Nonholonomic Ricci Flows and Running Cosmological Constant: I. 4D Taub–NUT Metrics

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Abstract
In this work we construct and analyze exact solutions describing Ricci flows and nonholonomic deformations of four dimensional (4D) Taub-NUT spacetimes. It is outlined a new geometric techniques of constructing Ricci flow solutions. Some conceptual issues on spacetimes provided with generic off–diagonal metrics and associated nonlinear connection structures are analyzed. The limit from gravity/Ricci flow models with nontrivial torsion to configurations with the Levi-Civita connection is allowed in some specific physical circumstances by constraining the class of integral varieties for the Einstein and Ricci flow equations.

Keywords: Ricci flows, exact solutions, Taub-NUT spaces, anholonomic frame method.

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1 Introduction

Hawking has suggested [1] that the Euclidean Taub-NUT metric might give rise to the gravitational analogue of the Yang-Mills instanton. Also, in the long-distant limit, neglecting radiation, the relative motion of two monopoles is described by the geodesics of this space [2, 3]. The Kaluza-Klein monopole was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional Kaluza-Klein theory [4, 5] (there are various classes of solitonic/monopole solutions, in brief termed KK monopoles; see further developments and reviews of results in pseudo-classical models [6, 7]).

On the other hand, the recent experimental data seem to indicate that the universe does indeed possess a small positive cosmological constant. This induced a program of researches on generalized solutions in asymptotically (anti)–de Sitter (AdS) spacetimes (KK–AdS–Taub-NUT solutions [8, 9, 10] and new cosmological Taub–NUT like solutions [11, 12]).

The problem of extending the gravitational monopole solutions to extra dimensions and/or on spaces with nontrivial cosmological constants is related to a more general problem of constructing and interpretation of solutions defining physical objects self–consistently embedded in arbitrary nontrivial gravitational backgrounds. The usual techniques for generating new classes of monopole solutions with nonzero cosmological constants is to add the time like coordinate, to generalize the metric to time dependencies (for instance, to a cosmological model) and than to perform the Kaluza–Klein compactification on the fifth dimension (in brief, 5D).

There is a more general approach of constructing exact solutions in gravity following the so–called ‘anholonomic frame method’ elaborated and developed in a series of works (see Refs. [13, 14, 15]). The idea is to use certain classes of nonholonomic (equivalently, anholonomic) deformations of the frame, metric and connection structures and superpositions of generalized conformal maps in order to generate a class of off–diagonal metric ansatz solving exactly the vacuum or nonvacuum Einstein equations. The method was considered, for instance, for constructing locally anisotropic Taub-NUT solutions [16] and investigating self–consistent propagations of three dimensional Dirac and/or solitonic waves in such spacetimes [17].

A general nonholonomic transform of a ”primary” metric (it can be an-

\footnote{Such ansatz can not be diagonalized by coordinate transforms but can be effectively diagonalized with respect to certain systems of nonholonomic local frames with associated nonlinear connection structures, see details in [15]. This allows us to apply a well developed geometric techniques in order to define generalized symmetries and to integrate exactly the corresponding systems of field equations. For instance, for 5D spacetimes, such solutions depend on sets of integrating functions on four and three variables.}
exact solution or, for instance, a conformal transform of a known exact solution) into a generic off–diagonal exact solution does not preserve the properties of former metric. Nevertheless, if there are satisfied certain smooth limits to already known solutions, boundary (asymptotic) and deforming symmetry conditions, one may consider that, for instance, a primary Taub-NUT configuration became locally anisotropic with polarized constants and/or imbedded self–consistently in a nontrivial, for instance, solitonic background. The new classes of solutions can be used for testing various type of physical theories (string/brane gravity, noncommutative and gauge gravity, Finsler like generalizations ...) when the physical interactions are described by generic off–diagonal metrics and nonholonomic constraints; in general, with nontrivial topological configurations and new type symmetries (non–Killing ones, for instance, with generalized Lie algebra symmetries); the extra dimensions are not subjected to the Kaluza Klein restrictions.

The anholonomic frame method seems to be effective in constructing exact solutions of the Ricci flow equations [18]. Such equations were introduced by R. Hamilton [19] in order to describe the geometric evolution of a Riemannian manifold \((V, g)\), where by \(g\) we denote the metric, in the direction of its Ricci tensor \(Ric(g)\), see reviews of results and applications in [20, 21, 22]. There is a vast potential use of this geometric approach in theoretical physics and cosmology [23, 24, 26, 27, 28, 29, 30].

It is known that there is a unique solution for the Ricci flow with bounded curvature on a complete noncompact manifold [31]. A very important task is to construct exact solutions for the Ricci flow equations and to investigate their physical implications. The purpose of this work is two–fold. The first one is to elaborate a general method of constructing exact Ricci flow solutions for certain classes of off–diagonally deformed metric ansatz, both for connections with trivial and nontrivial torsion. The other is to construct explicit examples of such 4D exact solutions, for Taub-NUT like metrics, to analyze their physical properties and to show that they may possess nontrivial limits to the Einstein spaces.

The structure of the paper is as follows: In section 2, we define a new geometric approach to constructing exact solutions for Ricci flows of generic off–diagonal metrics. The approach is based on the so–called anholonomic frame method with associated nonlinear connection structure. We construct a general class of integral varieties of Ricci flow equations and define the constraints for the Levi-Civita configurations. In Section 3, we apply the formalism to 4D Taub–NUT configurations and their off–diagonal Ricci flows. The last section is devoted to conclusions and discussion. In Appendix A, we outline the necessary material from the geometry of nonlinear connections and related nonholonomic deformations.
2 Nonholonomic Ricci flows

In this section we introduce an off–diagonal ansatz for metrics depending on three coordinates and consider the anholonomic frame method of constructing exact solutions for the system of equations defining Ricci flows of metrics subjected to nonholonomic constraints.

2.1 Geometric preliminaries

The normalized Ricci flows \[19, 20, 21, 22, 23, 24\], with respect to a coordinate base \( \partial_\alpha = \partial/\partial u^\alpha \), are defined by the equations

\[
\frac{\partial}{\partial \tau} g_{\alpha\beta} = -2R_{\alpha\beta} + \frac{2r}{5} g_{\alpha\beta},
\]

(1)

where \( R_{\alpha\beta} \) is the Ricci tensor of a metric \( g_{\alpha\beta} \) and corresponding Levi–Civita connection\(^2\) and the normalizing factor \( r = \int R dV/dV \) is introduced in order to preserve the volume \( V \). In this work we shall consider flows of generic off–diagonal four dimensional (4D) metrics

\[
g = g_{\alpha\beta}(u) du^\alpha \otimes du^\beta,
\]

(2)

where

\[
g_{\alpha\beta} = \left[ \begin{array}{cc} g_{ij} + N_i^a N_j^b h_{ab} & N_i^e h_{ae} \\ N_j^e h_{be} & h_{ab} \end{array} \right],
\]

with the indices of type \( \alpha, \beta = (i, a), (j, b) \)... running the values \( i, j, ... = 1, 2 \) and \( a, b, ... = 3, 4 \) (we shall omit underlying of indices for the components of arbitrary basis or even with respect to coordinate basis if that will not result in ambiguities) and local coordinates labelled in the form \( u = (x, y) = \{ u^\alpha = (x^i, y^a) \} \).

Applying the frame transforms

\[
e_\alpha = e_\alpha^\alpha \partial_\alpha \quad \text{and} \quad c_\alpha = e_\alpha^\alpha du^\alpha
\]

for \( e_\alpha | c^\beta = \delta_\alpha^\beta \), where "|" denotes the inner product and \( \delta_\alpha^\beta \) is the Kronecker symbol, with

\[
e_\alpha^\alpha = \left[ \begin{array}{c} \delta_i^i \\ 0 \end{array} \right], \quad e_\alpha^\beta = \left[ \begin{array}{c} 0 \\ \delta_\beta^\alpha \end{array} \right],
\]

(3)

\(^2\)By definition this connection is torsionless and metric compatible.
we represent the metric (2) in an effectively diagonalized \((n + m)\)-distinguished form, see the metric (A.5) in Appendix,

\[
g = g_\alpha(u) \, c^\alpha \otimes c^\alpha = g_i(u) \, b^i \otimes b^i + h_a(u) \, b^a \otimes b^a, \\
c^\alpha = (b^i = dx^i, b^a = dy^a + N^a_i(u)dx^i).
\] (4)

Such metrics and frame transforms are considered in the geometry of nonholonomic manifolds with associated N–connection structure defined by the set \(N = \{N^b_k\}\) stating a nonintegrable distribution on a 4D manifold \(V\). In Appendix A, we outline the geometry of such spaces. In this paper, we shall consider classes of solutions with ansatz of type (4) (and (A.1) or (A.5)), when

\[
g_i = g_i(x^k), h_a = h_a(x^k, v), N^3_i = w_i(x^k, v), N^4_i = n_i(x^k, v),
\] (5)

for \(y^3 = v\) being the so–called ”anisotropic” coordinate. Such metrics are very general off–diagonal ones, with the coefficients depending on 2 and 3 coordinates but not depending on the coordinate \(y^4\).

To consider flows of metrics related both to the Einstein and string gravity (in the last case there is a nontrivial antisymmetric torsion field), it is convenient to work with the so–called canonical distinguished connection (in brief, d–connection) \(\hat{\mathbf{D}} = \{\hat{\mathbf{\Gamma}}_{\alpha\beta}\}\) which is metric compatible but with nontrivial torsion (see formulas (A.15) and related discussions). Imposing certain restrictions on the coefficients \(N^b_k\), see (A.17), (A.18) and (A.19), we can satisfy the conditions that the coefficients of the canonical d–connection of the Levi–Civita \(\nabla = \{\Gamma^\alpha_{\beta\gamma}\}\) are defined by the same nontrivial values \(\hat{\mathbf{\Gamma}}_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}\) with respect to \(N\)–adapted basis (A.7) and (A.8).

The Ricci flow equations (1) can be written for the Ricci tensor of the canonical d–connection (A.13) and metric (4), as it was considered in Ref. [18],

\[
\frac{\partial}{\partial \tau} g_{ij} = -2R_{ij} + 2\lambda g_{ij} - h_{cd} \frac{\partial}{\partial \tau} (N^c_i N^d_j),
\] (6)

\[
\frac{\partial}{\partial \tau} h_{ab} = -2S_{ab} + 2\lambda h_{ab},
\] (7)

\[R_{\alpha\beta} = 0\] and \(g_{\alpha\beta} = 0\) for \(\alpha \neq \beta\),

where \(\lambda = r/5\), \(y^3 = v\) and \(\tau\) can be, for instance, the time like coordinate, \(\tau = t\), or any parameter or extra dimension coordinate. The equations (6) and (7) are just the nonholonomic frame transform with the matrices (3) of the equations (1). The aim of this section is to show how the anholonomic frame method developed in [18] (for Ricci flows) and in [13, 14, 15, 16, 17]
when the polarizations \( \eta \) metric cation in extra/lower dimension spaces.

like solutions, or some their conformal transforms and/or trivial embedding / compactification (which is not an exact solution).

lution of the Einstein equations, or any conformal transform of a such one nonholonomically constrained coordinate. The metric may be an exact solution of the Einstein equations, or any conformal transform of a such one (which is not an exact solution)\(^3\). An anholonomic transform, \( \tilde{\mathbf{N}} \rightarrow \mathbf{N} \) and \( \tilde{\mathbf{g}} \rightarrow \mathbf{g} \rightarrow \mathbf{g} = (g, h) \), can defined by formulas (A.6),

\[
g_i = \eta_i(x^k, v, y^4)\tilde{g}_i, h_a = \eta_a(x^k, v, y^4)\tilde{h}_a, N_i^a = \eta_i^a(x^k, v, y^4)\tilde{N}_i^a
\]

when the polarizations \( \eta \) and \( \eta^a \) are chosen in a form that the "target" metric \( \mathbf{g} \) has coefficients of type \( \tilde{\mathbf{g}} \), i.e. it is parametrized in the form

\[
g = g_1(x^k)(dx^1)^2 + g_2(x^k)(dx^2)^2 + h_3(x^k, v) (b^3)^2 + h_4(x^k, v) (b^4)^2,
\]

\[
b^3 = dv + \omega_3(x^k, v) \, dx^i, \quad b^4 = dy^4 + n_i(x^k, v) \, dx^i,
\]

defining an exact solution for the nonholonomic Ricci flow equations (6) and (7), when \( \tau \) is one of the coordinates \( x^k \), or \( v \).

The nontrivial components of the Ricci tensor \( R_{\alpha\beta} \) (A.13) (see details of a similar calculus in Ref. [15]) are

\[
R^1_{\ 1} = R^2_{\ 2} = \frac{1}{2g_1g_2} \left[ \frac{g_1 \ddot{g}_2}{2g_1} + \frac{(g_2)^2}{2g_2} - g_2 \frac{\ddot{g}_1}{2g_1} + \frac{(g_1)^2}{2g_2} - g_1'' \right],
\]

\[
S^3_{\ 3} = S^4_{\ 4} = \frac{1}{2h_3h_4} \left[ -h_{4}^{**} + \frac{(h_{4}^{*})^2}{2h_4} + \frac{h_{4}^{*}h_{3}^{*}}{2h_4} \right],
\]

\[
R_{3i} = -\frac{1}{2h_4} (w_i \beta + \alpha_i),
\]

\[
R_{4i} = -\frac{h_4}{2h_3} (n_i^{**} + \gamma n_i^*)
\]

\(^3\)In this work, we shall consider the primary metrics to be related to certain Taub-NUT like solutions, or some their conformal transforms and/or trivial embedding / compactification in extra/lower dimension spaces.
where
\[
\alpha_i = \partial_i h_4^* - h_4^* \partial_i \ln |h_3 h_4|, \quad \beta = h_4^{**} - h_4^* \left( \ln |h_3 h_4| \right)^*, \quad (14)
\]
\[
\gamma = 3h_4^*/2h_4 - h_3^*/h_3, \quad \text{for } h_3^* \neq 0, h_4^* \neq 0,
\]
defined by \( h_3 \) and \( h_4 \) as solutions of equations \((7)\). In the above presented formulas, it was convenient to write the partial derivatives in the form \( a^* = \partial a/\partial x^1, a' = \partial a/\partial x^2 \) and \( a^* = \partial a/\partial v \).

We consider a general method of constructing solutions of the Ricci flows equations related to the so-called Einstein spaces with nonhomogeneously polarized cosmological constant, when
\[
R_{ij} = \lambda_{[h]}(x^k) \delta_{ij}, \quad S_{ab} = \lambda_{[v]}(x^k, v) \delta_{ab},
\]
\[
R_{\alpha\beta} = 0 \quad \text{and} \quad g_{\alpha\beta} = 0 \quad \text{for } \alpha \neq \beta,
\]
when \( \lambda_{[h]} \) and \( \lambda_{[v]} \) are induced by certain string gravity ansatz \((A.27)\), or matter field contributions. In particular case, we can fix \( \lambda_{[h]} = \lambda_{[v]} \) and consider off–diagonal metrics of the usual Einstein spaces with cosmological constant.

The nonholonomic Ricci flows equations \((6)\) and \((7)\) for the Einstein spaces with nonhomogeneous cosmological constant defined by ansatz of type \((9)\) transform into the following system of partial differential equations consisting from two subsets of equations: The first subset of equations consists from those generated by the Einstein equations for the off–diagonal metric,
\[
\frac{g_1'g_2'}{2g_1} + \frac{(g_2')^2}{2g_2} - g_2^* + \frac{g_1'g_2^*}{2g_1} + \frac{(g_1')^2}{2g_2} - g_1^* = 2g_1 g_2 \lambda_{[h]}(x^k), \quad (15)
\]
\[-h_4^{**} + \frac{(h_4^*)^2}{2h_4} + \frac{h_4^* h_3^*}{2h_4} = 2h_3^* h_4 \lambda_{[v]}(x^k, v), \quad (16)
\]
\[
w_i \beta + \alpha_i = 0, \quad (17)
\]
\[
n_i^{**} + \gamma n_i^* = 0. \quad (18)
\]
The second subset of equations is formed just by those describing flows of the diagonal, \( g_{ij} = \text{diag}[g_1, g_2] \) and \( h_{ab} = \text{diag}[h_3, h_4] \), and off diagonal, \( w_i \) and \( n_i \), metric coefficients,
\[
\frac{\partial}{\partial \tau} g_{ij} = 2\lambda_{[h]}(x^k) g_{ij} - h_3 \frac{\partial}{\partial \tau} (w_i w_j) - h_4 \frac{\partial}{\partial \tau} (n_i n_j), \quad (19)
\]
\[
\frac{\partial}{\partial \tau} h_a = 2\lambda_{[v]}(x^k, v) h_a. \quad (20)
\]
The aim of the next section is to show how we can integrate the equations \((15)–(20)\) in a quite general form.
2.3 Integral varieties of solutions of Ricci flow equations

We emphasize that the system of equations (15)–(18) was derived and solved in general form for a number of 4D and 5D metric ansatz of type (9), or (A.5), in Refs. [13, 14, 15, 16, 17] for various models of gravity theory. The idea of work [18] was to use the former method and some integral varieties of those solutions in order to subject the metric and N–connection coefficients additionally to the conditions (19) and (20) and generate Ricci flows of off–diagonal metrics. Here, we briefly outline the method of constructing such general solutions.

We begin with equation (15) for the metric coefficients 
\[ g_i(x^k) \] 
on a 2D subspace. By a corresponding coordinate transform 
\[ x^e_i \rightarrow x^e_i(x^i) \], such metrics can be always diagonalized and represented in conformally flat form, 
\[ g_i(x^k)(dx^i)^2 = e^{2\psi} \left[ \epsilon_1(dx^1)^2 + \epsilon_2(dx^2)^2 \right], \]
where the values \( \epsilon_i = \pm 1 \) depend on chosen signature. The equation (15) transforms into 
\[ \epsilon_1 \psi'' + \epsilon_2 \psi'' = 2\lambda h_i(x^k). \] (21)
Such equations and their equivalent 2D coordinate transform can be written in three alternative ways convenient for different types of nonholonomic deformations of metrics. For instance, we can prescribe that \( g_1 = g_2 \) and write the equation for \( \psi = \ln |g_1| = \ln |g_2| \). Alternatively, we can suppose that \( g'_1 = 0 \) for a given \( g_1(x^1) \) (or \( g'_2 = 0 \), for a prescribed \( g_2(x^2) \)) and get from (15) the equation
\[ \frac{g'_1 g'_2}{2g_1} + \frac{(g'_i)^2}{2g_2} - g'' = 2g_1 g_2 \lambda [h_i(x^k)] \] (or
\[ \frac{g'_1 g'_2}{2g_1} + \frac{(g'_i)^2}{2g_2} - g'' = 2g_1 g_2 \lambda [h_i(x^k)], \]
in the inverse case). In general, we can prescribe that, for instance, \( g_1(x^k) \) is defined by any solution of the 2D Laplace/D’Alambert/solitonic equation and try to define \( g_2(x^k) \) constrained to be satisfied one of the above equations related to (15). We conclude that such 2D equations can be solved always in explicit or non explicit form.

Equation (16) relates two nontrivial v–coefficients of the metric coefficients \( h_3(x^k, v) \) and \( h_4(x^k, v) \) depending on three coordinates but with partial derivatives only on the third (anisotropic) coordinate. As a matter of
principle, we can fix $h_3$ (or, inversely, $h_4$) to describe any physically interesting situation being, for instance, a solution of the 3D solitonic, or pp–wave equation, and than we can try to define $h_4$ (inversely, $h_3$) in order to get a solution of (16). Here we note that it is possible to solve such equations for any $\lambda_{[v]}(x^k, v)$, in general form, if $h_4^* \neq 0$ (for $h_4 = 0$, there are nontrivial solutions only if $\lambda_{[v]} = 0$). Introducing the function

$$\phi(x^i, v) = \ln \left| h_4^*/\sqrt{|h_3h_4|} \right|,$$

we write that equation in the form

$$\left( \sqrt{|h_3h_4|} \right)^{-1} (e^\phi)^* = -2\lambda_{[v]}.$$  

Using (22), we express $\sqrt{|h_3h_4|}$ as a function of $\phi$ and $h_4^*$ and obtain

$$|h_4^*| = -(e^\phi)^*/4\lambda_{[v]}$$

which can be integrated in general form,

$$h_4 = h_{4[0]}(x^i) - \frac{1}{4} \int dv \frac{[e^{2\phi(x^i, v)}]^*}{\lambda_{[v]}(x^i, v)}, \quad (25)$$

where $h_{4[0]}(x^i)$ is the integration function. Having defined $h_4$ and using again (22), we can express $h_3$ via $h_4$ and $\phi$,

$$|h_3| = 4e^{-2\phi(x^i, v)} \left[ \left( \sqrt{|h_4|} \right)^* \right]^2.$$  

The conclusion is that prescribing any two functions $\phi(x^i, v)$ and $\lambda_{[v]}(x^i, v)$ we can always find the corresponding metric coefficients $h_3$ and $h_4$ solving (16). Following (26), it is convenient to represent such solutions in the form

$$h_4 = \epsilon_4 \left[ b(x^i, v) - b_0(x^i) \right]^2$$

$$h_3 = 4\epsilon_4 e^{-2\phi(x^i, v)} \left[ b^*(x^i, v) \right]^2$$

where $\epsilon_a = \pm 1$ depending on fixed signature, $b_0(x^i)$ and $\phi(x^i, v)$ can be arbitrary functions and $b^*(x^i, v)$ is any function when $b^*$ is related to $\phi$ and $\lambda_{[v]}$ as stated by the formula (24). Finally, we note that if $\lambda_{[v]} = 0$, we can relate $h_3$ and $h_4$ solving (23) as $(e^\phi)^* = 0$.

For any couples $h_3$ and $h_4$ related by (16), we can compute the values $\alpha_i, \beta$ and $\gamma$ (14). This allows us to define the off–diagonal metric (N–connection) coefficients $w_i$ solving (17) as algebraic equations,

$$w_i = -\alpha_i/\beta = -\partial_i\phi/\phi^*,$$  

\[ \text{(27)} \]
We emphasize, that for the vacuum Einstein equations one can be solutions of (16) resulting in \( \alpha_i = \beta = 0 \). In such cases, \( w_i \) can be arbitrary functions on variables \((x^i, v)\) with finite values for derivatives in the limits \( \alpha_i, \beta \to 0 \) eliminating the "ill-defined" situation \( w_i \to 0/0 \). For the Ricci flow equations with nonzero values of \( \lambda_{[v]} \), such difficulties do not arise. The second subset of N-connection (off-diagonal metric) coefficients \( n_i \) can be computed by integrating two times on variable \( v \) in (18), for given values \( h_3 \) and \( h_4 \). One obtains

\[
\hat{n}_k(x^i, v) = \int h_3(\sqrt{|h_4|})^{-3} dv, \quad h_4^* \neq 0;
\]

\[
= \int h_3 dv, \quad h_4^* = 0;
\]

\[
= \int (\sqrt{|h_4|})^{-3} dv, \quad h_3^* = 0,
\]

and \( n_{k[1]}(x^i) \) and \( n_{k[2]}(x^i) \) are integration functions.

We conclude that any solution \( (h_3, h_4) \) of the equation (16) with \( h_4^* \neq 0 \) and non vanishing \( \lambda_{[v]} \) generates the solutions (27) and (28), respectively, of equations (17) and (18). Such solutions (of the Einstein equations) are defined by the mentioned classes of integration functions and prescribed values for \( b(x^i, v) \) and \( \psi(x^i) \). Further restrictions on \( (g_1, g_2) \) and \( (h_3, h_4) \) are necessary in order to satisfy the equations (19) and (20) relating flows of the metric and N-connection coefficients in a compatible manner. It is not possible to solve in a quite general form such equations, but in the next section we shall give certain examples of such solutions defining flows of the Taub-NUT like metrics.

### 2.4 Extracting solutions for the Levi-Civita connection

The method outlined in the previous section allows us to construct integral varieties for the Ricci flow equations (15)–(20) derived for the canonical d–connection with nontrivial torsion, see formulas (A.15) and (A.11) in Appendix. We can restrict such integral varieties (constraining the off–diagonal metric, equivalently, N–connection coefficients \( w_i \) and \( n_i \) and related integration functions) in order to generate solutions for the Levi-Civita connection. The conditions \( \Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma} \) (i.e. the coefficients of the Levi-Civita connection are equal to the coefficients of the canonical d–connection, both classes of coefficients being computed with respect to the N–adapted bases (A.7)
and (A.8) hold true if there are satisfied the equations (A.17), (A.18) and (A.19).

Introducing the coefficients for the ansatz (9), we can see that the constraints (A.18) are trivially satisfied and the equations (A.19) are written in the form

\[
\frac{\partial h_3}{\partial x^k} - w_k h_3^* - 2w_k^* h_3 = 0, \quad (29)
\]

\[
\frac{\partial h_4}{\partial x^k} - w_k h_4^* = 0, \quad (30)
\]

\[n_k^* h_4 = 0. \quad (31)
\]

The relations (29) and (30) are equivalent for the general solutions \(h_3, h_4\), see (26), \(h_4, h_4, n_k^*\), see (25) and \(w, w\), see (27), generated by a function \(\phi(x^i, v)\) (22) if \(\phi \to \phi - \ln 2\), when

\[\phi = \ln |(\sqrt{|h_4|})^*| - \ln |(\sqrt{|h_3|})|\]

and

\[w_k = (h_4^*)^{-1} \frac{\partial h_4}{\partial x^k} = - (\phi^*)^{-1} \frac{\partial \phi}{\partial x^k},\]

where \(\phi = \text{const}\) is possible only for the vacuum Einstein solutions. The condition (31) for \(h_4 \neq 0\) constrains \(n_k^* = 0\) which holds true if we put the integration functions \(n_k[2] = 0\) in (28), when \(n_k = n_k[1](x^i)\). These values of \(w, n_k\) have to be constrained one more again in order to solve the equations (A.17), which for our ansatz are of type

\[w'_1 - w'_2 + w_2 w'_1 - w_1 w'_2 = 0, \quad (32)\]

\[n'_1 - n'_2 = 0, \quad (33)\]

stating integrable (pseudo) Riemannian foliations. We have to take such integration functions when (33) is satisfied from the very beginning for some two functions \(n_k[1](x^i)\) depending on two variables. In a particular case, we can consider any parametrization of type \(w_k = \hat{w}_k(x^i)q(v)\) for some functions \(\hat{w}_k(x^i)\) and \(q(v)\) defining a class of solutions of (32).

The final conclusion in this section is that taking any solution of equations (16), (17) and (18) we can restrict the integral varieties to integration functions satisfying the conditions (32) and (33) allowing us to extract torsionless configurations for the Levi-Civita connection.

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4We emphasize that connections on manifolds are not defined as tensor objects. If the coefficients of two different connections are equal with respect to one frame, they can be very different with respect to other frames.
3 Nonholonomic Ricci Flows and 4D Taub-NUT Spaces

The techniques elaborated in previous section can be applied in order to generate Ricci flow solutions for various classes 4D metrics. In this section, we examine such configurations derived from a primary Taub-NUT metric.

We begin with the primary quadratic element

$$d\tilde{s}^2 = F^{-1}(r^2 + n^2)dr^2 - F(r)[dt - 2nw(\vartheta)d\varphi]^2 + (r^2 + n^2)a(\vartheta)d\varphi^2$$  (34)

for the so-called topological Taub–NUT–AdS/dS spacetimes [37, 38, 12] with NUT charge $n$. There are three possibilities:

$$F(r) = \frac{r^4 + (\varepsilon l^2 + n^2)r^2 - 2\mu rl^2 + \varepsilon n^2(l^2 - 3n^2) + (1 - |\varepsilon|)n^2}{l^2(n^2 + r^2)}$$

for $\varepsilon = 1, 0, -1$ defining respectively

$$\begin{cases} U(1) \text{ fibrations over } S^2; \\ U(1) \text{ fibrations over } T^2; \\ U(1) \text{ fibrations over } H^2; \end{cases} \quad \begin{cases} a(\vartheta) = \sin^2 \vartheta, w(\vartheta) = \cos \vartheta, \\ a(\vartheta) = 1, w(\vartheta) = \vartheta, \\ a(\vartheta) = \sinh^2 \vartheta, w(\vartheta) = \cosh \vartheta. \end{cases}$$

The ansatz (34) for $\varepsilon = 1, 0, -1$ but $n = 0$ recovers correspondingly the spherical, toroidal and hyperbolic Schwarzschild–AdS/dS solutions of 4D Einstein equations with cosmological constant $\lambda = -3/l^2$ and mass parameter $\mu$. For our further purposes, it is convenient to consider a coordinate transform

$$(r, \vartheta, t, \varphi) \rightarrow (r, \vartheta, p(\vartheta, t, \varphi), \varphi)$$

with a new time like coordinate $p$ when

$$dt - 2nw(\vartheta)d\varphi = dp - 2nw(\vartheta)d\vartheta.$$

and $t \rightarrow p$ are substituted in (34). This is possible if

$$t \rightarrow p = t - \int \nu^{-1}(\vartheta, \varphi)d\xi(\vartheta, \varphi)$$

with

$$d\xi = -\nu(\vartheta, \varphi)d(p - t) = \partial_\vartheta \xi d\vartheta + \partial_\varphi \xi d\varphi$$

when

$$d(p - t) = 2nw(\vartheta)(d\vartheta - d\varphi).$$
The last formulas state that the functions \( \nu(\vartheta, \varphi) \) and \( \xi(\vartheta, \varphi) \) should be taken to solve the equations

\[
\partial_\vartheta \xi = -2nw(\vartheta)\nu \quad \text{and} \quad \partial_\varphi \xi = 2nw(\vartheta)\nu.
\]

The solutions of such equations can be generated by any

\[
\xi = e^{f(\varphi - \vartheta)} \quad \text{and} \quad \nu = \frac{1}{2nw(\vartheta)} \frac{df}{dx} e^{f(\varphi - \vartheta)}
\]

for \( x = \varphi - \vartheta \).

The primary ansatz (34) can be written in a form similar to (8)

\[
\tilde{g} = \eta_1(r, \vartheta) \tilde{g}_1(r) + \eta_2(r, \vartheta) \tilde{g}_2(r),
\]

\[
\tilde{h}_3(r) = -F(r), \quad \tilde{h}_4(r, \vartheta) = (r^2 + n^2)a(\vartheta),
\]

\[
\tilde{w}_1(\vartheta) = -2nw(\vartheta), \quad \tilde{w}_2 = 0, \quad \tilde{n}_i = 0.
\]

An anholonomic transform \( \tilde{N} \rightarrow N \) and \( \tilde{g}=(\tilde{g}, \tilde{h}) \rightarrow g=(g, h) \) can be defined by formulas of type (35)

\[
\tilde{g} = \tilde{g}_1(x^1, v, y^4)(dx^1)^2 + \tilde{g}_2(x^k, v, y^4)(dx^2)^2 + \tilde{h}_3(x^k, v, y^4)(\tilde{b}^3)^2 + \tilde{h}_4(x^k, v, y^4)(\tilde{b}^4)^2,
\]

\[
\tilde{b}^3 = dv + \tilde{w}_1(x^k, v, y^4) dx^i, \quad \tilde{b}^4 = dy^4 + \tilde{n}_i(x^k, v, y^4) dx^i,
\]

following the parametrizations

\[
x^1 = r, x^2 = \vartheta, y^3 = v = p, y^4 = \varphi
\]

\[
\tilde{g}_1(r) = F^{-1}(r), \quad \tilde{g}_2(r) = (r^2 + n^2),
\]

\[
\tilde{h}_3(r) = -F(r), \quad \tilde{h}_4(r, \vartheta) = (r^2 + n^2)a(\vartheta),
\]

\[
\tilde{w}_1(\vartheta) = -2nw(\vartheta), \quad \tilde{w}_2 = 0, \quad \tilde{n}_i = 0.
\]

Our aim is to state the coefficients when this off–diagonal metric ansatz define solutions of the nonholonomic Ricci flow equations (15)–(20) for \( \tau = p \).

We construct a family of exact solutions of the Einstein equations with polarized cosmological constants following the same steps used for deriving
formulas (21), (22), (26), (24) and (28). By a corresponding 2D coordinate transform \( x^i \rightarrow \tilde{x}^i(r, \vartheta) \), such metrics can be always diagonalized and represented in conformally flat form,

\[
g_1(r, \vartheta)(dr)^2 + g_2(r, \vartheta)(d\vartheta)^2 = e^{\psi(x^i)} \left[ \epsilon_1(dx^1)^2 + \epsilon_2(dx^2)^2 \right],
\]

where the values \( \epsilon_i = \pm 1 \) depend on chosen signature and \( \psi(x^i) \) is a solution of

\[
\epsilon_1\psi^{**} + \epsilon_2\psi'' = 2\lambda_{[\epsilon]}(x^i).
\]

For other metric coefficients, one obtains the relations

\[
\phi(r, \vartheta, p) = \ln \left| h^*_k / \sqrt{|h_3h_4|} \right|,
\]

for

\[
(e^\phi)^* = -2\lambda_{[\epsilon]}(r, \vartheta, p) \sqrt{|h_3h_4|},
\]

\[
|h_3| = 4e^{-2\phi(r, \vartheta, p)} \left[ \left( \sqrt{|h_4|} \right)^* \right]^2, |h_4| = -(e^\phi)^*/4\lambda_{[\epsilon]}.
\]

It is convenient to represent such solutions in the form, see (26),

\[
h_4 = \epsilon_4 [b(r, \vartheta, p) - b_0(r, \vartheta)]^2, \quad h_3 = 4\epsilon_3 e^{2\phi(r, \vartheta, p)} [b^*(r, \vartheta, p)]^2
\]

where \( \epsilon_n = \pm 1 \) depend on fixed signature, \( b_0(r, \vartheta) \) and \( \phi(r, \vartheta, p) \) can be arbitrary functions and \( b(r, \vartheta, p) \) is any function when \( b^* \) is related to \( \phi \) and \( \lambda_{[\epsilon]} \) as stated by the formula (24).

The N–connection coefficients are of type

\[
n_k = n_{k[1]}(r, \vartheta) + n_{k[2]}(r, \vartheta)\hat{n}_k(r, \vartheta, p),
\]

where

\[
\hat{n}_k(r, \vartheta, p) = \int h_3(\sqrt{|h_4|})^{-3} dp,
\]

and \( n_{k[1]}(r, \vartheta) \) and \( n_{k[2]}(r, \vartheta) \) are integration functions and \( h^*_k \neq 0 \).

The above constructed coefficients for the metric and N–connection depend on arbitrary integration functions. One have to constrain such integral varieties in order to construct Ricci flow solutions. Let us consider possible solutions of the equation (19) for \( n_i = 0 \) as a possible solution of (33) (necessary for the Levi–Civita configurations). One obtains a matrix equation for matrices \( \tilde{g}(r, \vartheta) = [2\lambda_{[\epsilon]}(r, \vartheta) \ g_{ij}(r, \vartheta)] \) and \( \tilde{w}(r, \vartheta, p) = [w_i(r, \vartheta, p) \ w_j(r, \vartheta, p)] \)

\[
\tilde{g}(r, \vartheta) = h_3(r, \vartheta, p) \frac{\partial}{\partial p} \tilde{w}(r, \vartheta, p).
\]
This equation can be compatible for such 2D systems of coordinates when \( \tilde{g} \) is not diagonal (when it is more easy to contract a solution (21) for the diagonalized case) because \( \tilde{w} \) is also not diagonal. For 2D subspaces the coordinate and frame transforms are equivalent but such configurations should be corresponding adapted to the nonholonomic structure defined by \( \tilde{w}(r, \vartheta, p) \) which is possible for a general 2D coordinate system. We can consider the transforms

\[
g_{ij} = e_i^r(x^k(r, \vartheta))e_j^r(x^k(r, \vartheta))g_{ij}^r(x^k)
\]

and

\[
w_{ij}(x^k) = e_i^r(x^k(r, \vartheta))w_i((r, \vartheta, p))
\]

associated to a coordinate transform \((r, \vartheta) \to x^k(r, \vartheta)\) with \(g_{ij}^r(x^k)\) defining, in general, a symmetric but non–diagonal \((2 \times 2)\)–dimensional matrix. We can integrate the equation (39) in explicit form by separation of variables in \(\phi, b, h_3\) and \(w_{ij}\) when

\[
\phi = \tilde{\phi}(x^i)\hat{\phi}(p), h_3 = \tilde{h}_3(x^i)\hat{h}_3(p),
\]

\[
w_{ij}(x^k) = \tilde{w}_{ij}(x^k)q(p), \text{ for } \tilde{w}_{ij} = -\partial_r \ln |\tilde{\phi}(x^k)|, q = (\partial_p \hat{\phi}(p))^{-1}
\]

where separation of variables for \(h_3\) is related to a similar separation of variables \(b = \tilde{b}(x^i)\hat{b}(p)\) as follows from (38). One obtains the matrix equation

\[
\tilde{g}(x^k) = \alpha_0 \tilde{h}_3(x^i)\tilde{w}_0(x^k)
\]

where the matrix \(\tilde{w}_0\) has components \((\tilde{w}_{ij}\tilde{w}_{kj})\) and constant \(\alpha_0 \neq 0\) is chosen from any prescribed relation

\[
\tilde{h}_3(p) = \alpha_0 \partial_p \left[ \partial_p \hat{\phi}(p) \right]^{-2}.
\]  

(40)

We conclude that any given functions \(\tilde{\phi}(x^k), \hat{\phi}(p)\) and \(\tilde{h}_3(x^i)\) and constant \(\alpha_0\) we can generate solutions of the Ricci flow equation (19) for \(n_i = 0\) with the metric coefficients parametrized in the same form as for the solution of the Einstein equations (15)–(18). In a particular case, we can take \(\phi(p)\) to be a periodic or solitonic type function.

The last step in constructing flow solutions is to solve the equation (20) for the ansatz (37) redefined for coordinates \(x^k = x^k(r, \vartheta)\),

\[
\frac{\partial}{\partial p} h_a = 2\lambda_w(x^k, p) h_a.
\]

This equation is compatible if \(h_4 = \varsigma(x^k)h_3\) for any prescribed function \(\varsigma(x^k)\). We can satisfy this condition by corresponding parametrizations of
function \( \phi = \hat{\phi}(x') \) \( \hat{\phi}(p) \) and/or \( b = \hat{b}(x')\hat{b}(p) \), see (38). As a result we can compute the effective cosmological constant for such Ricci flows,

\[
\lambda_{[\phi]}(x'k, p) = \partial_p \ln |h_3(x'k, p)|
\]

which for solutions of type (40) is defined by a polarization running in time,

\[
\lambda_{[\phi]}(p) = \alpha_0 \partial^2_p [\partial_p \hat{\phi}(p)]^{-2}.
\]

In this case we can identify \( \alpha_0 \) with a cosmological constant \( (\lambda = -3/l^2, \text{for primary Taub-NUT configurations, or any } \lambda = \lambda_H^2/4, \text{for string configurations, see formula (A.28))} \) if we choose such \( \hat{\phi}(p) \) that \( \partial^2_p [\partial_p \hat{\phi}(p)]^{-2} \to 1 \) for \( p \to 0 \).

Putting together the coefficients of metric and N–connection, one obtains

\[
g = \alpha_0 \hat{h}_3(x') \{ \partial_{x'} \ln |\hat{\phi}(x')| \partial_{x'} \ln |\hat{\phi}(x')| dx'dx' + \partial_p [\partial_p \hat{\phi}(p)]^{-2} \times \left[ [dp - (\partial_p \hat{\phi}(p))^{-1} \left( dx' \partial_{x'} \ln |\hat{\phi}(x')| \right)]^2 + \varsigma(x')(d\varphi)^2 \right]. \tag{41}
\]

This metric ansatz depends on certain type of arbitrary integration and generation functions \( \hat{h}_3(x'), \hat{\phi}(x'), \varsigma(x') \) and \( \hat{\phi}(p) \) and on a constant \( \alpha_0 \) which can be identified with the primary cosmological constant. It was derived by considering nonholonomic deformations of some classes of 4D Taub-NUT solutions (35) by considering polarizations functions (36) deforming the coefficients of the primary metrics into the target ones for corresponding Ricci flows. The target metric (41) model 4D Einstein spaces with "horizontally" polarized, \( \lambda_{[\phi]}(x'k) \) and "vertically" running, \( \lambda_{[\phi]}(p) \), cosmological constant managed by the Ricci flow solutions.

Finally, we conclude that if the primary 4D topological Taub–NUT–AdS/dS spacetimes have the structure of \( U(1) \) fibrations over 2D hypersurfaces (sphere, torus or hyperboloid) than their nonholonomic deformations to Ricci flow solutions with effectively polarized/running cosmological constant defines the generalized 4D Einstein spaces as certain foliations on the corresponding 2D hypersurfaces. This holds true if the nonholonomic structures is chosen to be integrable and for the Levi-Civita connection. In more general cases, with nontrivial torsion, for instance, induced from string gravity, we deal with "nonintegrable" foliated structures, i.e. with nonholonomic Riemann–Cartan manifolds provided with effective nonlinear connection structure induces by off–diagonal metric terms.
4 Outlook and Discussion

In summary, we have developed the method of anholonomic frames in order to construct exact solutions describing Ricci flows of 4D Taub-NUT like metrics. Toward this end, we applied a program of study and applications to physics which is based on the geometry of nonholonomic/ foliated spaces with associated nonlinear connection structure induced by generic off–diagonal metric terms. The premise of this methodology is that one was possible to generate exact nonholonomic solutions for three, four and five dimensional spacetimes (in brief, 3D, 4D, 5D) in the Einstein and low/extra dimension gravity with cosmological constant (possibly induced by some ansatz for the antisymmetric torsion in string gravity, or other models of gravity and effective matter field interactions) [13, 14, 15, 16, 17]. The validity of this approach in constructing Ricci flow solutions was substantiated by generating certain examples of Ricci flow of solitonic pp–waves [18].

In this paper, we elaborated the geometric background for generalized Einstein spaces with effectively polarized (anisotropically on some space coordinates and running on time like coordinate) cosmological ”constants” arising naturally if we consider generic off–diagonal gravitational (vacuum and with nontrivial matter field interactions or extra dimension corrections) and nonholonomic frame effects. Such spacetimes are distinguished by corresponding nontrivial nonholonomic (foliated, if the integrability conditions are satisfied) structures (examined in different approaches to Lagrange–Finsler geometry [33], semi–Riemannian and Finsler foliations [36] and, for instance, in modern gravity and noncommutative geometry [15]). In searching for physical applications of such geometric methods, we addressed to the geometry and physics of Taub-NUT spacetimes [1, 2, 3, 4, 5, 6, 7] (see also more recent developments in Refs. [8, 9, 10, 11, 12]) being interested in an analysis of the Ricci flows of Taub-NUT spaces.

There are a number of mathematical results and certain applications in modern physics related to Ricci flow geometry [19, 20, 21, 22, 23, 24]. Nevertheless, possible applications to gravity theories are connected to quite cumbersome approximated methods in constructing solutions of the Ricci flow equations. Perhaps the first attempt to construct explicit exact solutions was performed following a linearization approach [27] allowing to generate exact solutions for lower dimensions. Another class of Ricci flow solutions was related to solitonic pp–waves in five dimensional gravity [18]. The program of constructing exact Ricci flow solutions can be naturally extending to an analysis of such flows of some physically valuable metrics describing exact solutions in gravity.

The case of flows of the so–called Taub-NUT–AdS/dS spacetimes presents
a special interest. They describe a number of very interesting physical situations with nontrivial cosmological constant. Certain classes of nonholonomic deformations of such solutions by generic off–diagonal metric terms and associated anholonomic frame structures result in another classes of exact solutions of the Einstein equations defining the so–called locally anisotropic Taub-NUT spacetimes, see a detailed study in Refs. [16, 17]. One of the important result of those investigations was that the occurrence of effective polarizations on space like coordinates and running of cosmological constant can be considered in a manner resulting in exactly integrable systems of partial equations (to which the Einstein equations transform for very general ansatz in 3D–5D gravity).

The main idea of this work is to prove that effectively induced nonhomogeneous Einstein spaces may describe Ricci flows of Taub-NUT like metrics, for certain parametrizations of polarizations of the metric coefficients and of the cosmological constant. We considered nonholonomic Ricci flows of 4D Taub-NUT spaces primarily defined as fibrated structures on 2D spherical/toroidal/hyperbolic hypersurfaces. Such 4D off–diagonal flows can be also effectively modeled by families of nonhomogeneous Einstein spaces with corresponding nonholonomic frame structure (which transform into a foliated structure for the Levi-Civita configurations). Here, we note that the integral varieties of the Ricci flow solutions depend on very general classes of generating functions and integration functions depending on three, two, one variables and on integration constants (for 3D configurations, such functions depend on two and one variables). This is a general property of the systems of partial equations to which the Ricci flow and/or Einstein equations reduce for very general off–diagonal metric and non–Riemannian linear connection ansatz.

The ansatz for usual (non–deformed) Taub-NUT spaces transform the Einstein equations into certain systems of nonlinear ordinary differential equations on one variable. From the very beginning such solutions were constrained to depend only on integration constants (like the mass parameter, NUT constant which were defined from certain physical considerations). It is a very difficult technical task to construct exact off–diagonal solutions depending both holonomically and anholonomically on two, three variables and defining 4–5D spacetimes. The anholonomic frame method offers a number of such possibilities but it results in a more sophisticate conceptual problem for possible geometrical and physical meaning/interpretations of various classes of integration functions and constants. In the view of such considerations, we can argue that by imposing certain constraints on classes of integration functions we select certain new or prescribed physical situations when the solutions are classified by new nonlinear symmetries (noncommutative or Lie
algebra generalizations to Lie algebroid configurations) and self–consisting embedding in solitonic and/or pp–wave backgrounds, with anisotropic polarization of constants, deformation of horizons and so on, see detailed discussions in Refs. \[13, 14, 16, 17, 18\]. The approach appear to be promising in constructing exact solutions for the Ricci flows of physically important metrics and connections when the generation and integration functions are constrained, for instance, to define Levi-Civita configurations being derived effectively by nonhomogeneous cosmological constants.

In conclusion, we emphasize that the nonholonomic Ricci flow solutions for Taub–NUT like metrics continue to have a number of issues when they are viewed from the perspective of black hole flows and cosmological solutions, generalizations to extra dimensions non–Riemann theories of gravity. We shall also analyze nonholonomic Ricci flows in string gravity in our further works.

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Appendix A
The Geometry of N–connections and Anholonomic Deformations

We outline the geometry of anholonomic deformations of geometric structures on a Riemann–Cartan manifold. The geometric constructions will be related to Ricci flows on such spaces and possible reductions to the Einstein spaces. For integrable frame structures and Levi-Civita connections such spaces transform into usual (pseudo) Riemannian fibrations.\footnote{The geometry of fibrations is considered in details in Ref. \[36\] following a different class of linear connections on nonholonomic manifolds not imposing the conditions that those connections are solutions of the Einstein, or Ricci flow, equations. In this work, we develop for Ricci flows the approach outlined for exact solutions in Einstein and string gravity, for instance, in Ref. \[15\].}
A.1 Nonholonomic transforms of vielbeins and metrics

We consider a \((n + m)\)-dimensional manifold (spacetime)\(V\), \(n \geq 2, m \geq 1\), of arbitrary signature enabled with a "primary" metric structure \(\tilde{g} = \tilde{g} \oplus_N \tilde{h}\) distinguished in the form

\[
\tilde{g} = \tilde{g}_i(u)(dx^i)^2 + \tilde{h}_a(u)(\tilde{b}^a)^2, \quad \tilde{b}^a = dy^a + \tilde{N}_a^i(u)dx^i.
\]  

(A.1)

The local coordinates are parametrized \(u = (x, y) = \{u^\alpha = (x^i, y^a)\}\), for the indices of type \(i, j, k, ... = 1, 2, ..., n\) (in brief, horizontal, or \(h\)-indices/components) and \(a, b, c, ... = n + 1, n + 2, ... n + m\) (vertical, or \(v\)-indices/components).\(^6\) The off–diagonal coefficients \(\tilde{\kappa}_a^i(u)\) in (A.1) state, in general, a nonintegrable \((n + m)\)-splitting \(\oplus N\) in any point \(u \in V\) and define a class of ‘\(N\)-adapted’ local bases (frames, equivalently, vielbeins) \(\hat{e} = (\hat{e}, e)\), when

\[
\hat{e}_\alpha = \left(\hat{e}_i = \frac{\partial}{\partial x^i} - \tilde{\kappa}_a^i(u) \frac{\partial}{\partial y^a}, e_a = \frac{\partial}{\partial y^a}\right), \quad \text{(A.2)}
\]

and local dual bases (co–frames) \(\hat{b} = (b, \hat{b})\), when

\[
\hat{b}^a = \left(b^i = dx^i, \hat{b}^b = dy^b + \tilde{N}_b^i(u) dx^i\right), \quad \text{(A.3)}
\]

for \(\text{\(\hat{b}|\hat{e} = I\)}\), i.e. \(\text{\(\hat{e}_\alpha | \hat{b}^\beta = \delta_\alpha^\beta\)}\), where the inner product is denoted by ‘\(\cdot\)’ and the Kronecker symbol is denoted by \(\delta_\alpha^\beta\). The nonintegrability of the frame structure and corresponding \(h\)- and \(v\)-splitting in (A.2) results in the nonholonomy (equivalently, anholonomy) relations

\[
\hat{e}_\alpha \hat{e}_\beta - \hat{e}_\beta \hat{e}_\alpha = \hat{w}_{\alpha \beta}^\gamma \hat{e}_\gamma
\]

with nontrivial anholonomy coefficients

\[
\hat{w}_{ij}^a = -\hat{w}_{ij}^a = \hat{\Omega}_{ij}^a \hat{N}_i^a - \hat{e}_i \left(\tilde{N}_j^a\right), \quad \hat{w}_{ia}^b = -\hat{w}_{ia}^b = e_a(\tilde{N}_j^b). \quad \text{(A.4)}
\]

A metric \(g = g \oplus_N h\) parametrized in the form

\[
g = g_i(u)(b^i)^2 + g_a(u)(b^a)^2, \quad b^a = dy^a + \tilde{N}_a^i(u)dx^i.
\]  

(A.5)

\(^6\)In this paper we shall consider four dimensional constructions and possible reductions to three dimensions, when \(n = 2\) and \(m = 2\), or \(m = 1\).

\(^7\)The Einstein’s summation rule on ‘up–low’ indices will be applied if the contrary case will be not emphasized.
is a nonholonomic transform (deformation), preserving the $(n + m)$–splitting, of the metric $\hat{g} = \hat{g} \oplus N \hat{h}$ if the coefficients of (A.1) and (A.5) are related by formulas

$$g_i = \eta_i(u) \hat{g}_i, \quad h_a = \eta_a(u) \hat{h}_a \quad \text{and} \quad N^a_i = \eta_i^a(u) \hat{N}^a_i,$$

(A.6)

(the summation rule is not considered for the indices of ‘polarizations’ $\eta^a_i = (\eta_i, \eta_a)$ and $\eta^a_i$ in (A.6)). Under anholonomic deformations, for nontrivial values $\eta^a_i(u)$, the nonholonomic frames (A.2) and (A.3) transform correspondingly into

$$e^a = \left( e^i = \frac{\partial}{\partial x^i} - N^a_i(u) \frac{\partial}{\partial y^a}, e^a = \frac{\partial}{\partial y^a} \right)$$

(A.7)

and

$$b^a = \left( b^i = dx^i, b^c = dy^c + N^c_i(u) \ dx^i \right)$$

(A.8)

with the anholonomy coefficients $w^a_{\alpha\beta}$ defined by $N^a_i$ substituted in formulas (A.4). We adopt the convention to use "bold" symbols for any geometric object adapted/defined with respect to $N$–elongated bases and corresponding $N$–connection structures.

A set of coefficients $\mathbf{N} = \{ \hat{N}^a_i \}$ states a $N$–connection structure on a manifold $\mathbf{V}$ if it defines a locally nonintegrable (nonholonomic) distribution $T\mathbf{V}|_u = h\mathbf{V}|_u \oplus N v\mathbf{V}|_u$ in any point $u \in \mathbf{V}$ which can be globalized to a Whitney sum

$$T\mathbf{V} = h\mathbf{V} \oplus N v\mathbf{V}. \quad (A.9)$$

We say conventionally that a $N$–connection decomposes the tangent space $T\mathbf{V}$ into certain horizontal (h), $h\mathbf{V}$, and vertical (v), $v\mathbf{V}$, subspaces. With respect to a $N$–adapted’ base (A.7), any vector field $\mathbf{X}$ splits into its h- and v–components,

$$\mathbf{X} = h\mathbf{X} + v\mathbf{X} = X^i \hat{e}_i + X^a e_a$$

with $X^i \doteq \mathbf{X}|b^i$ and $X^a \doteq \mathbf{X}|\hat{b}^a$. A similar decomposition holds for a co–vector (1–form) $\tilde{\mathbf{X}}$,

$$\tilde{\mathbf{X}} = h\tilde{\mathbf{X}} + v\tilde{\mathbf{X}} = X_i d^i + X_a \hat{b}^a.$$

It should be noted that the ’interior product’ $\doteq$ is defined by the metric structure but the ’h- and v–splitting’ are stated by the $N$–connection coefficients $N^a_i$, which in this work are related to generic off–diagonal metric coefficients defined with respect to a usual coordinate basis.

---

For simplicity, we shall omit (inverse) hats on symbols if this does not result in confusion.
The N–connection curvature $\Omega$ of a N–connection $\mathbf{N}$ is by definition just the Neijenhuis tensor

$$\Omega(\mathbf{X}, \mathbf{Y}) \doteq [v \mathbf{X}, v \mathbf{Y}] + v [\mathbf{X}, \mathbf{Y}] - v [v \mathbf{X}, \mathbf{Y}] - v [\mathbf{X}, v \mathbf{Y}]$$

where, for instance, $[\mathbf{X}, \mathbf{Y}]$ denotes the commutator of vector fields $\mathbf{X}$ and $\mathbf{Y}$ on $TV$. The coefficients $\Omega_{ij}$ of a 'N–curvature' $\tilde{\Omega}$ stated with respect to the bases (A.2) and (A.3) are computed following the first formula in (A.4).

We can diagonalize the metric (A.5) by certain coordinate transforms if all $w^\gamma_{\alpha \beta}$ vanish, i.e. the N–connection structure became trivial (integrable) with $\Omega = 0$ and $e_a(N^b_j) = 0$. The subclass of of linear connections is selected as a particular case by parametrizations of type $N^b_a(x, y) = A^b_a(x^k)y^a$ (for instance, in the Kaluza–Klein gravity, the values $A^b_a(x^k)$ are associated to the gauge fields after extra dimension compactifications on $y^a$).

An anholonomic transform $\tilde{\mathbf{N}} \to \mathbf{N}$ and $\tilde{\mathbf{g}}=(\tilde{g}, \tilde{h}) \to \mathbf{g}=(g, h)$, defined by formulas (A.6), deforms correspondingly the nonholonomic frame (A.2) and metric structures (A.1). In this paper we consider maps (transforms) of spaces (manifolds) provided with nonlinear connection (N–connection) structure [15, 33] when invariant conventional $h$– and $v$–splitting. A manifold $V$ is called N–anholonomic if it is provided with a preferred anholonomic frame structure induced by the generic off–diagonal coefficients of a metric [13, 14, 15].

### A.2 Torsions and curvatures of d–connections

A linear connection (1–form) $\Gamma^\gamma_\alpha = \Gamma^\gamma_\alpha \mathbf{b}^\beta$ on $V$ defines an operator of covariant derivation,

$$D = \{D_a = (D_i, D_a)\}.$$ 

The coefficients

$$\Gamma^\gamma_\alpha \doteq (D_a e_\beta)] \mathbf{b}^\gamma = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})$$

can be computed in N–adapted form (i.e. with $h$– and $v$–splitting) with respect to the local bases (A.2) and (A.3) following the formulas

$$L^i_{jk} \doteq (D_k e_j)] \mathbf{b}^i, L^a_{bk} \doteq (D_k e_b)] \mathbf{b}^a,$$

$$C^i_{jc} \doteq (D_c e_j)] \mathbf{b}^i, C^a_{bc} \doteq (D_c e_b)] \mathbf{b}^a.$$ 

Following the terminology from [33, 15], we call $\Gamma^\gamma_\alpha$ to be N–distinguished (equivalently, a d–connection) if it preserves the $(n + m)$–splitting $\oplus_N$, i.e. the decomposition

$$D \doteq X^a D_a = X^i D_i + X^a D_a$$
holds for any vector field $X = X^i e_i + X^a e_a$, under parallel transports on $V$.

The torsion $T^\alpha$ of a d–connection $D$ is defined in standard form

$$T^\alpha \equiv Db^\alpha = b^\alpha + \Gamma^\gamma_{\alpha} \wedge b^\beta.$$  \hspace{1cm} \text{(A.10)}

There are five types of N–adapted components of $T^\alpha_{\beta \gamma}$ computed with respect to (A.7) and (A.8),

$$T^i_{\ jk} = L^i_{\ jk} - L^i_{\ kj}, \quad T^\alpha_{\ bc} = C^\alpha_{\ bc} - C^\alpha_{\ cb},$$

$$T^a_{\ ja} = C^a_{\ ja}, \quad T^a_{\ bi} = \partial N^a_{\ i} - L^a_{\ bi}, \quad T^a_{\ ji} = e_j N^a_{\ i} - e_i N^a_{\ j} = \Omega^a_{\ ij},$$

which for some physical models can be related to a complete antisymmetric tensor $H^\alpha_{\beta \gamma} = e\left[B^\alpha_{\beta \gamma}\right]$ in string gravity \cite{34,35} or to certain torsion fields in (non)commutative gauge gravity \cite{15}.

The curvature

$$R^\alpha_{\ \beta \gamma} = Db^\alpha = d\Gamma^\gamma_{\beta} \wedge \Gamma^\alpha_{\gamma},$$

of a d–connection $D$, splits into six types of N–adapted components with respect to (A.7) and (A.8),

$$R^\alpha_{\ \beta \gamma \delta} = \left(R^i_{\ hjk}, R^a_{\ bjk}, P^i_{\ hja}, P^c_{\ bja}, S^i_{\ jbc}, S^a_{\ bcd}\right),$$

where

$$R^i_{\ hjk} = e_k L^i_{\ hj} - e_j L^i_{\ hk}, \quad R^a_{\ bjk} = e_k L^a_{\ bj} + L^c_{\ bj} L^a_{\ ck} - L^c_{\ bk} L^a_{\ cj} - C^a_{\ bc} \Omega^i_{\ kj},$$

$$R^i_{\ jka} = e_a L^i_{\ jk} - D_k C^i_{\ ja} + C^i_{\ jb} T^b_{\ ka},$$

$$R^c_{\ bka} = e_a L^c_{\ bk} - D_k C^c_{\ ba} + C^e_{\ bd} T^c_{\ ka},$$

$$R^i_{\ jbc} = e_c C^j_{\ cb} + C^h_{\ jbc} C^i_{\ hc} - C^k_{\ jbc} C^i_{\ hc},$$

$$R^a_{\ bcd} = e_c C^a_{\ bc} + C^e_{\ bcd} C^a_{\ ed} - C^e_{\ bdc} C^a_{\ ec}.$$  \hspace{1cm} \text{(A.12)}

Contracting respectively the components of \text{(A.12)}, $R_{\alpha \beta} \equiv R_{\alpha \beta \gamma \delta}$, one computes the h– v–components of the Ricci d–tensor (there are four N–adapted components)

$$R_{ij} \equiv R^k_{\ ijk}, \quad R_{ia} \equiv -R^k_{\ ika}, \quad R_{ai} \equiv R^b_{\ aib}, \quad S_{ab} \equiv R^c_{\ abc},$$

The scalar curvature is defined by contracting the Ricci d–tensor with the inverse metric $g^\alpha_{\ \beta}$,

$$\bar{R} \equiv g^\alpha_{\ \beta} R_{\alpha \beta} = g^{ij} R_{ij} + h^{ab} S_{ab}. \hspace{1cm} \text{(A.14)}$$
There are two types of preferred linear connections uniquely determined by a generic off–diagonal metric structure with \( n+m \) splitting, see \( \mathbf{g} = g \oplus_{N} h \) \((A.5)\):

1. The Levi-Civita connection \( \nabla = \{ \Gamma_{\beta\gamma}^{\alpha} \} \) is by definition torsionless, \( \mathcal{T} = 0 \), and satisfies the metric compatibility condition, \( \nabla \mathbf{g} = 0 \) (we shall use a left low label ”\( \mid \)” in order to emphasize that some geometric objects are constructed just for the Levi-Civita connection).

2. The canonical d–connection \( \hat{\Gamma}_{\alpha\beta}^{\gamma} = \left( \hat{L}_{jk}^{i}, \hat{L}_{bk}^{a}, \hat{C}_{jc}^{i}, \hat{C}_{bc}^{a} \right) \) is also metric compatible, i. e. \( \hat{\mathbf{D}} \mathbf{g} = 0 \), but the torsion vanishes only on h– and v–subspaces, i.e. \( \hat{T}_{jk} = 0 \) and \( \hat{T}_{bc} = 0 \), for certain nontrivial values of \( \hat{T}_{ja}, \hat{T}_{bi}, \hat{T}_{ji} \). In this paper we shall work only with the canonical d–connection. For simplicity, we shall omit hats on symbols and write, for simplicity, \( L_{jk}^{i} \) instead of \( \hat{L}_{jk}^{i} \), \( T_{ja}^{i} \) instead of \( \hat{T}_{ja}^{i} \) and so on...but preserve the general symbols \( \hat{\mathbf{D}} \) and \( \hat{\Gamma}_{\alpha\beta}^{\gamma} \).

By a straightforward calculus with respect to \( N \)–adapted frames \((A.7)\) and \((A.8)\), one can verify that the requested properties for \( \hat{\mathbf{D}} \) are satisfied if

\[
\begin{align*}
L_{jk}^{i} &= \frac{1}{2} g^{ir} (e_{k}g_{jr} + e_{j}g_{kr} - e_{r}g_{jk}), \\
L_{bk}^{a} &= e_{b}(N_{k}^{a}) + \frac{1}{2} h^{ac} (e_{k}h_{bc} - h_{dc}e_{b}N_{k}^{d} - h_{db}e_{c}N_{k}^{d}), \\
C_{jc}^{i} &= \frac{1}{2} g^{ik} e_{k}g_{jk}, \quad \hat{C}_{bc}^{a} = \frac{1}{2} h^{ad} (e_{c}h_{bd} + e_{d}h_{cd} - e_{d}h_{bc}).
\end{align*}
\]

We note that these formulas are computed for the components of the metric \( \mathbf{g} = g \oplus_{N} h \) \((A.5)\) but in a similar form, using symbols with ”inverse hats” we can compute the components \( \hat{\mathbf{D}} \) of \( \hat{\mathbf{g}} = \hat{g} \oplus_{N} \hat{h} \) \((A.1)\) with respect to \((A.2)\) and \((A.3)\).

The Levi-Civita linear connection \( \nabla = \{ \Gamma_{\beta\gamma}^{\alpha} \} \), uniquely defined by the conditions \( \mathcal{T} = 0 \) and \( \nabla \mathbf{g} = 0 \), is not adapted to the distribution \((A.9)\) and its nonholonomic deformations. Let us parametrize the coefficients in the form

\[
\Gamma_{\beta\gamma}^{\alpha} = \left( L_{jk}^{i}, L_{bk}^{a}, L_{bk}^{i}, L_{bk}^{i}, C_{jc}^{i}, C_{bc}^{a}, C_{bc}^{a} \right),
\]

where with respect to \( N \)-adapted bases \((A.7)\) and \((A.8)\)

\[
\begin{align*}
\nabla_{e_{k}}(e_{j}) &= L_{jk}^{i}e_{i} + \frac{1}{2} g^{ir} (e_{k}g_{jr} + e_{j}g_{kr} - e_{r}g_{jk}), \\
\nabla_{e_{b}}(e_{j}) &= C_{jc}^{i}e_{i} + \frac{1}{2} g^{ik} e_{k}g_{jk}.
\end{align*}
\]
A straightforward calculus shows that the coefficients of the Levi-Civita connection can be expressed in the form

\begin{align*}
\bar{L}^i_{jk} &= L^i_{jk}, \quad \bar{L}^a_{jk} = -C^i_{jk}g_{ik}h^{ab} - \frac{1}{2}\Omega^a_{jk}, \quad (A.16) \\
\bar{L}^i_{bk} &= \frac{1}{2}\Omega^c_{jk}h_{cb}g^{ji} - \frac{1}{2}(\delta^i_j\delta^h_k - g_{jk}g^{ih})C^j_i, \\
\bar{L}^a_{bk} &= L^a_{bk} + \frac{1}{2}(\delta^a_c\delta^b_d + h_{cd}h^{ab})[L^c_{bk} - e_b(N^c_j)], \\
\bar{C}^i_{kb} &= C^i_{kb} + \frac{1}{2}\Omega^a_{jk}h_{cb}g^{ji} + \frac{1}{2}(\delta^i_j\delta^h_k - g_{jk}g^{ih})C^j_i, \\
\bar{C}^a_{jb} &= -\frac{1}{2}(\delta^a_c\delta^d_b - h_{cb}h^{ad})[L^c_{dj} - e_d(N^c_j)], \quad \bar{C}^a_{bc} = C^a_{bc}, \\
\bar{C}^i_{ab} &= -\frac{\Omega^a_{jk}h_{cb} - e_a(N^c_j)}{2} \left\{ [L^c_{aj} - e_a(N^c_j)]h_{cb} + [L^c_{bj} - e_b(N^c_j)]h_{ca} \right\},
\end{align*}

where \(\Omega^a_{jk}\) are computed as in the first formula in (A.4) but for the coefficients \(N^c_j\).

For our purposes, it is important to state the conditions when both the Levi–Civita connection and the canonical d–connection may be defined by the same set of coefficients with respect to a fixed frame of reference. Following formulas (A.15) and (A.16), we conclude that one holds the component equality

\begin{equation}
\bar{\Gamma}^\alpha_\beta\gamma = \hat{\Gamma}^\gamma_\alpha\beta \quad \text{if} \quad \Omega^c_{jk} = 0 \quad \text{(A.17)}
\end{equation}

(there are satisfied the integrability conditions and our manifold admits a foliation structure),

\begin{equation}
\bar{C}^i_{kb} = C^i_{kb} = 0 \quad \text{(A.18)}
\end{equation}

and

\begin{equation}
L^c_{aj} - e_a(N^c_j) = 0 \quad \text{(A.19)}
\end{equation}

which, following the second formula in (A.15), is equivalent to

\[ e_kh_{bc} - h_{dc} e_bN^d_k - h_{db} e_cN^d_k = 0. \]

We conclude this section with the remark that if the conditions (A.17), (A.18) and (A.19) hold true for the metric (A.5), the torsion coefficients (A.11) vanish. This results in respective equalities of the coefficients of the Riemann, Ricci and Einstein tensors.

\footnote{Such results were originally considered by R. Miron and M. Anastasiei for vector bundles provided with N–connection and metric structures, see Ref. [33]. Similar proofs hold true for any nonholonomic manifold provided with a prescribed N–connection structures.}
A.3 Gravity on N–anholonomic manifolds and foliations

Contracting with the inverse to the d–metric (A.5) in \( V \), we can introduce the scalar curvature of a d–connection \( \tilde{D} \),

\[
\tilde{R} \doteq g^{\alpha \beta} R_{\alpha \beta} \doteq R + S,
\]

where \( R \doteq g^{ij} R_{ij} \) and \( S \doteq h^{ab} S_{ab} \) and compute the Einstein tensor

\[
G_{\alpha \beta} \doteq R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} \tilde{R}.
\]

In the vacuum case, \( G_{\alpha \beta} = 0 \), which mean that all Ricci d–tensors (A.13) vanish.

The Einstein equations for the canonical d–connection \( \Gamma_{\alpha \beta}^{\gamma} \) (A.15),

\[
R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} \tilde{R} = \kappa \Upsilon_{\alpha \beta},
\]

are defined for a general source of matter fields and, for instance, possible string corrections, \( \Upsilon_{\alpha \beta} \). It should be emphasized that there is a nonholonomically induced torsion \( T_{\gamma}^{\alpha \beta} \) with d–torsions computed by introducing consequently the coefficients of d–metric (A.5) into (A.15) and than into formulas (A.11). The gravitational field equations (A.22) can be decomposed into h– and v–components following formulas (A.13) and (A.20),

\[
R_{ij} - \frac{1}{2} g_{ij} (R + S) = \Upsilon_{ij},
\]

\[
S_{ab} - \frac{1}{2} h_{ab} (R + S) = \Upsilon_{ab},
\]

\[
1 P_{ai} = \Upsilon_{ai},
\]

\[
- 2 P_{ia} = \Upsilon_{ia}.
\]

The vacuum equations, in terms of the Ricci tensor \( R_{\alpha \beta}^{\gamma} = g^{\alpha \gamma} R_{\gamma \beta} \), are

\[
R_{ij} = 0, S_{ab} = 0, 1 P_{ai} = 0, 2 P_{ia} = 0.
\]

If the conditions (A.17), (A.18) and (A.19) are satisfied, the equations (A.23) and (A.24) are equivalent to those derived for the Levi-Civita connection. In such cases, the spacetime is modelled as foliated manifold with generic off–diagonal metric (A.5) if the anholonomy coefficients \( w_{\alpha \beta}^{\tilde{\gamma}} \), defined by \( N_{i}^{a} \) substituted in formulas (A.4), are not zero.
In string gravity the nontrivial torsion components (A.11) and source \( \kappa \gamma_{\alpha \beta} \) can be related to certain effective interactions with the strength (torsion)

\[
H_{\mu \nu \rho} = e_{\mu} B_{\nu \rho} + e_{\rho} B_{\mu \nu} + e_{\nu} B_{\rho \mu}
\]

of an antisymmetric field \( B_{\nu \rho} \), when

\[
R_{\mu \nu} = -\frac{1}{4} H_{\mu}^{\ \nu \rho} H_{\nu \lambda \rho}
\]  

(A.25)

and

\[
D_{\lambda} H^{\lambda \mu \nu} = 0,
\]  

(A.26)

see details on string gravity, for instance, in Refs. [34, 35]. The conditions (A.25) and (A.26) are satisfied by the ansatz

\[
H_{\mu \nu \rho} = \hat{Z}_{\mu \nu \rho} + \hat{H}_{\mu \nu \rho} = \lambda_{[H]} \sqrt{|g_{\alpha \beta}|} \varepsilon_{\nu \lambda \rho}
\]  

(A.27)

where \( \varepsilon_{\nu \lambda \rho} \) is completely antisymmetric and the distortion (from the Levi–Civita connection) and

\[
\hat{Z}_{\mu \alpha \beta} c^\mu = e_{\beta} |T_\alpha - e_\alpha |T_\beta + \frac{1}{2} (e_\alpha |e_{\beta} |T_\gamma) c^\gamma
\]

is defined by the torsion tensor (A.10) with coefficients (A.11). We emphasize that our \( H \)-field ansatz is different from those already used in string gravity when \( \hat{H}_{\mu \nu \rho} = \lambda_{[H]} \sqrt{|g_{\alpha \beta}|} \varepsilon_{\nu \lambda \rho} \). In our approach, we define \( H_{\mu \nu \rho} \) and \( \hat{Z}_{\mu \nu \rho} \) from the respective ansatz for the \( H \)-field and nonholonomically deformed metric, compute the torsion tensor for the canonical distinguished connection and, finally, define the ’deformed’ \( H \)-field as

\[
\hat{H}_{\mu \nu \rho} = \lambda_{[H]} \sqrt{|g_{\alpha \beta}|} \varepsilon_{\nu \lambda \rho} - \hat{Z}_{\mu \nu \rho}.
\]

Such spacetimes are both nonholonomic with nontrivial torsion related to that in string gravity and with sources induced by string corrections via an effective cosmological constant \( \lambda_{[H]} \), when

\[
R^\alpha_{\ \beta} = -\frac{\lambda_{[H]}^2}{4} \delta^\alpha_{\ \beta}.
\]  

(A.28)

In order to generate solutions for such equations it is more convenient to work directly with the canonical d–connection (A.15).

References


