Gravitational Backreaction of Matter Inhomogeneities

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Abstract

The non-linearity of Einstein’s equations makes it possible for small-scale matter inhomogeneities to affect the Universe at cosmological distances. We study the size of such effects using a simple heuristic model that captures the most important backreaction effect due to nonrelativistic matter, as well as several exact solutions describing inhomogeneous and anisotropic expanding universes. We find that the effects are $O(H^2l^2/c^2)$ or smaller, where $H$ is the Hubble parameter and $l$ the typical size scale of inhomogeneities. For virialized structures this is of order $v^2/c^2$, where $v$ is the characteristic peculiar velocity.
1 Introduction

The Friedmann-Robertson-Walker (FRW) homogeneous solution to Einstein’s equations gives an excellent approximation to the large-scale structure of our universe (for a survey, see e.g. [1]). This seemingly uncontroversial assumption is less obvious than it appears because the distribution of matter—visible or otherwise—is inhomogeneous already at $\sim 10$ Mpc scales. At that scale the density contrast $\delta \rho / \rho$ becomes $O(1)$ and the non-linearities of Einstein’s equations makes it conceivable that this may affect some of the properties of the universe even at cosmological scales.

This possibility has been examined in many papers. Extreme effects were advocated in [2], which argued that primordial CMB inhomogeneities coupled with the non-linear nature of Einstein’s equations could account for the late-time acceleration of the universe without the need for dark energy. Convincing arguments against this were advanced in [3, 4], yet the less extreme claim that inhomogeneity can affect the very large-scale behavior of the universe is worth investigating. Indeed, even if these effects turn out to be small, they may still be relevant for interpreting data of the next generation of cosmological probes. For instance the study of Ref. [5] suggests that the backreaction effect of inhomogeneities at the Hubble scale $H$ is of the order of $10^{-5}$.

When studying the effect of inhomogeneities, one must be careful in not mistaking the onset of large matter inhomogeneities with the breakdown of the linear approximation for the gravitational field itself. Intuitively, the difference is that gravity is so weak that a linear equation for the gravitational field holds everywhere outside black holes, so, in particular even when $\delta \rho / \rho \gg 1$ (as on the surface of the Earth). Yet making this intuition into a precise statement is difficult. One would need a systematic expansion that takes into account matter non-linearities to all orders, but treats the gravitational backreaction perturbatively. That is, one would need an appropriate perturbative series in $G \delta \rho$.

We will study this problem heuristically in section 2, by presenting a “paradox” that arises already at second order in the $G \delta \rho$ expansion, together with its resolution. Namely, we shall consider point-like particles of mass $m$, distributed on a regular lattice of size $l$. At scales larger than $l$, one would expect to find a uniform FRW solution with Hubble constant $H^2 = 8\pi G m / 3 l^3$. Yet as we will see the backreaction of the point-like sources is formally infinite. It becomes finite and of order $l^2 H^2$ only after an appropriate renormalization of the mass $m$. Physically, this renormalization arises because one cannot separate a “bare” mass from its gravitational energy.

Section 2 is heuristic in that the only backreaction term kept there is the Newto-

\footnote{$G$ is the Newton constant and everywhere in this paper $c = 1$.}
nian gravitational energy. This is not a rigorous procedure, because effects of similar size are ignored. To better study backreaction effects we proceed in sections 3 and 4 to study simple exactly soluble cosmologies: an array of equally spaced parallel two-dimensional dust walls in section 3, and in section 4 solutions for arbitrary arrangements of parallel cosmic strings and the “Swiss Cheese” model of ref. [6]. The symmetries of these toy models allow us to find explicit solutions of the full Einstein’s equations that again only show effects of $O(l^2 H^2)$, consistent with our general picture.

The toy models will also allow us to address another problem, namely that whenever inhomogeneities are present, the very definition of average cosmological parameters becomes ambiguous and deserves a thorough re-examination. In the literature on backreaction effects, the most commonly used prescription for averages is spatial averages over surfaces of constant proper time [7]. This definition is clearly not physical. The synchronous gauge may not even exist in a general cosmological solution, and even when it does it becomes singular whenever caustics in the matter flow develop. Moreover, it requires averaging over regions outside our past light cone. Finally, it gives rise to pitfalls clearly and succinctly described in [4].

In the toy model of Section 3 we will define several observable quantities that can be identified with a physical Hubble parameter. Significantly, these definitions differ from each other and from the homogeneous result exactly by terms $O(l^2 H^2)$.

In Section 4 we will find arbitrarily inhomogeneous cosmologies for which the Hubble flow receives no corrections at all, and briefly discuss the Swiss Cheese model, another inhomogeneous and anisotropic cosmology in which the local Hubble flow is uncorrected.

Tellingly, the scale we firmly associate to the large scale effect of inhomogeneities is $O(v^2)$, with $v$ the typical peculiar velocity. This result is both physically sensible and large enough to risk to become a factor in future precision cosmology.

Some technical material omitted in the body of the paper is collected in two appendices.

## 2 Backreaction and Mass Renormalization

Consider an FRW universe with metric

$$ds^2 = dt^2 - a^2(t) \gamma_{ij} dx^i dx^j, \quad i, j = 1, \ldots, 3.$$  \hspace{1cm} (1)

In this section we will often choose a flat homogeneous space metric $\gamma_{ij} = \delta_{ij}$ to make our (heuristic) equations simpler, but our results could be easily extended to other homogeneous metrics.
Linearized perturbations of this FRW universe reduce to the well-known Newtonian limit whenever the stress energy tensor $T_{mn}$ can be decomposed into a homogeneous piece, $T^0_0 = \bar{\rho}$, $T^i_j = \delta^i_j \bar{\rho}$, sourcing the Friedmann’s equations, plus an inhomogeneous “dust” component $t^0_0(\bar{x}, t) = \delta \rho(\bar{x}, t)$, $t^i_j \approx 0$ [1]. Here an over-bar will denote space-averaged quantities (to simplify the discussion we will for the moment ignore ambiguities in the averaging procedure, which will be addressed below). To first order in $G \delta \rho$, the metric is

$$ds^2 = (1 + 2\phi)dt^2 - a^2(t)(1 - 2\phi)\gamma_{ij}dx^i dx^j,$$

while $\phi$ obeys the Poisson equation [1]

$$a^{-2}D_i D^i \phi = 4\pi G \delta \rho.$$

Here $D_i$ is the covariant derivative w.r.t. the metric $\gamma_{ij}$, which is also used to raise and lower the indices $i, j$. The dominant non-linear correction to this equation has a simple physical meaning: gravity couples to all forms of energy, including the energy of the gravitational field itself. Thus, to second order in $\phi$, equation (3) becomes

$$a^{-2}D_i D^i \phi = 4\pi G \delta \rho - \frac{1}{2}a^{-2}D_i \phi D^i \phi.$$

Though this equation is heuristic, it does capture the main effect of non-linearities since a more complete derivation of backreaction effects leads to a similar formula [5, 8].

Consider now the case that the background space metric is flat $\gamma_{ij} = \delta_{ij}$, and that the “dust” making up $\delta \rho$ is composed of very compact objects of radius $r$ distributed on a cubic lattice of physical size $l$; by compact we mean that $r \ll l$. Then

$$\delta \rho(\bar{x}) = \sum_{\bar{n} \in \mathbb{Z}^3} \rho_r(a\bar{x} - l\bar{n}) - \bar{\rho}.$$

Here $\rho_r(\bar{x})$ is an arbitrary positive function vanishing for $|\bar{x}| > r$ and normalized to

$$\int d^3x \rho_r(\bar{x}) = m.$$

The backreaction equation [1] does not contain any time derivative, so, when studying the effect of inhomogeneities at any given time, as we will do next, we can set the scale factor $a = 1$ to avoid needless complications.

The average background density $\bar{\rho}$ is a function of the cluster mass $m$ and the volume of the fundamental cell of the lattice $\bar{\rho} = m/l^3$.

To understand the large scale effect of the non-linear term we average eq. [1] over a cube of side much larger than $l$ at constant time $t$. The averaging procedure itself
introduces an ambiguity, since we could have chosen to average over a different space-like surface. In this section, this ambiguity will not matter, since here we merely want to show how to avoid much larger, indeed divergent, unphysical effects.

The average $\overline{\delta \rho}$ is zero by construction, so eq. (4) averages to

$$\triangle \overline{\phi} = -\frac{1}{2} \nabla \phi \nabla \phi,$$

(7)

The gradient and the Laplacian are the standard flat-space ones. To second order in $G\delta \rho$, the potential $\phi$ appearing in the right hand side of this equation is the solution of the Poisson eq. (4).

A brief computation then shows that the average gravitational energy density is

$$-\frac{1}{8\pi G} \nabla \phi \nabla \phi = -\frac{G}{l^6} \sum_{\vec{m} \in \mathbb{Z}^3, \vec{m} \neq \overline{0}} \frac{l^2}{2\pi m^2} \left| \tilde{\rho}_r \left( \frac{2\pi}{l} \vec{m} \right) \right|^2,$$

(9)

where $\tilde{\rho}_r(\vec{k}) \equiv \int d^3 x \rho_r(\vec{x}) \exp(i\vec{k} \cdot \vec{x})$.

Here we encounter a (pseudo) paradox: when the “clusters” are point-like, the distribution $\tilde{\rho}_r$ is a constant and sum in eq. (9) diverges. If we regulate the divergence—say by making $\rho_r(\vec{x}) = 3/4\pi r^3$ for $|\vec{x}| \leq r$ and zero otherwise—we may still end up with a gravitational energy density as large as the background energy $\bar{T}_0$. This happens in particular if the radius of the sphere is close to its Schwarzschild radius. Physically, this would mean that if all matter in a universe at critical density was distributed into black holes, corrections to the linear approximation would be $O(1)$ no matter how small the lattice step be! Even when $r \gg Gm$ this contribution is suspicious because it depends strongly on the size of the cluster.

An estimate of the $r$ dependence of the sum is given in the Appendix, here we only quote the result, and we give a simple argument to justify it:

$$-\frac{1}{8\pi G} \nabla \phi \nabla \phi = -Gm^2 \left[ \frac{C_1}{r^3} + \frac{C_2}{l^4} + O \left( \frac{r}{l^5} \right) \right],$$

(10)

where $C_1$, $C_2$ are dimensionless numbers of order unity which are determined by the density profile of compact objects $\rho_r$ and by the shape of the lattice. The first term in the expansion (10) comes from the classical gravitational energy of a body of mass $m$ and size $r$. Other terms, describing corrections, must be there because for a body of uniform density filling the entire lattice cell $\tilde{\rho}_0$ vanishes by definition. The divergence at small $r$ term scales with $l$ as pressureless mass density, while the finite term, independent of $r$, scales as ultra-relativistic energy density.
At this point, it is obvious that the divergent term is unphysical and it must be canceled by redefining the “bare” mass of the compact objects to first order in $G$ by

$$m = m_{\text{physical}} + C_1 \frac{G m^2}{r}. \quad (11)$$

This classical renormalization is not an option because the gravitational field is actually determined by the physical mass.

$$m - \frac{G}{2} \int d^3x \rho_r(0) \frac{1}{|\vec{x}|} \rho_r(\vec{x}) = m - G \int \frac{d^3k}{(2\pi)^3} \frac{2\pi}{k^2} \left| \tilde{\rho}_r(\vec{k}) \right|^2 \equiv m_{\text{physical}}. \quad (12)$$

Notice that corrections due to inhomogeneities do exist. They are due to the finite term in eq. (10), which cannot be eliminated by a renormalization of the mass, since it scales with $l$ as radiation. These corrections change the Friedmann equation by terms $O(H^2 l^2)$. If $l$ is interpreted as the non-linear length scale of the actual universe, then $H^2 l^2 \sim v^2$, where $v$ is the typical peculiar velocity. So, non-linear corrections to the Hubble parameter due to gravitational energy are of the same order as relativistic corrections due to peculiar velocities. We have actually neglected the latter as well as ambiguities in the definition of the Hubble parameter. Evidently, to make further progress we need to be more systematic, even if at the price of studying a vastly simplified model of inhomogeneities. This is what we will do next, starting by considering a distribution of pressureless matter that only breaks translational invariance in one direction, while preserving translations and rotation in two orthogonal directions.

### 3 Dust-Wall Universe

Our goal is to construct an exact solution of the Einstein’s equations describing an expanding inhomogeneous universe. A simple possibility is planar symmetry – the metric depending on $t$ and $x$ but not on $y$ and $z$, and isotropic in the $yz$ plane. Here we use Taub’s explicit expressions for the metric of vacuum plane-symmetric space-times to construct a universe of equidistant plane-parallel dust walls. Our treatment of the walls is similar to the single wall case \[10\] \[11\].

The plane-symmetric metric can be written as

$$ds^2 = e^{2u}(dt^2 - dx^2) - e^{2v}(dy^2 + dz^2), \quad (13)$$

where $u = u(t, x)$ and $v = v(t, x)$. The walls are located at $x = \pm b, \pm 3b, ...$ so that the metric coefficients $u$ and $v$ are $2b$-periodic functions of $x$. It also follows from the symmetry of the problem that $u$ and $v$ are even functions of $x$. 

The stress-energy tensor of a thin dust wall of proper surface density $\sigma(t)$ located at $x = b$ is

$$T^\mu_\nu(t, x) = \sigma(t)e^{-u(t, b)}\delta(x - b) \text{diag}(1, 0, 0, 0).$$  \hspace{1cm} (14)

For simplicity we consider walls with zero pressure, but it is easy to find the analogous solutions for an array of walls with general equation of state. Calculating the Einstein tensor $G^\mu_\nu$ for the metric (13), equating it to the stress-energy tensor of the dust walls, and using the symmetry of the metric, we get the following jump conditions at the $x = b$ wall: the $G^0_0$-component gives

$$\sigma = 4e^{-u}\partial_x v,$$  \hspace{1cm} (15)

the $G^0_1$ and $G^1_1$ components are non-singular at the wall, and the $G^2_2 = G^3_3$ components give

$$\partial_x(u + v) = 0,$$  \hspace{1cm} (16)

where the derivatives are calculated for $x = b - 0$.

Between the walls the metric is given by the Taub’s expressions:

$$e^{2u} = \frac{f'g'}{\sqrt{f + g}}, \quad e^{2v} = f + g, \quad f = f(t + x), \quad g = g(t - x),$$  \hspace{1cm} (17)

where $f$ and $g$ are arbitrary functions of one variable, and $f'$ and $g'$ are their derivatives. Since $u$ and $v$ are even functions of $x$ at all time, we must take $g(\xi) = f(\xi)$. Then (16) gives an equation for $f$:

$$\frac{f''(t + b)}{f'(t + b)} - \frac{f''(t - b)}{f'(t - b)} + \frac{1}{2}\frac{f'(t + b) - f'(t - b)}{f(t + b) + f(t - b)} = 0.$$  \hspace{1cm} (18)

The remaining jump condition (15) gives the proper surface density as a function of time in terms of $f$.

At large times $t \gg b$ there are many walls within the horizon and we recover the homogeneous matter-dominated universe. To lowest order in $b$, finite differences in equation (18) can be replaced by differentials, and we get

$$\left(\frac{f''}{f'}\right)' + \frac{1}{4}\frac{f''}{f} = 0,$$  \hspace{1cm} (19)

with the solution $f(t) \propto t^4$. To lowest order in $b$ this gives the metric $e^{2u} \propto e^{2v} \propto t^4$. This describes homogeneous matter-dominated universe with conformal time $t$.

As expected, there are finite $b$ corrections. In the next to leading order, we get $f(t) \propto t^4 + 40b^2t^2$. This gives the metric coefficients

$$e^{2u} \propto t^4 + t^2(20b^2 - 6x^2), \quad e^{2v} \propto t^4 + t^2(40b^2 + 6x^2).$$  \hspace{1cm} (20)
We see that the metric is close to that of a uniform matter dominated universe, with $b^2/t^2$ corrections. In physical units the corrections are $\sim H^2l^2$, where $l$ is the physical distance between the walls and $H$ is the Hubble constant.

Now we would like to see how the averaged expansion rate of the universe is changed due to inhomogeneities. To this end, we define parallel and perpendicular scale factors on the wall:

$$a_{\parallel}(t) \propto e^{v(t,b)}, \quad a_{\perp}(t) \propto e^{u(t,b)}.$$  

(21)

In the next to leading order we get

$$a_{\parallel}(t) \propto t^2 + 23b^2, \quad a_{\perp}(t) \propto t^2 + 7b^2.$$  

(22)

Further we define parallel and perpendicular Hubble constants

$$H_{\parallel} = \frac{d\log a_{\parallel}}{d\tau}, \quad H_{\perp} = \frac{d\log a_{\perp}}{d\tau},$$  

(23)

where $d\tau \propto a_{\perp}dt$ is the proper time on the wall. Both definitions of the Hubble constants are physical. $H_{\parallel}$ can be measured from the proper surface energy density as a function of proper time. $H_{\perp}$ can be measured by light propagation time between the walls. We then get corrections to the Friedmann equation for parallel and perpendicular Hubble constants

$$H_{\parallel}^2a_{\parallel}^3 \propto 1 + \frac{9}{16}H^2l^2, \quad H_{\perp}^2a_{\perp}^3 \propto 1 - \frac{7}{16}H^2l^2,$$  

(24)

where $l \approx 2be^u$ is the physical distance between the walls and $H \approx H_{\parallel} \approx H_{\perp} \approx 2/(3\tau)$.

4 Uncorrected Hubble Flow

In this section we will consider a broad class of examples of anisotropic cosmologies for which we can find exact solutions, this time sourced by arrangements of parallel relativistic cosmic strings rather than parallel walls. The dynamics of such objects—so long as they remain parallel—is identical to that of massive point particles in 2+1 dimensional gravity, and so for the rest of the section we will use that language. We will show that starting with any isotropic and homogenous solution of Einstein’s equations in 2+1 dimensions one can add point particles—or indeed an arbitrary distribution of dust—and (due to the special properties of gravity in 2+1 dimensions) the presence of the point particles and/or dust does not modify the Hubble expansion at all.
We begin with the metric
\[ ds^2 = dt^2 - a(t)^2 ds_2^2, \] (25)
where as usual \( ds_2^2 \) is the metric of a homogeneous and isotropic two-dimensional flat, spherical, or hyperbolic space. From this ansatz the 2+1 Friedmann equation follows:
\[ H^2 = \rho - k/a^2, \] (26)
where as usual \( k = \{-1, 0, +1\} \), \( H = \dot{a}/a \) and \( \rho = \rho_0 (a_0/a)^{2(1+w)} \) for the case of a perfect fluid with equation of state \( w \).

One peculiar feature special to the case of matter domination \( (w = 0) \) is that \( a(t) = t \), with zero acceleration. This indicates that uniformly expanding dust in 2+1 dimensions does not exert a force on itself, a result which might have been expected since there is no force between static particles in 2+1 dimensions. Note also that, since in this case the matter density scales the same way as curvature, the scale factor is the same for all three types of universes.

### 4.1 Point Particles

In order to consider an anisotropic cosmology we begin by reviewing the physics of a point particle with mass \( m \in (0, 2\pi) \) in a 2+1 dimensional space\(^3\). As is well-known the metric is simply flat space with a conical defect along the world-line of the particle: \( ds_2^2 = dt^2 - dr^2 - r^2 d\theta^2 \), where the range of \( \theta \) is from 0 to \( 2\pi - m \). Perhaps the simplest way to see this is to recall that in 2+1 dimensions Einstein’s equation in vacuum \( R_{\mu\nu} = 0 \) implies \( R_{\mu\nu\lambda\sigma} = 0 \), so that the space must be flat away from the point particle. The conical defect induces a delta function in the curvature of strength \( 2m \), from which it follows that the metric satisfies Einstein’s equations with a point-like source. Since any space is locally flat, all of the solutions we consider here are of this form very close to the singularity. Note that the mass of the particle should be smaller than \( 2\pi \) since the circumference of the circle goes to zero in that limit.

Let us consider a collection of point particles with masses \( m_i \) at locations \( z_i \) in flat space. Taking the metric ansatz
\[ ds^2 = dt^2 - e^{\phi(z, \bar{z})} dz d\bar{z}, \] (27)
\(^2w\) refers to the 2+1 dimensional equation of state. Lifting \( w = 0 \) here to 3+1 gives an anisotropic but homogeneous 3+1 dimensional universe filled with a uniform “gas” of parallel cosmic strings.
\(^3\)We will work in units where \( 8\pi G_3 = 1 \).
Einstein’s equations reduce to
\[ \partial_z \partial_{\bar{z}} \phi = - \sum_{i=1}^{N} m_i \delta^2(z - z_i). \]  
(28)

So long as the sum of the masses satisfies the constraint \( M = \sum_{i=1}^{N} m_i < 2\pi \) the solution is
\[ e^{\phi} = \prod_{i=1}^{N} |z - z_i|^{\frac{m_i}{\pi}}. \]  
(29)

It is easy to see where the constraint on the total mass comes from. Consider a large disk \( S \) containing all the masses. Recall that the Euler character \( \chi = 1 \) for a disk, that the space is flat away from the point particles, and that a point particle of mass \( m \) gives rise to a delta function in the scalar curvature \( R \) of strength \( 2m \). Then, since the integrated extrinsic curvature of a circle is the derivative of its circumference with respect to the radius, \( \partial_r c \), the Gauss-Bonnet theorem,
\[ \int_S R + 2 \int_{\partial S} K = 4\pi \chi, \]  
(30)
implies \( \sum m_i = 2\pi - \partial_r c \). If the total mass \( M \) exceeds \( 2\pi \) the space cannot be globally flat, since the circumference of the disk must shrink as the radius increases. If we insist that the curvature be zero, the solution \( \text{[29]} \) now implies the presence of a conical singularity at \( z = \infty \) equal to \( 4\pi - \sum_{i=1}^{N} m_i \). The metric then describes a compact space with spherical topology, with \( N + 1 \) conical singularities whose total mass adds up to \( 4\pi \).

From the Gauss-Bonnet theorem it follows that in order to accommodate point particles with total mass \( M > 4\pi \) some curvature is needed. We wish to consider the case of a large or infinite number of (positive) masses, and so the background 2-dimensional curvature must be negative. In fact the expansion of the universe automatically acts as a source for the negative spatial curvature.

Consider a metric of the form
\[ ds^2 = dt^2 - a(t)^2 e^{\phi(z, \bar{z})} dz d\bar{z}. \]  
(31)

If we now require that the energy-momentum tensor consists of point particles of mass \( m_i \) at fixed spatial locations \( z_i \) plus a homogeneous background matter density \( \rho_{\text{hom}} \), Einstein’s equations reduce to
\[ \partial_z \partial_{\bar{z}} \phi = -k e^{\phi} - \sum_{i=1}^{N} m_i \delta^2(z - z_i), \]  
\[ \left( \frac{\dot{a}}{a} \right)^2 = \rho_{\text{hom}} - \frac{k}{a^2}, \]  
(32)
where $k$ determines the (constant) spatial curvature away from the singularities. This is the main result of this section. As is clear from the second equation the evolution of the scale factor is not disturbed by the defects as long as the first equation can be satisfied. This in turn depends only on the masses $m_i$ (see appendix). The conclusions above hold even in the presence of general homogenous matter components. In this case the pressure of the homogenous background is given by,

$$p = -\frac{\ddot{a}}{a}$$

(33)

For the case $k = 0$ the first equation reduces to the Poisson equation, whose solution with point particle masses we discussed above. When $k = \pm 1$ eq. (32) is the Liouville equation with positive and negative curvature respectively. As we discuss in the appendix, solutions to this equation always exist, unless topological obstructions arise. From the Liouville equation it also follows that, as required by the Gauss-Bonnet theorem, the local density of particles cannot be too large to avoid “overclosing” the universe.

One case of interest is a square grid with a point mass at each vertex and with the background matter density $\rho_{\text{hom}} = 0$. Since the total mass in the universe is greater than $4\pi$ the background curvature must be negative. The mathematical problem we need to solve reduces to finding the metric with constant negative curvature and a single conical singularity on a square torus. Although in this case the Liouville equation cannot be solved in closed form, the solution is known to exist and to be unique. Physically, while the metric has locally negative spatial curvature away from the particles, it has precisely the correct density of positively curved defects to cancel the negative curvature when averaged over a large region (or over one tile of the grid). The solution approximates flat FRW at large scales (and it is for this reason that it is possible to arrange the masses in a square grid even though the metric locally is negatively curved).

In general there is a very large continuous infinity of solutions of this type (corresponding to the freedom of changing the positions and masses of the particles). In fact (as is clear since we can take the limite of very small strength $\delta$-functions closely spaced) if we replace the term $\sum_{i=1}^{N} m_i \delta^2(z - z_i)$ in equation (32) with a function $\rho_{\text{dust}}(z, \bar{z})$ we can obtain solutions with arbitrary distributions of dust.

It is remarkable that the function $a(t)$ in the metric (24) remains unchanged for any arrangement of point masses or distribution of dust, and that therefore the local Hubble expansion law is entirely uncorrected by the inhomogeneities. This is a physical statement confirmable by local experiments; for example an observer in this universe could measure the rate of change in the local matter density as a function of his proper time and would observe it to be determined only by the
average Hubble constant (averages here can be taken over surfaces of constant \( t \), for which the expansion is uniform), independent of his location and the locations of the anisotropies.

Let us mention that, even though the Hubble expansion is not modified, many experiments will be sensitive to the positions of the masses, for example test particles fired out along some trajectory will scatter off the point masses. For the same reason the observed brightness of a distribution of standard candles of fixed red-shift will vary.

### 4.2 Swiss Cheese

There exists in the literature another exact solution describing an inhomogeneous universe which gives no corrections to the Hubble parameter—the “Swiss Cheese” model of Schucking [6]. This is a universe in which spherical regions of homogeneous background matter are excised and replaced with vacuum, but with a spherical concentration of mass at the center of the void. If the mass at the center is chosen appropriately and the voids are non-intersecting the metric outside the voids is completely unaffected by their existence and the Hubble flow as measured by an (outside) observer is unchanged. Of course as before there are many experiments which would be sensitive to the inhomogeneities.

### 5 Conclusion

Both generic (but heuristic) as well as exact (but highly symmetric) models show that the effect of matter inhomogeneities on the large scale expansion of the universe is small, bounded from above by \( v^2 \sim 10^{-5} \). The numerical coefficient and sign of the effect depend on the model and the definition chosen for Hubble. What we have shown can also be interpreted as follows: there always exist gauges where the metric differs from the FRW metric only at order \( \sim v^2 \).

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Appendix A: Estimating the Gravitational Self-Energy Divergence

Here we estimate the behavior of the sum in eq. (9).

The solution of the Poisson eq. (4) is easily written in Fourier transform as

\[
\tilde{\phi}(\vec{k}) = -\frac{4\pi G}{k^2} \sum_{\vec{m} \in \mathbb{Z}^3, \vec{m} \neq 0} \frac{(2\pi)^3}{l^3} \delta^3 \left( \vec{k} - \frac{2\pi}{l} \vec{m} \right) \tilde{\rho}_r(\vec{k}).
\]  

(A.1)

For \( l \to \infty \), eq. (9) can be approximated by the integral

\[
\lim_{l \to \infty} \frac{1}{l^3} \sum_{\vec{m} \in \mathbb{Z}^3, \vec{m} \neq 0} \frac{l^2}{2\pi m^2} \left| \tilde{\rho}_r \left( \frac{2\pi}{l} \vec{m} \right) \right|^2 = \int \frac{d^3k}{(2\pi)^3} \frac{2\pi}{k^2} \left| \tilde{\rho}_r(\vec{k}) \right|^2.
\]  

(A.2)

Since \( \tilde{\rho}_r(\vec{k}) \) is analytic and square summable, we can also estimate the sum at finite \( l \) as

\[
\frac{1}{l^3} \sum_{\vec{m} \in \mathbb{Z}^3, \vec{m} \neq 0} \frac{l^2}{2\pi m^2} \left| \tilde{\rho}_r \left( \frac{2\pi}{l} \vec{m} \right) \right|^2 = \frac{m^2}{r} f \left( \frac{r}{l} \right).
\]  

(A.3)

By dimensional analysis, \( f(x) \) is a dimensionless function, independent of \( m, r, l \), and smooth in a neighborhood of \( x = 0 \).

Therefore we can see that the energy density diverges as \( 1/r \) for \( r \to 0 \), and we also recover the finite term of eq. (10).

Appendix B: More on the Liouville Equation

As we have shown in section 4, the problem of point particles (with zero peculiar velocities) in an homogenous FRW universe in 2+1 dimensions reduces to finding solutions to the Liouville equation,

\[
\partial_z \partial_{\bar{z}} \phi = -\frac{k}{2} e^\phi - \sum_{i=1}^N m_i \delta^2(z - z_i).
\]  

(B.1)

This equation describes a two dimensional surface of constant curvature \( k \) with conical singularities \( m_i \). The general solution of the Liouville equation is

\[
e^\phi = \frac{4|w'|^2}{[1 + k|w|^2]^2},
\]  

(B.2)
where $w(z)$ is a holomorphic function that must be chosen appropriately in order to reproduce the correct singularities at $z_i$,

$$\phi \sim -\frac{m_i}{\pi} \log |z - z_i|, \quad \text{as } z \to z_i. \quad (B.3)$$

In principle $w(z)$ can be determined by using the technology of the Fuchsian equations (see for example [12]), even though explicit solutions can only be found in special cases. The solutions of the Liouville equation are in general determined by the singularities and by the topology of the space. Since the locations $z_i$ of the singularities in eq. (B.1) are arbitrary (up to conformal transformations) there is actually a large moduli space of solutions. Note that even though the $z_i$’s are free, the position in physical space are constrained since on scales smaller than the curvature the total deficit angle cannot exceed $2\pi$. This is automatically guaranteed by solving the Liouville equation.

Let us now consider the case where the space is compact. By integrating both sides of eq. (B.1) one finds,

$$V = \frac{4\pi(1 - g) - \sum_{i=1}^{N} m_i}{k}, \quad (B.4)$$

where $V$ is the volume and $g$ is the genus of the surface. This formula follows from the fact that the left hand side of eq. (B.1) is proportional to $\sqrt{g}R$ and the Gauss-Bonnet formula for compact surfaces without boundaries,

$$\frac{1}{4\pi} \int_S \sqrt{g}R = 2 - 2g. \quad (B.5)$$

Since the volume must be positive, eq. (B.4) implies restrictions on the allowed singularities depending on the sign of the curvature.

$k = 1$

This situation can arise for example in the presence of background dust or positive cosmological constant. Since we only allow positive masses the volume formula (B.4) here determines that the only possible topology is the spherical one and that the sum of the deficit angles must be less than $4\pi$. The Liouville equation can be solved explicitly in the case of three singularities. In general a solution exists provided that a mild constraint on the deficit angles is satisfied, namely that the largest conical defect is less than the sum of the remaining ones [13].

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4This asymptotic behavior holds for $m_i < 2\pi$. When $k$ is negative singularities with $m_i = 2\pi$ (so called parabolic) are also acceptable in which case $\phi \sim -2 \log |z - z_i| - 2 \log |\log |z - z_i||$. 

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This is the case relevant for particles in vacuum when the sum of the deficit angles is greater than $4\pi$.

An important theorem dating back to Poincaré and Picard establishes that when the curvature is negative a solution of the Liouville equation with prescribed singularities always exists and is unique, once eq. (B.4) is satisfied. If the sum of the deficit angles is greater than $4\pi$ any topology is allowed and in particular we can choose the spherical topology.

Note that in hyperbolic space a deficit angle increases the volume of the space. A case relevant for our purposes is the torus with one conical singularity. This problem is equivalent to that of a grid of equal masses. From eq. (B.4) we derive that the volume of each tile equals the deficit angle. This ensures that the average curvature is zero as expected on physical grounds.

References


