in QE- and \( \Delta \)-regions have approximately the same forms as in the equations just mentioned, in which, however, the RFG scaling functions are changed by the CDFM scaling functions obtained in Sections 2 and 3.

In Figs. 6-15 we give results of calculations within the CDFM of inclusive electron scattering on \( ^{12}\text{C} \) at different incident energies and angles. The QE-contribution is calculated using the Fermi-type density distribution of \( ^{12}\text{C} \) with the same values of the parameters as in [13, 14]: \( R = 2.47 \text{ fm} \) and \( b = 0.42 \text{ fm} \) (which lead to a charge rms radius equal to the experimental one) and Fermi momentum \( k_F = 1.156 \text{ fm}^{-1} \). The delta-contribution is calculated using the necessary changes of the parameter values of the Fermi-type density (used
in Fig. 5: \( R_{\Delta} = 1.565 \) fm, \( b_{\Delta} = 0.42 \) fm and \( k_F = 1.20 \) fm\(^{-1}\). The coefficient \( c_1 \) used in the \( \Delta \)-region scaling function is fixed to be equal to 0.54 so that the maximum of the scaling function to be in agreement with the data. The scaling function \( f^\Delta(\psi_{\Delta}) \) is symmetric, its maximum is chosen to be 0.54 (but not 0.75) and it is normalized to unity by means of the fixed value of \( k_F = 1.20 \) fm\(^{-1}\). The inclusive electron-\(^{12}\)C scattering cross sections shown in Figs. 6-15 are the sum of the QE and \( \Delta \)-contribution. The results of the CDFM calculations are presented for two values of the coefficient \( c_1 \) in the QE case (noted further by \( c_{QE}^1 \)), namely for \( c_{QE}^1 \simeq 0.72 \) and \( c_{QE}^1 = 0.63 \). This is related to two types of experimental data. In the first one the transferred momentum in the position of the maximum of the QE peak extracted from data (\( \omega_{\text{exp}}^{QE} \)) is \( q_{\text{exp}}^{QE} \geq 450 \) MeV/c \( \approx 2k_F \), roughly corresponding to the domain where scaling is fulfilled \([19, 23]\). Such cases are presented in Figs. 6-12. In these cases we found by fitting to the maximum of the QE peak the value of \( c_{QE}^1 \) to be 0.72–0.73, i.e. it is not the same as in the RFG model case (case of symmetry of the RFG and of the CDFM scaling functions with \( c_{QE}^1 = 0.75 \), but is slightly lower. This leads to a weak asymmetry of the CDFM scaling function for cases in which \( q_{\text{exp}}^{QE} \geq 450 \) MeV/c. In the second type of experimental data \( q_{\text{exp}}^{QE} \) is not in the scaling region (\( q_{\text{exp}}^{QE} < 450 \) MeV/c). Such cases are given in Figs. 13-15. For them we found by fitting to the maximum of the QE peak extracted from data the value of \( c_{QE}^1 \) to be 0.63. For these cases the CDFM scaling function is definitely asymmetric. So, the results in Figs. 6-15 are presented for both almost symmetric \( (c_{QE}^1 \simeq 0.72) \) and asymmetric \( (c_{QE}^1 = 0.63) \) CDFM scaling functions. One can see that the results for the almost symmetric CDFM scaling function agree with the electron data in the region close to the QE peak in cases where \( q_{\text{exp}}^{QE} \geq 450 \) MeV/c and overestimate the data for cases where approximately \( q_{\text{exp}}^{QE} < 450 \) MeV/c. The results with asymmetric CDFM scaling function agree with the data in cases where \( q_{\text{exp}}^{QE} < 450 \) MeV/c and underestimate the data in cases where \( q_{\text{exp}}^{QE} \geq 450 \) MeV/c. Here we would like to emphasize that, in our opinion, the usage of asymmetric CDFM scaling function is preferable, though the results in some cases can underestimate the empirical data, because other additional effects, apart from QE and \( \Delta \)-resonance (e.g. meson exchange currents effects) could give important contributions to the cross section for some specific kinematics and minor for others. A similar situation occurs for the results obtained within the RMF approach \([25]\) particularly when the CC2 current operator is selected.

In Table 1 we list the energies, the angles, the values of \( c_{QE}^1 \) obtained by fitting the
magnitude of the QE peak, and the energy shifts in the QE and Δ-case, as well as the approximate values of the transfer momentum $q_{exp}^{QE}$ in the position of the maximum of the QE peak ($\omega_{exp}^{QE}$) for different cases. The values of the energy shifts $\epsilon_{shift}^{QE(\Delta)}$ for the QE- and Δ-regions are generally between 20 and 30 MeV. In the Figures we also present the QE-contribution (as well as Δ-contribution) for the value of $c_1^{QE}$ which fits approximately the magnitude of the QE peak.

FIG. 6: Inclusive electron scattering on $^{12}\text{C}$ at $\epsilon = 1299$ MeV and $\theta = 37.5^\circ$ ($q_{exp}^{QE} = 792$ MeV/c $> 2k_F$). The results obtained using $c_1^{QE} = 0.72$ in the CDFM scaling function for the QE cross section and the total result are given by dashed and thick solid line, respectively. Dotted line: using CDFM Δ-scaling function; thin solid line: total CDFM result with $c_1^{QE} = 0.63$. Dash-dotted line: result of ERFG method [19, 23]. The experimental data are taken from [59].

In Figs. 8 and 11 we present also the calculations of the electron cross sections using QE- and Δ-scaling functions obtained by using the nucleon momentum distributions obtained in the LFD approach (Section 3) which give a reasonable agreement with the empirical electron scattering data. In Figs. 8 and 14 we also give for comparison the results of the cross sections obtained within the ERFG method [19, 23]. In this method the response functions and differential cross sections are calculated using the scaling function fitted to the data.

It is interesting to note that for those kinematics where the overlap between the QE and
FIG. 7: Inclusive electron scattering on $^{12}$C at $\epsilon = 2020$ MeV and $\theta = 20.02^\circ$ ($q_{exp}^{QE} = 703$ MeV/c $> 2k_F$). The results obtained using $c_1^{QE} = 0.73$ in the CDFM scaling function for the QE cross section and the total result are given by dashed and thick solid line, respectively. Dotted line: using CDFM $\Delta$-scaling function. Thin solid line: total CDFM result with $c_1^{QE} = 0.63$. The experimental data are taken from [60].

$\Delta$ peaks is bigger (Figs. 6, 7 and 8), the asymmetric CDFM model ($c_1^{QE} = 0.63$) gives rise to an excess of strength in the transition region. This makes a difference with the ERFG model (see Fig. 6) which fits nicely the data in that region. This discrepancy between the two models, asymmetric CDFM and ERFG, can be explained by noting the different behavior presented by the two scaling functions in the region of $\psi'$ between 0.5 and 1.5, being the asymmetric CDFM one significantly larger.

Note on the other hand, that in the cases where the overlap between the QE and $\Delta$ peaks is weaker (Figs. 9, 12), the asymmetric CDFM model, compared to the almost symmetric CDFM one, reproduces better the data in the transition region although it underpredicts importantly the maximum of the QE peak. Concerning results in Figs. 13-15 (it can be also applied to Figs. 11 and 12), one observes that both CDFM approaches do not reproduce the strength of data located in the region between the QE and delta peaks. This is not the case for the ERFG model (see Fig. 14) which fits nicely the experiment for $\omega \geq 180$ MeV. This result is connected with the much bigger strength of the scaling function provided by
FIG. 8: The same as in Fig. 7 for $\epsilon = 1108$ MeV and $\theta = 37.5^\circ$ ($q_{\text{exp}}^{QE} = 675$ MeV/c $> 2k_F$). Dotted-dashed line: using QE- and $\Delta$-scaling functions obtained in the LFD approach. The experimental data are taken from [59].

the ERFG model for larger values of the scaling variable, $\psi' \geq 2$ (see Fig. 2).

From this whole analysis, one may conclude that the phenomenological procedure introduced in the CDFM model to get an asymmetric scaling function, gives rise to an excess of strength in the region $0.5 \leq \psi' \leq 1.5$, whereas the model lacks strength for larger $\psi'$-values, $\psi' \geq 2$.

B. Scaling functions and charge-changing neutrino-nucleus reaction cross sections

In this Subsection we will present applications of the CDFM and LFD scaling functions to calculations of charge-changing neutrino-nucleus reaction cross sections. We follow the description of the formalism given in [23]. The charge-changing neutrino cross section in the target laboratory frame is given in the form

$$
\left[ \frac{d^2\sigma}{d\Omega dk'} \right]_\chi \equiv \sigma_0 \mathcal{F}_\chi^2,
$$

(80)
FIG. 9: The same as in Fig. 7 for $\epsilon = 620$ MeV and $\theta = 60^\circ$ ($q_{\text{exp}}^{QE} = 552$ MeV/c > $2k_F$). The experimental data are taken from [61].

FIG. 10: The same as in Fig. 6 for $\epsilon = 2020$ MeV and $\theta = 15.02^\circ$ ($q_{\text{exp}}^{QE} = 530$ MeV/c > $2k_F$) for the CDFM results. The experimental data are taken from [60].

where $\chi = +$ for neutrino-induced reaction (e.g., $\nu_\ell + n \rightarrow \ell^- + p$, where $\ell = e, \mu, \tau$) and $\chi = -$ for antineutrino-induced reactions (e.g., $\bar{\nu}_\ell + p \rightarrow \ell^+ + n$),

$$\sigma_0 \equiv \frac{(G \cos \theta_c)^2}{2\pi^2} [k' \cos \tilde{\theta}/2]^2, \quad (81)$$

28
FIG. 11: The same as in Fig. 6 for $\epsilon = 500$ MeV and $\theta = 60^\circ$ ($q_{\text{QE}} = 450$ MeV/c $\geq 2k_F$). Here the dot-dashed line shows the result using QE- and $\Delta$-scaling functions obtained in the LFD approach. The experimental data are taken from [62].

$G = 1.16639 \times 10^{-5}$ GeV$^{-2}$ being the Fermi constant, $\theta_c$ being Cabibbo angle ($\cos \theta_c = 0.9741$),

$$\tan^2 \tilde{\theta}/2 \equiv \frac{|Q|^2}{v_0},$$  \hspace{1cm} (82)

$$v_0 \equiv (\epsilon + \epsilon')^2 - q^2 = 4\epsilon \epsilon' - |Q|^2.$$  \hspace{1cm} (83)

The quantity $F_\chi^2$ which depends on the nuclear structure is written in [23] as a generalized Rosenbluth decomposition having charge-charge, charge-longitudinal, longitudinal-longitudinal and two types of transverse responses. The nuclear response functions are expressed in terms of the nuclear tensor $W^{\mu \nu}$ in both QE and $\Delta$-regions using its relationships with the RFG model scaling functions. Following [23], in the calculations of the neutrino-nucleus cross sections the Höhler parametrization 8.2 [65] of the form factors in the vector sector was used, while in the axial-vector sector the form factors given in [23] were used.

In our work, instead of the RFG scaling functions in QE- and $\Delta$-regions, we use those obtained in the CDFM and LFD approach (Sections 2 and 3). In Fig. 16 we give the results of calculations for cross sections of QE neutrino ($\nu_\mu$, $\bar{\nu}_\mu$) scattering (Figs. 16a, c,
FIG. 12: The same as in Fig. \( \square \) for \( \epsilon = 730 \) MeV and \( \theta = 37.1^\circ \) \( (q_{\text{QE exp}}^{\text{QE}} = 442 \) MeV/c \( \leq 2k_F) \) for the CDFM results. The experimental data are taken from [63].

FIG. 13: The same as in Fig. \( \square \) for \( \epsilon = 1650 \) MeV and \( \theta = 13.5^\circ \) \( (q_{\text{QE exp}}^{\text{QE}} = 390 \) MeV/c \( \leq 2k_F) \) for the CDFM results. The experimental data are taken from [64].

d, e, f) on \(^{12}\text{C}\) and also antineutrino \((\overline{\nu}_\mu, \mu^+)\) scattering (Fig. \( \square \)) for energies of neutrino \( \epsilon_\nu = 1, 1.5 \) and \( 2 \) GeV and of antineutrino \( \epsilon_\overline{\nu} = 1 \) GeV. The presented cross sections are functions of muon kinetic energy. The energy shift is equal to 20 MeV. The calculations of the neutrino-nucleus cross sections in the \( \Delta \)-region will be a subject of a future work.
FIG. 14: The same as in Fig. for $\epsilon = 1500$ MeV and $\theta = 13.5^\circ$ ($q_{exp}^{QE} = 352$ MeV/c $\leq 2k_F$). The experimental data are taken from [64].

We give the results of our calculations using the CDFM scaling function which is almost symmetric (with $c_1 = 0.72$), as well as the asymmetric CDFM scaling function (with $c_1 = 0.63$). These values of $c_1$ correspond to the cases of inclusive electron scattering considered. As can be seen the results obtained by using the almost symmetric CDFM scaling function

FIG. 15: The same as in Fig. for $\epsilon = 537$ MeV and $\theta = 37.1^\circ$ ($q_{exp}^{QE} = 326$ MeV/c $\leq 2k_F$) for the CDFM results. The experimental data are taken from [63].
TABLE I: Values of energies $\epsilon$, angles $\theta$, the coefficient $c^Q_E$ obtained by fitting the magnitude of the QE peak, energy shifts $\epsilon^Q_E_{\text{shift}}$ and $\epsilon^\Delta_{\text{shift}}$, and transferred momenta $q^Q_E_{\text{exp}}$ for the cases of inclusive electron scattering cross sections considered. Energies are in MeV, angles are in degrees and momenta are in MeV/c.

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are close to the RFG model results. On the other hand, the results obtained with the use of asymmetric CDFM and LFD scaling functions are quite different from those in the RFG model, but are close to the predictions of the ERFG model [19, 23]. The basic difference from the ERFG model result is observed in the tail extended to small muon energy values, where the ERFG model gives more strength.

V. CONCLUSIONS

The results of the present work can be summarized as follows:

i) In Ref. [14] we extended the CDFM description of the quasielastic $\psi'$-scaling function from [13] by expressing it explicitly and equivalently by means of both density and nucleon momentum distributions. In [13, 14] our results on $f^Q_E(\psi')$ were obtained on the basis of the experimental data on the charge densities for a wide range of nuclei. In the present work we extended our approach to consider the scaling function $f^Q_E(\psi')$ for medium and heavy nuclei with $Z \neq N$ for which the proton and neutron densities are not similar. In
FIG. 16: The cross section of quasielastic charge-changing ($\nu_\mu, \mu^-$) reaction [(a), (c)-(f)] and of ($\bar{\nu}_\mu, \mu^+$) reaction (b) on $^{12}$C for $\epsilon = 1, 1.5$ and 2 GeV using QE-scaling functions in CDFM (thin solid line: with $c_1 = 0.63$; thin dashed line: with $c_1 = 0.72$). The results using QE-scaling functions in LFD (thick solid line: with $c_1 = 0.63$; thick dashed line: with $c_1 = 0.72$) are presented in (b) and (f). The RFG model result and ERFG result [19, 23] are shown by dotted and dash-dotted lines, respectively.
this case $f^{QE}(\psi')$ is a sum of the proton and neutron scaling functions calculated by means of the proton and neutron densities obtained from nonrelativistic self-consistent mean-field calculations. This concerns calculations, as examples, of nuclei like $^{197}$Au, $^{82}$Kr, as well as $^{62}$Ni and $^{118}$Sn [13]. The comparison with the data from [11, 12] shows superscaling for negative values of the QE $\psi'$ including $\psi' < -1$, whereas in the RFG model $f(\psi') = 0$ for $\psi' \leq -1$ (see Fig. 1).

ii) We introduce the asymmetry in the CDFM QE scaling function using the fact that the maximum value of $f(\psi')$ in RFG model is $3/4$ while the empirical scaling function reaches values smaller than 0.6. In relation with this and the normalization, we parametrize the RFG scaling function for $\psi' \geq 0$, thus simulating the role of all the effects which lead to asymmetry and imposing this to the CDFM QE scaling function. In this way, simulating phenomenologically the effects which violate the symmetry of $f^{QE}(\psi')$ for $\psi' \geq 0$ including the role of the FSI, one can obtain in the CDFM a reasonable agreement of $f^{QE}(\psi')$ with the empirical data also for positive values of $\psi'$ (Fig. 2).

iii) We obtain the QE scaling function also on the basis of calculations of nucleon momentum distribution $n(k)$ within an approach based on the light-front dynamics method [44, 45, 46] which improves that used in Ref. [14]. Here we include in the particle-state part of $n(k)$ not only a contribution of the function $f_5$ (as in [44] and [14]) but also a contribution of the function $f_2$. $f_5$ and $f_2$ are two of the six scalar components of the deuteron total wave function in the LFD [45, 46] and are the main contributions to the tail of $n_d(k)$. It can be seen in Fig. 3 the reasonable agreement of $n(k)$ in LFD with the $y_{CW}$-scaling data [7, 8]. This result made it possible to obtain a good description of the experimental QE scaling function (Fig. 4) at least up to $\psi' \simeq -1.2$.

iv) We extend our analysis within the CDFM and LFD to the $\Delta$-peak region which is the main contribution to the inelastic scattering. Here we emphasize that reasonable agreement with the experimental data (Fig. 5) was obtained using the empirical value of the coefficient in front of the RFG scaling function (0.54 instead of 0.75) in both CDFM and LFD. Also, the parameter $R_\Delta$ used in the Fermi-type density for $^{12}$C (necessary to calculate the weight function $|F_\Delta(R)|^2$ and thus the scaling function $f_\Delta(\psi'_\Delta)$) has a smaller value (1.565 fm) than that ($R=2.42$ fm) in the QE case, while the value of the diffuseness parameter $b_\Delta$ remains the same as $b$ in the QE case. We note that the use of $n_{LFD}(k)$ with the same values of $\beta$ and of $k_F$ ($\beta = 0.80$, $k_F = 1.20$ fm$^{-1}$) gives a reasonable agreement with results for both
QE- and $\Delta$-region scaling functions (Figs. 4 and 5).

v) The QE- and $\Delta$-region scaling functions obtained in the CDFM and in the LFD approach are applied to description of experimental data on differential cross sections of inclusive electron scattering by $^{12}\text{C}$ at large energies and transferred momenta (Figs. 6-15). The CDFM results are presented for both almost symmetric ($c_{1}^{QE} \approx 0.72$) and asymmetric ($c_{1}^{QE} = 0.63$) scaling functions. We observe that there are two regions of the value of $q_{exp}^{QE}$ in different experiments at which the above mentioned (almost symmetric and asymmetric) scaling functions work better. The almost symmetric scaling function leads to results in agreement with the data in the region of the QE peak in cases when the transferred momentum ($q_{exp}^{QE}$) in the position of maximum of the QE peak ($\omega_{exp}^{QE}$) is in the scaling region ($q_{exp}^{QE} \geq 450$ MeV/c $\approx 2k_F$), while the data are overestimated in cases where $q_{exp}^{QE} < 450$ MeV/c. The results obtained when asymmetric scaling function ($c_{1}^{QE} = 0.63$) is used agree with the data in cases when $q_{exp}^{QE} < 450$ MeV/c and underestimate them when $q_{exp}^{QE} \geq 450$ MeV/c in the region close to the QE peak, but differences emerge in the transition region.

In our opinion, the latter case is preferable because additional effects (apart from QE and $\Delta$-resonance), e.g. of the meson exchange currents could give additional important contributions to the inclusive electron cross sections for some specific kinematics and minor for others.

vi) The CDFM and LFD scaling functions are applied to calculations of QE charge-changing neutrino-nuclei reaction cross sections. We present in Fig. 16 the predicted cross sections for the reactions ($\nu_{\mu}, \mu^{-}$) and ($\overline{\nu}_{\mu}, \mu^{+}$) on the $^{12}\text{C}$ nucleus for energies of the incident particles from 1 to 2 GeV. Our results are compared with those from the RFG model and from the ERFG model [19, 23]. The results obtained by using the asymmetric CDFM scaling function are close to those of ERFG and are quite different from the RFG results while the almost symmetric CDFM scaling function leads to cross sections which are similar to the results of the RFG model.

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