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YANG-MILLS DUALITY IN HIGHER DIMENSIONS

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ABSTRACT

We discuss a higher dimensional generalization of the four-dimensional Yang-Mills self duality equation of the form $*F = F \wedge \dots \wedge F$. Some general properties of the equations are established. We show that the canonical connection on the real, complex and quaternionic projective spaces is a solution of these duality equations.

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1. Introduction

The notion of self duality has proven to be a very fruitful concept in the study of $d = 4$ Euclidean gauge theories. In this paper we extend this notion of self duality to more than four dimensions and study the following set of nonlinear algebraic relations between the field strength components on even dimensional Euclidean spaces

$$*F = F \wedge F \dots \wedge F .$$

This relation implies the Yang-Mills equations. In spite of the fact that the number of spacetime components of this relation is rather large, which puts severe restrictions on the space of solutions, we discuss various classes of solutions in detail and show their topological nature.

One of the main motivations to embark on this program (apart from its mathematical interest) is the promising revival of the Kaluza-Klein idea of dimensional reduction (compactification) in attempts to unify four dimensional gravity with the Yang-Mills interactions into a single geometric framework [1].

Also the study of explicit gauge fields on higher dimensional Kaluza-Klein spaces has received a lot of interest. It may provide the way in which parity violation of the ultimate $d = 4$ effective theory is realized [2-3], as for example is envisaged in superstring theories [4]. Solutions to the Yang-Mills equations in higher dimensions are also of interest as a starting point in the search for topological excitations in Kaluza-Klein theories [5]. Finally we like to mention another interesting application of the $d = 2n$ duality equations in four dimensional Yang-Mills-Higgs theories. By dimensional reduction of pure Yang-Mills theories one obtains $d = 4$ Yang-Mills-Higgs systems, where spontaneous symmetry breaking may occur [6]. The dimensional reduction of our nonlinear duality equations gives rise to a system of first order equations which imply the coupled Yang-Mills-Higgs equations. We consider these as the natural generalization of the Bogomolny equations to theories where the Higgs fields are not restricted to be in the adjoint representation of the gauge group [38].

The outline of this paper is as follows. The remainder of this section is devoted to a summary of the salient features of the four dimensional duality

which is useful because we will make comparisons with the four dimensional case throughout the paper. In the following section we introduce the nonlinear generalization for even dimensions, and discuss the generic properties of its solutions. Section 3 is concerned with a detailed discussion of some solutions on spheres and projective spaces. In the concluding section we outline further applications and open problems.

Let us recall the basic properties of (anti-) self duality in four dimensions. For Yang-Mills fields with gauge group G one considers the relation

$$* F = \pm F \quad (1.1)$$

on a manifold with an Euclidean signature. We recall the following facts:

a) Algebraic structure

This system of linear algebraic relations between the curvature components implies a set of $3 \times \dim G$ first order coupled nonlinear differential equations for the $4 \times \dim G$ components A_μ^a of the Yang-Mills potential. If one adds $\dim G$ conditions to fix the gauge, the number of equations and independent components match exactly.

b) Yang-Mills equations

If we combine the self duality condition (1.1) with the Bianchi identity

$$D_A F = 0 , \quad (1.2)$$

we find that a self dual field automatically satisfies the Yang-Mills equations

$$D_A * F = 0 . \quad (1.3)$$

c) Yang-Mills action

The equations (1.3) determine the critical points of the Yang-Mills action functional

$$S = \frac{1}{4} \int \text{tr} (F \wedge * F) . \quad (1.4)$$

Inserting (1.1) into (1.4) one finds

$$S = \mp \frac{1}{4} \int \text{tr } F \wedge F \propto c_2(F) , \quad (1.5)$$

i.e. the action is proportional to the Pontryagin number, a topological invariant of the corresponding principal fiber bundle. For self dual solutions one can directly show that they correspond to a minimum of the action (within each topological class):

$$S = \frac{1}{8} \int \text{tr } (F \pm * F) \wedge *(F \pm * F) \mp \frac{1}{4} \int \text{tr } F \wedge F . \quad (1.6)$$

So, self dual configurations are weakly stable (weakly in the sense that they may have zero modes).

Less straightforward is the converse of the above, namely the question whether all minima of the action correspond to self dual solutions [7].

d) Riemannian geometry

Relation (1.1) has its counterpart in four dimensional Riemannian geometry where one may impose

$$R = \pm * R \quad (1.7)$$

on the curvature 2-form. In this case one speaks of a (anti-) self dual manifold [8,30]. Relation (1.7) implies that the space is Ricci-flat:

$$R_{\mu\nu} = 0 \quad (1.8)$$

and hence it is a solution to the Euclidean Einstein equations with vanishing cosmological constant. That (1.7) is indeed a much simpler system than (1.8) follows from the fact that all solutions to (1.7) are gauge equivalent (with a local Lorentz transformation) to a solution of the duality equation for the spin connection 1-form

$$\omega_{ab} = \pm \frac{1}{2} \varepsilon_{abcd} \omega^{cd} \quad (1.9)$$

where $a, b, \dots$ are Lorentz indices. This is a tremendous simplification; the system (1.9) is first order (in terms of the Vierbein) whereas (1.8) is second

order.

e) Dimensional reduction

It has been shown that dimensional reduction (for example over certain coset spaces S/R , $R \subset G$ of a pure Yang-Mills theory with gauge group G yields a (spontaneously broken) Yang-Mills-Higgs theory in lower dimensions [6,9]. The lower dimensional gauge group H corresponds to the centralizer of R in G which may be broken spontaneously due to a nonvanishing vacuum expectation value of the Higgs field. This implies that by dimensional reduction of the four dimensional self duality equations, one obtains a special set of first order equations which imply solutions to $d = 3$ and $d = 2$ Yang-Mills-Higgs systems. A well known example for $d = 3$ are the Bogomolny equations [10,12]

$$F_{ij} = \epsilon_{ijk} D_k \Phi, \quad (1.10)$$

where the $d = 4$ gauge potential $A_\mu^a(x^\nu)$ reduces to $A_i^a(x^j)$ and $A_0^a(x^j) \equiv \Phi^a(x^j)$ where Φ is the Higgs field [F1]. Another interesting example is obtained by reducing over the two-sphere, which yields first order equations for the Abelian Higgs model in two dimensions [10-12]. It is interesting to note that the dimensional reduction of the topological charge density in (1.5) yields interesting topological charge densities in the resulting lower dimensional theories whose existence is closely tied in with the nonvanishing expectation value of the Higgs field (magnetic charge for $d = 3$ and flux for $d = 2$).

These observations lead one to expect that from higher dimensional duality conditions which imply the higher dimensional Yang-Mills equations one may obtain interesting first order systems for four dimensional Yang-Mills-Higgs systems. We have obtained interesting results along these lines which will be published elsewhere [38].

2. Duality in even dimensions

In this section we consider a generalization of the four dimensional self duality condition to higher, even dimensions and discuss its generic properties.

Attempts to generalize the four dimensional self duality condition to higher dimensions have been made before [13-15], where one tried to construct a four index totally antisymmetric tensor T such that the set of linear algebraic relations

$$F = T F \quad (2.1)$$

between the components of the Yang-Mills curvature imply a solution to the Yang-Mills equations. Various interesting solutions were obtained this way in particular for $d = 8$.

In this paper we pursue another direction. We propose to study the following system of nonlinear algebraic relations between the (Hermitian) Yang-Mills curvature 2-forms on a space with dimension $d = 2n$

$$\begin{aligned} * F &= c F \wedge \dots \wedge F \text{ ((n-1)times)} \\ &\equiv c F^{n-1} \end{aligned} \quad (2.2)$$

where c is some arbitrary (real) constant with dimension $\frac{n}{2} (n-2)$. This system appears to be a natural generalization of (1.1) from a geometric point of view. Indeed the choice (2.2) is inspired by Kähler geometry where the fundamental 2-form does satisfy (2.2), as we will show later. We emphasize however that we consider (2.2) as a pure Yang-Mills problem on any given manifold. When this work was finished, the papers of Tchrakian [39-40] were brought to our attention, in which to our knowledge the study of (2.2) is taken up for the first time. However, there is little overlap between the results presented here and those found in reference [39].

Our first objective is to establish a number of general properties of the system (2.2) and its solutions.

a) Yang-Mills equations

The proof that the duality equation (2.2) implies the $d = 2n$ Yang-Mills equations

$$D_A * F = 0 \quad (2.3)$$

is the same as for $d = 4$: taking the gauge covariant derivative of (2.2) we obtain

$$D_A * F = c D_A F^{n-1} = 0 ,$$

due to the Bianchi identity

$$D_A F = 0 .$$

Strictly speaking one only has to satisfy the duality relations

$$* F^a = c \operatorname{tr}(T^a F^{n-1}) \quad (2.4)$$

in order to guarantee a solution to the Yang-Mills equations. Whether (2.2) is more restrictive than (2.4) depends on whether the r.h.s. of (2.2) is entirely contained in the Lie algebra. As we will discuss shortly this depends on the group and representation considered.

We note that one may consider alternative but inequivalent duality relations in $d = 2n$ of the following type:

$$* F^k = F^{n-k} \quad (k=2, \dots, [\frac{n}{2}]) . \quad (2.5)$$

These relations are of comparable complexity but the important difference is of course that they do not imply the Yang-Mills equations. Some of the solutions we will discuss in the following sections do actually satisfy all duality relations (2.5).

There is yet another type of duality one may consider which does imply a solution to the Yang-Mills equations:

$$* F = F^k \wedge \Omega_{(k)} , \quad (k=1, 2, \dots) \quad (2.6)$$

where $\Omega_{(k)}$ is a $2(n-k-1)$ -form (not Lie algebra valued!) which is closed,

$$d \Omega_{(k)} = 0 .$$

These are of interest, and some of the solutions we will discuss in the following section do in fact satisfy equations of this type basically because they satisfy an even simpler identity namely

$$F^s = F \wedge Q^{s-1}$$

for some $s < n$ and Q^{s-1} a generator of the cohomology group $H^{2s-2}(M, \mathbb{R})$. It is interesting to note that for $k = 1$, (2.6) becomes of the type (2.1).

b) Compatibility with the Lie algebra structure

The Yang-Mills curvature F is a Lie algebra-valued 2-form. As the exterior product is symmetric for 2-forms we have to see under what conditions the equations (2.2) are compatible with the structure of the Lie algebra. Whereas the four dimensional case is insensitive to the representation in which F is chosen this is no longer true for $d > 4$ as the symmetric product of generators appearing in eq. (2.2) is in general representation dependent. One should realize that a solution to (2.2) for F in a particular representation implies a solution to the Yang-Mills equations for any representation as these equations are of course representation independent. From (2.2) we obtain the following expression for $*F^a$:

$$\begin{aligned}
 *F^a &= c \operatorname{tr} (T^a F^{n-1}) \\
 &= \frac{c}{(n-1)!} F^{a_1} \wedge \dots \wedge F^{a_{n-1}} \operatorname{tr} T_a T_{(a_1} \dots T_{a_{n-1})} \\
 &\propto F^{a_1} \wedge \dots \wedge F^{a_{n-1}} \operatorname{STr} T_a T_{a_1} \dots T_{a_{n-1}}. \quad (2.7)
 \end{aligned}$$

[F2].

We now realize that the r.h.s of (2.7) is nothing but the contribution to the gauge anomaly in $d=2n-2$ for fermions which transform under the representation generated by $\{T_a\}$. If the anomaly vanishes the group (or representation) is called "safe" [16], which in the present context implies that the $d=2n$ duality equations have no solutions.

The duality equations are only compatible in $d=2n$ dimensions for representations of groups which are anomalous (i.e. "unsafe") in $2n-2$ dimensions.

From the well known results on the anomaly free groups and representations for $n=3$ [17] we learn that in $d=6$ the duality equations can only have interesting solutions for certain complex representations of $SU(N)$ and E_6 (and $SO(6) \sim SU(4)$). Furthermore we know that the representation dependence of the r.h.s of (2.5) is very simple as it can be entirely absorbed in the constant c : we recall that $\operatorname{tr} T_a \{T_b, T_c\} = d_{abc} \cdot A(R)$, here d_{abc} is representation independent, $A(R)$ is called the anomaly of the representation [18]. This implies that all solutions to (2.4) in $d = 6$ can be obtained from a solution in the fundamental (or spinor) representation after an appropriate scaling of the constant c . This simple property does not seem to persist in higher dimen-

sions. In the context of Kaluza-Klein theories the problem of gauge anomalies in higher dimensions has received attention recently [19-20]. From the work of for example Frampton and Kephart [20] for the groups SU(N), it is clear that the representation dependence does no longer factorize in such a simple way and can no longer be absorbed in the single constant c, i.e. (2.7) becomes in general representation dependent in an essential way.

For $d = 4k + 2$ (i.e. $n = 2k + 1$) we can prove that if F is in a real representation (2.7) has no solutions. We use the same trick as in [16a]:

for a real representation

$$T_a = - U T_a^* U^\dagger \quad (2.8)$$

[F3]

and so

$$\text{tr } T_a T_{a_1} \dots T_{a_{n-1}} = (-1)^n \text{tr } T_a^* T_{a_1}^* \dots T_{a_{n-1}}^* . \quad (2.9)$$

However, $\text{tr } T_{(a} T_{a_1} \dots T_{a_{n-1})}$ is real i.e.

$$\text{tr } T_{(a} T_{a_1} \dots T_{a_{n-1})} = \text{tr } T_{(a}^* T_{a_1}^* \dots T_{a_{n-1})}^* . \quad (2.10)$$

Combining this we find that

$$\text{tr } T_{(a} T_{a_1} \dots T_{a_{n-1})} = 0 .$$

In $d = 4k$ (i.e. $n = 2k$) we know from [21] that for simple Lie groups

$$\text{tr } T_{(a} T_{a_1} \dots T_{a_{n-1})} \neq 0 \quad (2.11)$$

for some subset of indices and in principle (2.7) has a nontrivial solution as far as the Lie algebra structure of the equation is concerned.

It is beyond the scope of this paper to tackle the problem of explicitly evaluating

$$\text{tr } T_{(a} T_{a_1} \dots T_{a_{n-1})}$$

for arbitrary n ($n > 3$) in full generality. It is a well defined problem in representation theory. We restrict ourselves to the discussion of the compa-

tibility of (2.2) when F is chosen to be in the defining (or fundamental) representation of a classical group.

- i) For the unitary groups $U(N)$, the Lie algebra consists of all $N \times N$ Hermitian matrices. The algebra is closed under both the antisymmetric (commutator) and the symmetric (anticommutator) product. For $U(N)$ therefore, the problem of finding solutions to (2.2) is well posed. If one makes the restriction to $SU(N)$, one will in general generate a nonvanishing trace at the r.h.s of (2.2), which implies an extra constraint. However, to obtain a solution to the Yang-Mills equations this constraint does not have to be satisfied and one may restrict oneself to the traceless part of (2.2) i.e. (2.4).
- ii) For the defining representation of $SO(N)$ groups we may consider the set $\{A\}$ of real antisymmetric $N \times N$ -matrices. It is not hard to show that a fully symmetrized product of matrices A_i satisfies

$$A_{(1}A_2, \dots, A_{2m+1)} \in \{A\} , \quad (2.12)$$

$$A_{(1}A_2, \dots, A_{2m)} \in \{S\} . \quad (2.13)$$

where S is the set of real symmetric $N \times N$ -matrices.

The relations (2.12 - 2.13) imply that for the defining representation of $SO(N)$ (2.2) has only the trivial solution if the dimension equals $d=4k+2$, because the l.h.s of (2.2) is symmetric while the r.h.s is antisymmetric. At the other hand, if $d=4k$ left- and right-hand sides of (2.2) are compatible and in contrast with the $SU(N)$ case no traces or other additional components arise.

- iii) A basis for the Lie algebra of $Sp(2N)$ in the defining representation is given by the set

$$P = \{i\sigma \otimes S , 1 \otimes A\} , \quad (2.14)$$

where σ are the Pauli matrices, S and A are real symmetric respectively antisymmetric $N \times N$ -matrices. To analyze the symmetrized product we first add the set

$$Q = \{1 \otimes S, i\sigma \otimes A\} . \quad (2.15)$$

and note that the algebra $R = \{P, Q\}$ is closed under the symmetric product.

It is easy to show that

$$P_{(1} P_{2)} \in Q, \quad P_{(1} Q_{2)} \in P \quad \text{and} \quad Q_{(1} Q_{2)} \in Q .$$

We therefore conclude that

$$P_{(1} \dots P_{2m+1)} \in P$$

and

$$P_{(1} \dots P_{2m)} \in Q$$

and the situation is the same as for $SO(N)$. We thus conclude that for $Sp(2N)$ system (2.2) has only the zero solutions for $d=4k+2$ and is compatible if $d=4k$ where no additional components arise on the r.h.s..

c) Counting the number of components

So far we considered the Lie algebra structure of the system (2.2). Let us now consider the number of spacetime components. Equations (2.2) form a system of $\binom{2n}{2} \times \dim G$ coupled first order equations for the $2n \times \dim G$ potentials A_μ^a , where we should add another $\dim G$ gauge conditions. If $d=4$ we see that the number of equations and constraints exactly matches the number of gauge potentials; because F appears linear at both sides of the equations each equation appears twice. However, the number of equations grows quadratically with n and we expect fewer solutions. As we will see in the following sections however, solutions do exist. Although the geometry underlying them is well known their duality properties were never explored.

d) The action functional

The Yang-Mills action for a self dual configuration becomes

$$S = \frac{1}{4} \int_M \text{tr } F \wedge *F = \frac{c}{4} \int_M \text{tr } F^n \propto c_n$$

and is proportional to the Chern class which characterizes the corresponding principal fiber bundle (assuming F to be in some irreducible representation of G). This is just like what we found in $d=4$ (see (1.5)). However, it is no longer true that a configuration satisfying our duality equation automatically corresponds to a minimum of the action. In fact,

$$\begin{aligned} S &= \frac{1}{4} \int_M \text{tr } F \wedge *F \\ &= \frac{1}{4} \int_M \text{tr } (F - c *F^{n-1}) \wedge * (F - c *F^{n-1}) \\ &\quad - \frac{c^2}{4} \int_M \text{tr } F^{n-1} \wedge *F^{n-1} + \frac{c}{2} \int_M \text{tr } F^n. \end{aligned} \quad (2.17)$$

If $d=4$ the second term at the r.h.s equals $-c^2 S$ and bringing it to the l.h.s of the equation we see that the self dual configurations saturates a lower bound. This is no longer true in general, for a deviation from self duality the negative contribution from the second term may dominate the positive contribution of the first. Hence the solution, though related to a topological

invariant is not necessarily stable. Equation (2.17) suggests another, higher order action:

$$S' = \frac{1}{4} \int_M \text{tr } F \wedge *F + \frac{c^2}{4} \int_M \text{tr } F^{n-1} \wedge *F^{n-1}$$

for which solutions of (2.2) do correspond to a minimum. On a classical level (2.17) may be of interest, on the quantum level the higher order terms make the theory ill-defined (see also [22]), except if this structure would be part of a finite theory.

e) Duality in Riemannian Geometry

Duality equations of the type (2.2) have occurred at various places in Riemannian geometry. Indeed, our original motivation to study (2.2) in general, stems from some of those examples.

Most well known is the example of Kähler geometry [23]. A Kähler manifold is a Hermitian manifold for which the fundamental 2-form

$$K \equiv \frac{i}{2} g_{\mu\bar{\nu}} dx^\mu \wedge d\bar{x}^\nu$$

is closed. The Kähler form K satisfies (2.2). Furthermore for compact Kähler manifolds, K is harmonic and therefore a nontrivial element of the second cohomology group. This implies the existence of a nontrivial regular solution of Maxwell's equations where the field strength is proportional to K . For the complex projective spaces $\mathbb{CP}^n$ which are Kähler manifold, the corresponding connection is most easily obtained from the bundle S^{2n+1} over $\mathbb{CP}^n$ with fiber $U(1)$. However, as we will show in the next section there is also a natural $SU(n)$ -gauge field on $\mathbb{CP}^n$ satisfying (2.2).

Somewhat similar is the case of quaternionic geometry. One may show that there exists a fundamental 4-form Ω which satisfies an equation like (2.2). This 4-form can be considered as the wedge product of an $Sp(2)$ -valued 2-form H with itself

$$\Omega = \text{tr } H \wedge H .$$

This 2-form H is the field strength of an $Sp(2)$ -valued gauge field. For the

case of the quaternionic projective spaces the corresponding connection is the canonical connection on the sphere bundle S^{4n+3} over $\mathbb{H}P^n$ with fiber $SU(2) \sim Sp(2)$. This is discussed in the next section where we also give a nontrivial $Sp(2n)$ -gauge field on $\mathbb{H}P^n$ which satisfies (2.2). Finally there are the tangent bundles of the spheres S^{2n} where the curvature 2-form provides a solution to the duality equations for the spinor representation of $SO(2n)$. We conclude therefore that on the projective spaces a number of interesting nontrivial solutions to the duality equations can be rather easily obtained.

3. Duality on S^{2n} , $\mathbb{CP}^n$ and $\mathbb{HP}^n$

In this section we show that the duality equations have nontrivial solutions. We consider canonical Yang-Mills connections on the real, complex and quaternionic projective spaces, and prove that they do satisfy the duality equations. The aim of this section is not so much as to solve the duality equation in full generality but rather to show that although the duality equations are very restrictive they certainly do admit nontrivial solutions. Furthermore, the calculations will elucidate many of the remarks made in section 2.

As is well known it is possible to explicitly construct solutions to the Yang-Mills equations on coset spaces. We only review the basic elements of this construction and refer to the literature where more details can be found [24]. A group manifold G can be considered as a fiber bundle with fiber H , where H is a Lie subgroup of G , and base space G/H . From $g \in G$ we can construct the Maurer-Cartan form which is the left-invariant 1-form $g^{-1}dg$ which can be expanded as follows

$$\begin{aligned} g^{-1}dg &= e^A T_A \\ &= e^m T_m + e^a T_a \end{aligned}$$

where T_a are the (anti-Hermitian) generators of H and T_m generators of the coset space G/H ($A=1, \dots, \dim G$).

The canonical connection on the bundle is given by [37]

$$\omega_A = e^a T_a \quad (3.1)$$

and the corresponding curvature is

$$\begin{aligned} \Omega_A &= d\omega_A + \omega_A \wedge \omega_A \\ &= -\frac{1}{2} c^a_{kl} e^k \wedge e^l T_a \end{aligned} \quad (3.2)$$

(c^A_{BC} are the structure constants on G).

It can be shown that the canonical connection satisfies the Yang-Mills equations on G/H where the gauge group is H . This can be understood by referring

to the Kaluza-Klein theory. The Yang-Mills equations of the dimensional reduced theory are the off-diagonal components of Einstein's equations on G . Since a Lie group is an Einstein space these equations are automatically satisfied.

When the decomposition of G into H and the coset space is symmetric the curvature 2-form is given by

$$\Omega_A = -\frac{1}{2} \omega_M \wedge \omega_M \quad (3.3)$$

where $\omega_M = e^m T_m$.

In order for this method of obtaining solutions to be applicable to the case of interest we must exhibit S^{2n} , $\mathbb{CP}^n$ and $\mathbb{HP}^n$ as base spaces of principal fiber bundles. We consider the following realizations

$$\begin{array}{ccccc} SO(2n) & \longrightarrow & SO(2n+1) & U(n) & \longrightarrow & SU(n+1) & Sp(2n) \times Sp(2) & \longrightarrow & Sp(2n+1) \\ & & \downarrow & & & \downarrow & & & \downarrow \\ & & S^{2n} & & & \mathbb{CP}^n & & & \mathbb{HP}^n \end{array}$$

In subsection A we will consider the first series of bundles and show that the canonical connection, which is nothing but the Levi-Civita connection on S^{2n} , satisfies the duality equation. We will discuss two special and well known solutions viz. the Dirac monopole ($S^2 \simeq \mathbb{CP}^1$) and the BPST instanton ($S^4 \simeq \mathbb{HP}^1$) which we will encounter in a more general form in subsections B and C when we discuss the complex and quaternionic projective spaces respectively.

A. Real projective spaces (and spheres) [25]

The relevant bundle is

$$\begin{array}{c} SO(k) \longrightarrow SO(k+1) \\ \downarrow \\ S^k \end{array}$$

which is the orthonormal frame bundle of S^k [23b][F3]. The canonical connection on this bundle is the Levi-Civita connection. To see this we write

$$g^{-1}dg = e^m D_{m,k+1} + \frac{1}{2} \omega^{ml} D_{ml} \quad (m, l = 1, 2, \dots, k) \quad (3.4)$$

where D_{MK} ($M, K = 1, \dots, k+1$) are the anti-symmetric generators of $SO(k+1)$ in the adjoint representation. They satisfy

$$[D_{MK}, D_{LN}] = \delta_{KL} D_{MN} - \delta_{MN} D_{LK} - \delta_{KN} D_{ML} + \delta_{LM} D_{NK}, \quad (3.5)$$

$\{D_{ml}\}$ are the generators of $SO(k)$ and $\{D_{m,k+1}\}$ generate the coset S^k . The Maurer-Cartan equation for $g^{-1}dg$ yields

$$de^m + \omega^m_i \wedge e^i = 0, \quad (3.6)$$

$$d\omega^m_\lambda + \omega^m_i \wedge \omega^i_\lambda = e^m \wedge e^1.$$

The first of these equations is nothing but the zero torsion condition for the spin connection 1-form ω^m_λ (the metricity condition is fulfilled because D_{ml} are anti-symmetric matrices). The curvature 2-form can be read off from the second equation

$$R_{ml} = e^m \wedge e^1. \quad (3.7)$$

We now interpret the Levi-Civita connection as a Yang-Mills connection so

$$\omega_A = \frac{1}{2} \omega^{ml} D_{ml} \quad (3.8)$$

and from this the $SO(k)$ -valued field strength is found to be

$$F = \frac{1}{2} R^{ml} D_{ml} = \frac{1}{2} e^m \wedge e^1 D_{ml}. \quad (3.9)$$

Let us now turn to the duality properties of these solutions on even dimensional spheres S^{2n} . We use expression (3.9) for F

$$F = \frac{i}{2} e^m \wedge e^1 D_{ml} \quad (3.10)$$

where we have inserted i to make F Hermitian (cf. section 2). Furthermore we take the generators in the spinor representation of $SO(2n+1)$ (see Appendix A for an explicit representation):

$$i D_{mk} \longrightarrow \frac{1}{2} \Sigma_{mk} = \frac{1}{4i} [\gamma_m, \gamma_k]$$

so

$$\begin{aligned} F &= -i \frac{1}{8} [\gamma_m, \gamma_k] e^m \wedge e^k \\ &= -i Q \wedge Q \end{aligned} \quad (3.11)$$

where

$$Q = \frac{1}{2} \gamma_m e^m. \quad (3.12)$$

From (A.3) we conclude that Q^2 and hence F is block-diagonal so we may split F in two parts F^+ and F^-

$$F = F^+ + F^- = \begin{pmatrix} F^+ & 0 \\ 0 & F^- \end{pmatrix}. \quad (3.13)$$

Q has the property that Q^{2n} is proportional to the volume element on S^{2n} : (use (A.4))

$$\begin{aligned} Q^{2n} &= \frac{1}{2^{2n}} (\gamma_{m_1} e^{m_1}) \wedge \dots \wedge (\gamma_{m_{2n}} e^{m_{2n}}) \\ &= \frac{1}{2^{2n}} (2n)! \gamma_1 e^1 \wedge \gamma_2 e^2 \dots \wedge \gamma_{2n} e^{2n} \\ &= \frac{(2n)!}{2^n} i^{-n} (\sigma_3 \otimes 1) V(S^n). \end{aligned}$$

The duality equations are now easily obtained. For example,

$$*\gamma_1 e^1 \wedge \gamma_2 e^2 = \gamma_1 \gamma_2 e^3 \wedge e^4 \dots \wedge e^{2n}$$

but we know that, using (A.4) again

$$\begin{aligned}
(\gamma_1 \gamma_2)^\pm &= \pm i^{-n} (\gamma_{2n} \cdots \gamma_4 \gamma_3)^\pm \\
&= \pm i^{-n} (-1)^{n+1} (\gamma_3 \cdots \gamma_{2n})^\pm
\end{aligned}$$

and we therefore find (for the + -part)

$$*\gamma_1 e^1 \wedge \gamma_2 e^2 = -i^n (\gamma_3 e^3 \wedge \cdots \gamma_{2n} e^{2n}) .$$

This is used to obtain

$$*Q^2 = -i^n \cdot \frac{2^{n-1}}{(2n-2)!} Q^{2n-2} \quad (3.14)$$

and finally

$$*F_\pm = \pm \frac{2^{n-1}}{(2n-2)!} (-1)^n F_\pm^{n-1} . \quad (3.15)$$

We should note here that F itself does not satisfy the duality equations. We also note that it is easy to show that

$$*Q^4 \propto Q^{2n-4} \quad \text{etc.} \quad (3.16)$$

So, in this case all the other duality equations (eq. (2.6)) are also satisfied (the solution of [26] on S^8 satisfies $*F^2 = F^2$, see also [15]).

To end this section we discuss two well known examples of special solutions on spheres viz. the Dirac monopole ($n=1$, $S^2 \simeq \mathbb{CP}^1$) and the BPST instanton ($n=2$, $S^4 \simeq \mathbb{HP}^1$). These are related to the famous Hopf fibrations [27], $(S^3 \rightarrow S^2, S^1)$, $(S^7 \rightarrow S^4, S^3)$. We consider these because they will turn up in a more general context when we discuss the complex and quaternionic projective spaces. The last Hopf map, $(S^{15} \rightarrow S^8, S^7)$ has been discussed in detail in [26].

i) Dirac monopole [28]

The set of " γ "-matrices consists of σ_1 and σ_2 and we obtain

$$\begin{aligned}
F &= \frac{1}{4} e^m \wedge e^k \Sigma_{mk} \quad m, k=1, 2. \\
&= \frac{1}{2} \sigma_3 e^1 \wedge e^2 \\
&= \frac{1}{2} e^1 \wedge e^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}$$

and we see

$$\begin{aligned}
F^\pm &= \pm \frac{1}{2} e^1 \wedge e^2 \\
&= \pm \frac{1}{4} e \wedge \bar{e}
\end{aligned} \tag{3.17}$$

where we introduced complex coordinates

$$e = e^1 + ie^2, \quad \bar{e} = e^1 - ie^2.$$

Equation (3.17) tells us that $F^\pm$ is nothing but the Kähler 2-form on S^2 .

ii) BPST instanton [29]

The set of γ matrices in this case is the usual set of Dirac γ -matrices. We find

$$\begin{aligned}
\Sigma_{ij} &= \epsilon_{ijk} 1 \otimes \sigma_k \\
\Sigma_{k4} &= \sigma_3 \otimes \sigma_k,
\end{aligned}$$

consequently

$$\Sigma_{ij}^\pm = \pm \epsilon_{ijk} \Sigma_{k4}^\pm. \tag{3.19}$$

F takes the form

$$F = \frac{1}{4} \Sigma_{MK} e^M \wedge e^K \quad M = (m, 4).$$

For $*F$ we find

$$* F = \frac{1}{4} \left(\sum_{mk} \epsilon_{mki}^{mk} e^i \wedge e^4 + \sum_{m4} \epsilon_{ki}^m e^k \wedge e^i \right)$$

$$* F^\pm = \frac{1}{4} \left(\pm 2 \sum_{i4}^\pm e^i \wedge e^4 \pm \sum_{ki}^\pm e^k \wedge e^i \right)$$

$$= \pm F^\pm . \quad (3.20)$$

Using the η -symbol (Appendix B) we can write

$$\begin{aligned} F^+ &= \frac{1}{4} \left(2 e^m \wedge e^4 + \epsilon_{ki}^m e^k \wedge e^i \right) \sigma_m \\ &= \frac{1}{4} \eta_{MK}^a e^M \wedge e^K \sigma_a \quad (a = 1, 2, 3) . \end{aligned} \quad (3.21)$$

Similarly

$$F^- = \frac{1}{4} \eta_{MK}^{-a} e^M \wedge e^K \sigma_a . \quad (3.22)$$

One should note here that the splitting of the spinor representation results in a solution which has a smaller gauge group ($SU(2)$) than the one we started with ($SO(4)$). This is due to the fact that locally $SO(4)$ is $SU(2) \otimes SU(2)$. This situation is special since it only occurs in four dimensions.

B. Complex projective spaces

The bundles

$$\begin{array}{ccc} SU(n) \times U(1) & \longrightarrow & SU(n+1) \\ & & \downarrow \\ & & \mathbb{CP}^n \end{array}$$

will provide us with $SU(n) \times U(1)$ -gauge fields on $\mathbb{CP}^n$.

The Cartan subalgebra of $SU(n+1)$ we represent by traceless, $(n+1) \times (n+1)$ diagonal matrices and the other generators by the $(n+1) \times (n+1)$ - matrices E_{JK} ($J \neq K$) which have a single entry equal to one in the J th row and K th column.

The $SU(n)$ generators E_{jk} satisfy the useful relations

$$E_{jk} E_{lm} = \delta_{kl} E_{jm} . \quad (3.23)$$

The coset generators are

$$E_k = E_{k,n+1} \quad \text{and} \quad E_{\bar{k}} = E_{n+1,k} . \quad (3.24)$$

We furthermore define

$$\hat{E} \equiv E_{n+1,n+1} . \quad (3.25)$$

The metric on CP^n is

$$\begin{aligned} ds^2_{CP^n} &= \text{tr}(e^m E_m + \bar{e}^{\bar{m}} E_{\bar{m}} \otimes e^k E_k + \bar{e}^{\bar{k}} E_{\bar{k}}) \\ &= 2e^m \otimes \bar{e}^{\bar{m}} . \end{aligned} \quad (3.26)$$

To calculate F we use (3.3)

$$\begin{aligned} F &= - (e^k E_k + \bar{e}^{\bar{k}} E_{\bar{k}}) \wedge (e^m E_m + \bar{e}^{\bar{m}} E_{\bar{m}}) \\ &= - e^k \wedge \bar{e}^{\bar{m}} (E_{km} - \delta_{km} \hat{E}) . \end{aligned} \quad (3.27)$$

This F is anti-Hermitian so let us take instead

$$F = e^k \wedge \bar{e}^{\bar{m}} iE_{km} - K \hat{E} \quad (3.28)$$

where

$$K = i e^k \wedge \bar{e}^{\bar{k}} \quad (k=1, \dots, n) \quad (3.29)$$

is the Kähler 2-form on CP^n .

The proper $U(1)$ -generator U is given by

$$U = \alpha \text{diag} (1, 1, \dots, 1, -n)$$

where $\alpha = \frac{1}{\sqrt{n(n+1)}}$.

In terms of $\hat{E}$ it is written as

$$U = -\alpha n \hat{E} + \alpha E_{mm} . \quad (3.30)$$

Substituting this in (3.27) yields

$$F = (e^k \wedge \bar{e}^m iE_{km} - \frac{1}{n} K E_{mm}) + \frac{1}{\alpha n} K U . \quad (3.31)$$

The last part is the U(1)-part of the gauge field, and the corresponding field strength is the Kähler 2-form K. This form is harmonic (with respect to the exterior derivative) and generates the cohomology of $\mathbb{CP}^n$. In physical terms it is a solution to Maxwell's equations. The corresponding circle bundle is

$$\begin{array}{ccc} S^1 & \longrightarrow & S^{2n+1} \\ & & \downarrow \\ & & \mathbb{CP}^n \end{array}$$

The Kähler form satisfies the duality equations

$$* K = \frac{1}{(n-1)!} K^{n-1} \quad (3.32)$$

and also

$$* K^2 = \frac{1}{(n-2)!} K^{n-2} \quad \text{etc.} . \quad (3.33)$$

The Kähler 2-form clearly is a generalization of the monopole field strength (see (3.17)). The other part of the field strength corresponds to an SU(n)-gauge field. The corresponding bundle sits on top of the circle bundle

$$\begin{array}{ccc} SU(n) & \longrightarrow & SU(n+1) \\ & & \downarrow \\ U(1) & \longrightarrow & S^{2n+1} \\ & & \downarrow \\ & & \mathbb{CP}^n \end{array} .$$

To verify the duality equations we use (3.27) to compute F^2 .
One finds rather easily by using (3.23) that

$$F \wedge F = - F \wedge K . \quad (3.34)$$

We find from this that

$$\begin{aligned} F^{n-1} &= (-1)^n F \wedge K^{n-2} \\ &= (-1)^n (e^k \wedge \bar{e}^m i E_{km} - K \hat{E}) \wedge K^{n-2} \\ &= (-1)^n (e^1 \wedge \bar{e}^1 \wedge K^{n-2} i E_{11} + e^1 \wedge \bar{e}^2 \wedge K^{n-2} i E_{12} + \dots \\ &\quad - \hat{E} K^{n-1}) . \end{aligned} \quad (3.35)$$

We now use

$$\begin{aligned} * e^1 \wedge \bar{e}^1 \wedge K^{n-2} &= - i (n-2)! (K - e^1 \wedge \bar{e}^1) \\ * e^1 \wedge \bar{e}^2 \wedge K^{n-2} &= - (n-2)! e^1 \wedge \bar{e}^2 \end{aligned} \quad (3.36)$$

(and similar expressions for the other possible combinations), which together with (3.32) yield

$$\begin{aligned} * F^{n-1} &= (-1)^n [i E_{11} (-i)(n-2)! (K - i e^1 \wedge \bar{e}^1) + \dots \\ &\quad i E_{12} \cdot - (n-2)! e^1 \wedge \bar{e}^2 + \dots \\ &\quad - \hat{E} \cdot (n-1)! K] \\ &= (-1)^n (n-2)! [E_{11} K - i E_{km} e^k \wedge \bar{e}^m \\ &\quad - \hat{E} (n-1) K] . \end{aligned}$$

Again introducing the proper $U(1)$ -generator we find

$$\begin{aligned}
* F^{n-1} &= (-1)^n (n-2)! \left[E_{11} K - i E_{km} e^k \wedge \bar{e}^m \right. \\
&\quad \left. - (n-1) \cdot \frac{1}{n} \cdot \left(E_{11} - \frac{1}{\alpha} U \right) K \right] \\
&= (-1)^n (n-2)! \left[- \left(i E_{km} e^k \wedge \bar{e}^m - \frac{1}{n} E_{11} K \right) + (n-1) \frac{1}{\alpha n} U K \right] . \quad (3.37)
\end{aligned}$$

We can now read off the duality properties:

$$\begin{aligned}
(* F^{n-1})_{SU(n)} &= (-1)^{n+1} (n-2)! F_{SU(n)} , \\
(* F^{n-1})_{U(1)} &= (-1)^{n+1} (n-2)! \cdot -(n-1) F_{U(1)} . \quad (3.38)
\end{aligned}$$

We see that we have obtained a $U(n)$ -solution to the duality equations, where it is understood that the $U(1)$ - and $SU(n)$ -components of the duality equations can and do have different constants c : as mentioned before the $U(1)$ -component of F satisfies the duality equation all by itself; however, the $SU(n)$ -component of F does not, in fact it equals the $SU(n)$ -component of $*F^{n-1}$ where in this expression F is taken to be the full $U(n)$ -field strength. The $U(1)$ -component is essential to take care of the nonvanishing trace contribution from the product of $SU(n)$ generators. Nevertheless, the $SU(n)$ -part of F is a solution to (2.5) and therefore satisfies the $SU(n)$ Yang-Mills equations. Observe that from (3.35) one can see that F satisfies a linear duality equation of type (2.1) as well.

C. Quaternionic projective spaces

The discussion of this section parallels in many respects that of the previous section. The bundle which is relevant here is

$$\begin{array}{ccc}
 \text{Sp}(2n) & \longrightarrow & \text{Sp}(2n+2) \\
 & & \downarrow \\
 \text{Sp}(2) & \longrightarrow & S^{4n+3} \\
 & & \downarrow \\
 & & \mathbb{HP}^n
 \end{array}$$

$\text{Sp}(2n+2)$ is generated by

$$i\sigma_i \otimes S \quad (i=1,2,3) \quad , \quad 1 \otimes A$$

where S and A are symmetric and antisymmetric $(n+1) \times (n+1)$ matrices respectively.

The coset is generated by $T_k^{(\mu)} \quad (\mu=1,2,3,4)$

$$\begin{aligned}
 T_k^{(1)} &= i\sigma_i \otimes S_k \\
 T_k^{(4)} &= 1 \otimes A_k
 \end{aligned} \tag{3.39}$$

$$\text{with} \quad S_k = E_{k,n+1} + E_{n+1,k} ,$$

$$A_k = E_{k,n+1} - E_{n+1,k} . \tag{3.40}$$

The metric on $\mathbb{HP}^n$ is

$$\begin{aligned}
 ds_{\mathbb{HP}^n}^2 &= - \text{tr} (T_k^{(\mu)} e_\mu^k \otimes T_l^{(\nu)} e_\nu^l) \\
 &= 4 e_\mu^k \otimes e_\mu^k .
 \end{aligned} \tag{3.41}$$

The field strength F is obtained from (3.2)

$$\begin{aligned}
F &= - T_k^{(\mu)} e_\mu^k \wedge T_m^{(\nu)} e_\nu^m \\
&= + \frac{1}{2} [1 \otimes (E_{km} - E_{mk})(e_i^k \wedge e_i^m + e_4^k \wedge e_4^m) \\
&\quad + i\sigma_m \otimes (E_{kl} + E_{lk})(\varepsilon_{ijm} e_i^k \wedge e_j^l + 2e_m^k \wedge e_4^l) \\
&\quad + i\sigma_m \otimes 2 \hat{E} (\varepsilon_{ijm} e_i^k \wedge e_j^k - 2e_m^k \wedge e_4^k)]
\end{aligned}$$

and introducing the η -symbol [Appendix B]

$$\begin{aligned}
F &= 1 \otimes i E_{kl} \delta_{\mu\nu} e_\mu^k \wedge e_\nu^l \\
&\quad + i\sigma_a \otimes i E_{kl} \eta_{\mu\nu}^a e_\mu^k \wedge e_\nu^l \\
&\quad + i\sigma_a \otimes i \hat{E} \bar{\eta}_{\mu\nu}^a e_\mu^k \wedge e_\nu^l
\end{aligned} \tag{3.42}$$

where we have inserted an extra i to make F Hermitian. Since $i\sigma_a \otimes i \hat{E}$ are the generators of the $SU(2)$ -subgroup of $Sp(2n+n)$ we may identify

$$i\sigma_a \otimes i \hat{E} \bar{\eta}_{\mu\nu}^a e_\mu^k \wedge e_\nu^k \equiv H \otimes \hat{E} \tag{3.43}$$

as the $SU(2)$ -gauge field, while

$$f \equiv 1 \otimes i E_{kl} \delta_{\mu\nu} e_\mu^k \wedge e_\nu^l + i\sigma_a \otimes i E_{kl} \eta_{\mu\nu}^a e_\mu^k \wedge e_\nu^l \tag{3.44}$$

is the $Sp(2n)$ -gauge field.

This result has many similarities with what we found for the complex projective spaces. We name only a few of the correspondences.

Instead of the pairs of Vielbeine (e^k and $\bar{e}^k$) we now have sets consisting of four Vielbeine (e_μ^k , $\mu = 1, 2, 3, 4$).

The $U(1)$ -gauge field represented by the Kähler form K is a direct generalization of the Dirac monopole field. Here, the $SU(2)$ -gauge field is in precisely the same sense a generalization of the (anti-self dual) BPST instanton solution.

To make the analogy even more striking we show that H is related to the metric on $\mathbb{H}P^n$ in exactly the same way as the Kähler form is to the metric on $\mathbb{C}P^n$. To see this introduce

$$\begin{aligned}\bar{\sigma}^\mu &= (\underline{1}, i\sigma_a) , \\ \underline{\sigma}^\mu &= (\underline{1}, -i\sigma_a) .\end{aligned}\tag{3.45}$$

Using (B.3) we find that

$$\begin{aligned}H &= - \sigma_a \bar{\eta}_{\mu\nu}^a e_\mu^k \wedge e_\nu^k \\ H &= - \underline{\sigma}^\mu e_\mu^k \wedge \bar{\sigma}^\nu e_\nu^k = i \underline{e}^k \wedge \bar{e}^k ,\end{aligned}\tag{3.46}$$

where we have defined

$$\begin{aligned}\underline{e}^k &= \underline{\sigma}^\mu e_\mu^k , \\ \bar{e}^k &= \bar{\sigma}^\mu e_\mu^k .\end{aligned}$$

In terms of $\underline{\sigma}^\mu$ and $\bar{\sigma}^\mu$ the metric on $\mathbb{HP}^n$ reads

$$\begin{aligned}ds_{\mathbb{HP}^n}^2 &= 4 e_\mu^k \otimes e_\mu^k \\ &= 2 \underline{e}^k \otimes \bar{e}^k ,\end{aligned}\tag{3.47}$$

which shows the above mentioned correspondence between the metric on $\mathbb{HP}^n$ and the 2-form H .

H may be split as follows

$$\begin{aligned}H &= i \underline{e}^k \wedge \bar{e}^k \\ &= H_1 + H_2 + \dots + H_n\end{aligned}$$

$$\text{where } H_1 = i \underline{e}^1 \wedge \bar{e}^1 = - 2 \vec{\sigma} \cdot \vec{h}_1 \tag{3.48}$$

with

$$\begin{aligned}
h_1^1 &= e_2^1 \wedge e_3^1 - e_1^1 \wedge e_4^1 \\
h_1^2 &= e_3^1 \wedge e_1^1 - e_2^1 \wedge e_4^1 \\
h_1^3 &= e_1^1 \wedge e_2^1 - e_3^1 \wedge e_4^1 .
\end{aligned}$$

It is easily verified that

$$\begin{aligned}
H_1^2 &= 4 \parallel \vec{h}_1 \parallel^2 \\
&= 4! e_1^1 \wedge e_2^1 \wedge e_3^1 \wedge e_4^1 \\
&= 4! V_1
\end{aligned} \tag{3.49}$$

where V_1 is the volume form on the space spanned by e_μ^1 ($\mu=1,\dots,4$).
The volume form on $\mathbb{HP}^n$ is given by

$$\begin{aligned}
V &= V_1 \dots V_n \\
&= \left(\frac{1}{4!}\right)^n H_1^2 \dots H_n^2 .
\end{aligned}$$

For H^2 we find

$$\begin{aligned}
H^2 &= (H_1 + H_2 + \dots H_n)^2 \\
&= H_1^2 + H_2^2 + H_1 H_2 + H_2 H_1 + \dots ,
\end{aligned}$$

or using the notation of (3.48)

$$H^2 = 4(\parallel \vec{h}_1 \parallel^2 + 2(\vec{h}_1 \cdot \vec{h}_2) + \dots) .$$

Using the fact that

$$(\vec{h}_1 \cdot \vec{h}_2)(\vec{h}_2 \cdot \vec{h}_3) = \frac{1}{3} \parallel \vec{h}_2 \parallel^2 (\vec{h}_1 \cdot \vec{h}_3)$$

we obtain

$$\begin{aligned} (H^2)^n &\propto H_1^2 \dots H_n^2 \\ &\propto V. \end{aligned} \quad (3.50)$$

We now prove that H satisfies the duality equations:

$$\begin{aligned} H_1 &= -2 \sigma_1 (e_2^1 \wedge e_3^1 - e_1^1 \wedge e_4^1) + \dots \\ * H_1 &= -2 \sigma_1 (e_1^1 \wedge e_4^1 - e_2^1 \wedge e_3^1) \wedge V_2 \wedge V_3 \dots \wedge V_n + \dots \\ &= -H_1 \wedge V_2 \wedge V_3 \wedge \dots \wedge V_n \end{aligned}$$

or

$$\begin{aligned} * H &= -H \wedge V_2 \wedge \dots \wedge V_n \\ &\propto -H^{2n-1}. \end{aligned}$$

It is easy to verify that in fact H satisfies all kinds of duality equations:

$$* H^k \propto H^{2n-k}. \quad (3.51)$$

We note that H is an $SU(2)$ -valued 2-form which is harmonic with respect to the $SU(2)$ -covariant derivative (this follows from the $SU(2)$ -component of the Yang-Mills equation). Since H^2 does not carry $SU(2)$ indices we find that

$$D_A H^2 = dH^2,$$

on the other hand we have

$$D_A H^2 = D_A H \wedge H + H \wedge D_A H = 0$$

so H^2 is closed. Furthermore,

$$\begin{aligned}
D_A * H^2 &\propto D_A H^{2n-2} \propto (D_A H) \wedge H^{2n-3} = 0 \\
&= d * H^2
\end{aligned}$$

so H^2 is also co-closed. The same argument holds for $H^4, H^6, \dots$ etc. We conclude that H^2 is the generator of the cohomology group of $\mathbb{H}P^n$

$$(H^{4i}(\mathbb{H}P^n; \mathbb{R}) = \mathbb{Z}, i=1, \dots, n) .$$

Let us finally prove that the $Sp(2n)$ -part of the field strength satisfies the duality equation.

$$\begin{aligned}
f &= 1 \otimes i E_{kl} \delta_{\mu\nu} e_\mu^k \wedge e_\nu^1 + i \sigma_a \otimes i E_{kl} \eta_{\mu\nu}^a e_\mu^k \wedge e_\nu^1 \\
&= \underline{e}^k \wedge \underline{e}^1 \otimes i E_{kl} .
\end{aligned} \tag{3.52}$$

From this we obtain for f^2

$$\begin{aligned}
f^2 &= - \underline{e}^k \wedge \underline{e}^1 \wedge \underline{e}^m \wedge \underline{e}^n \otimes E_{kn} \delta_{lm} \\
&= \underline{e}^k \wedge H \wedge \underline{e}^n \otimes i E_{kn} .
\end{aligned}$$

Now note that H cannot be interchanged with $\underline{e}^n$ since it carries $SU(2)$ indices. However, this is circumvented by computing f^3 . We arrive at the key property (cf. (3.34))

$$\begin{aligned}
f^3 &= \underline{e}^k \wedge H^2 \wedge \underline{e}^n \otimes i E_{kn} \\
&= f \wedge H^2 .
\end{aligned}$$

By iterating this there results

$$\begin{aligned}
f^{2n-1} &= f \wedge H^{2n-2} \\
&= \underline{e}^1 \wedge \underline{e}^1 \wedge H^{2n-2} \otimes i E_{11} \\
&\quad + \underline{e}^1 \wedge \underline{e}^2 \wedge H^{2n-2} \otimes i E_{12} + \dots .
\end{aligned} \tag{3.53}$$

Now H^{2n-2} is proportional to (with positive proportionality constant)

$$\begin{aligned} & \|\vec{h}_2\|^2 \|\vec{h}_3\|^2 \dots \|\vec{h}_n\|^2 + \dots \\ & + 2(\vec{h}_1 \cdot \vec{h}_2) \|\vec{h}_3\|^2 \dots \|\vec{h}_n\|^2 + \dots, \end{aligned}$$

substituting this in (3.53) and using that

$$\bar{e}^1 \wedge \underline{e}^1 (\vec{h}_1 \cdot \vec{h}_2) = 0,$$

which is easily verified to be true by a direct computation, we obtain

$$\begin{aligned} * f^{2n-1} & \propto * \bar{e}^1 \wedge \underline{e}^1 \wedge \|\vec{h}_2\|^2 \|\vec{h}_3\|^2 \|\vec{h}_n\|^2 \otimes iE_{11} \\ & + * 2 \bar{e}^1 \wedge \underline{e}^2 \wedge (\vec{h}_1 \cdot \vec{h}_2) \|\vec{h}_3\|^2 \dots \|\vec{h}_n\|^2 \otimes iE_{12} + \dots \end{aligned}$$

A tedious calculation shows that

$$\begin{aligned} & * \bar{e}^1 \wedge \underline{e}^1 \wedge \|\vec{h}_2\|^2 (\|\vec{h}_3\|^2 \dots \|\vec{h}_n\|^2) \\ & = -6 \bar{e}^1 \wedge \underline{e}^1 \cdot \tilde{K} \end{aligned} \tag{3.54a}$$

$$\begin{aligned} & * \bar{e}^1 \wedge \underline{e}^2 (\vec{h}_1 \cdot \vec{h}_2) (\|\vec{h}_3\|^2 \dots \|\vec{h}_n\|^2) \\ & = -3 \bar{e}^1 \wedge \underline{e}^2 \cdot \tilde{K}, \end{aligned} \tag{3.54b}$$

where $\tilde{K}$ is a constant.

Putting all this together finally yields

$$\begin{aligned} * f^{2n-1} & \propto - [\bar{e}^1 \wedge \underline{e}^1 \otimes iE_{11} + \bar{e}^1 \wedge \underline{e}^2 \otimes iE_{12} + \dots] \\ & = -f. \end{aligned}$$

Note that in contrast with the $\mathbb{CP}^n$ -case we find here that the $Sp(2n)$ - and $Sp(2)$ -gauge fields separately satisfy the duality equations. Furthermore because of property (3.53) we see that again the Yang-Mills curvature satisfy a linear duality relation as well (cf. (2.6) with $k=1$).

4. Discussion and Outlook

We pursued a nonlinear form of duality this in contrast with certain linear versions which were suggested before [13-15]. We analyzed a number of general properties of this duality condition. In particular we studied the representation dependence of the duality equations in $d=2n$ and showed that there are only nonzero solutions if the representation is "anomalous" in $d=2n-2$. We also explicitly demonstrated that certain canonical Yang-Mills connections on the (real, complex and quaternionic) projective spaces correspond to particular "one-instanton" solutions to the duality relations. The relevance of the suggested duality equations for four dimensional Yang-Mills-Higgs systems was indicated but explicit results will be published elsewhere [38]. There are a number of as yet unanswered questions. In the first place there is the problem of parameter counting and stability. In four dimensions (in particular on S^4) the number of parameters for a general gauge group is known [30-31], and is directly related to the index of the Dirac operator of the background field [32]. Also self dual solutions are weakly stable and moreover (at least on S^4) all finite-action stable solutions are self dual [7]. We have not considered the fluctuation problem associated with the duality equation but convinced ourselves that there is no direct relation with the Dirac operator for $d>4$. There are however partial results on solutions to the full Yang-Mills equations given in [7]. It is shown there that on spheres the canonical connection we discussed in section 3A is in fact unstable ($d>6$), however the unstable mode is not present if we consider the connection as a solution on S^n/Γ where Γ is some discrete subgroup of $SO(n+1)$ (e.g. on $RP^n = S^n/\mathbb{Z}_2$). It is also shown that these one-instanton solutions do not have any zero modes, i.e. they correspond to isolated critical points of the action. It would be interesting to investigate this problem for the other spaces we have considered as well as for the multi-instanton sectors.

Further results on these solutions on S^n have been obtained by Schellekens [3,33], who determined the stability of the $SO(n)$ -connection when embedded in larger groups.

As such the actual construction of multi-instanton solutions is still an open problem for $d>4$ and one may hope that the methods of Atiyah et al. [34] can be generalized [35].

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Appendix A. Spinor representation of $SO(2n)$

We construct by induction γ -matrices in even dimensions.

Pauli matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

In $d=2$ σ_1 and σ_2 are the elements of the Clifford algebra:

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

and we know that

$$\sigma_3 = -i\sigma_1\sigma_2$$

We construct from this a Clifford algebra for $d=4$:

$$\gamma_i = \sigma_1 \otimes \sigma_i \quad i=1,2,3.$$

$$\gamma_4 = \sigma_2 \otimes 1$$

and define

$$\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4 .$$

Denote the algebra in $d=2n$ by γ_i and define in $d=2n+2$

$$\Gamma_k = \sigma_1 \otimes \gamma_k , \quad k=1,2,\dots,2n+1$$

$$\Gamma_{2n+2} = \sigma_2 \otimes 1 .$$

Furthermore define

$$\Gamma_{2n+3} = i^{-(n+1)} \Gamma_1 \dots \Gamma_{2n+2} .$$

It is easily verified that $\{\Gamma_k\}$ forms a Clifford algebra.

The dimension of these γ -matrices in $d=2n$ is $2^n \times 2^n$.

The generators Σ_{ij} ($i, j=1, \dots, 2n$) of $SO(2n)$ defined as

$$\Sigma_{ij} = \frac{1}{2i} [\gamma_i, \gamma_j] \quad (A.1)$$

furnish a reducible spinor representation of $SO(2n)$.

Indeed, the Σ -matrices are all block-diagonal:

$$\begin{aligned} \Sigma_{ij} &= \frac{1}{2i} [\sigma_1 \otimes \gamma_i, \sigma_1 \otimes \gamma_j] \\ &= 1 \otimes \frac{1}{2i} [\gamma_i, \gamma_j] \end{aligned} \quad (A.3a)$$

$$\begin{aligned} \Sigma_{i, 2n+2} &= \frac{1}{2i} [\sigma_1 \otimes \gamma_i, \sigma_2 \otimes 1] \\ &= \sigma_3 \otimes \gamma_i. \end{aligned} \quad (A.3b)$$

If one adds the generators $\Sigma_{i, n+1} = \gamma_i$ we obtain the irreducible spinor representation of $SO(2n+1)$ and the above construction is nothing but the branching rule of the 2^n -spinor representation of $SO(2n+1)$ into the $2^{n-1} + 2^{n-1}$ spinor representations of $SO(2n)$.

Since the γ -matrices are Hermitian Σ_{ij} is also Hermitian. These generators satisfy the commutation relations:

$$[\Sigma_{ij}, \Sigma_{kl}] = -2i(\delta_{jk}\Sigma_{il} - \delta_{jl}\Sigma_{ik} + \delta_{ik}\Sigma_{lj} - \delta_{il}\Sigma_{kj}). \quad (A.2)$$

We now prove that " γ_5 " is always block-diagonal:

$$\text{in } d=2n \quad \gamma_{2n+1} = \sigma_3 \otimes 1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

This is certainly true for $n=1, 2$. We prove by induction that it is always true:

$$\begin{aligned} \Gamma_{2n+3} &= i^{-(n+1)} \Gamma_1 \dots \Gamma_{2n+2} \\ &= i^{-(n+1)} (\sigma_1 \otimes \gamma_1) \dots (\sigma_1 \otimes \gamma_{2n+1}) (\sigma_2 \otimes 1) \\ &= i^{-(n+1)} (1 \otimes i^n \gamma_{2n+1}) (\sigma_1 \otimes \gamma_{2n+1}) (\sigma_2 \otimes 1) \\ &= \sigma_3 \otimes 1. \end{aligned} \quad (A.4)$$

Appendix B. η -symbol

In this appendix we collect some relevant information on the 't Hooft's η -symbol [36].

$$\eta_{mk}^a = \epsilon_{amk} \quad \overline{\eta}_{mk}^a = \epsilon_{amk}$$

$$\eta_{m4}^a = \delta_{am} \quad \overline{\eta}_{m4}^a = -\delta_{am}.$$

η and $\overline{\eta}$ are antisymmetric in the last two indices.
 η is self dual, i.e.

$$\eta_{mk}^a = \frac{1}{2} \epsilon_{mkl n} \eta_{ln}^a$$

while $\overline{\eta}$ is anti-self dual;

$$\overline{\eta}_{mk}^a = -\frac{1}{2} \epsilon_{mkl n} \overline{\eta}_{ln}^a.$$

It is easy to check that $\overline{\sigma}^\mu = (1, i\sigma_a)$ and $\underline{\sigma}^\mu = (1, -i\sigma^a)$ satisfy the following equations

$$\overline{\sigma}^\mu \underline{\sigma}^\nu - \overline{\sigma}^\nu \underline{\sigma}^\mu = 2i\eta_{\mu\nu}^a \sigma_a \quad (B.1)$$

$$\underline{\sigma}^\mu \overline{\sigma}^\nu - \underline{\sigma}^\nu \overline{\sigma}^\mu = 2i\overline{\eta}_{\mu\nu}^a \sigma_a \quad (B.2)$$

$$\overline{\sigma}^\mu \underline{\sigma}^\nu + \overline{\sigma}^\nu \underline{\sigma}^\mu = 2\delta_{\mu\nu}. \quad (B.3)$$

Footnotes

- [F1] $1 \leq \mu \leq 4$, $1 \leq i \leq 3$ and a runs from 1 to $\dim H$
- [F2] STr denotes the full symmetrized trace, see e.g. [19]
- [F3] We use Hermitian generators. Had we used anti-Hermitian generators (2.8) and (2.9) would be with a plus sign, and the minus sign would turn up in (2.10)
- [F4] Note that $\mathbb{RP}^n = S^n/\mathbb{Z}_2$

References

- [1] Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin, Math. Phys. Kl (1921) 966
 O. Klein, Z. Physik 37 (1926) 859
 B.S. De Witt, in Lectures at 1963 Les Houches School, Relativity, Groups and Topology, ed. B.S. DeWitt and C. DeWitt, Gordon and Breach, New York (1964)
 R. Kerner, Ann. Inst. H. Poincaré, 9 (1968) 143
 A. Trautman, Rep. Math. Phys. 1 (1970) 29
 Y.M. Cho and P.G.O. Freund, Phys. Rev. D12 (1975) 1711
 Y.M. Cho and P.S. Jang, Phys. Rev. D12 (1975) 3789
 E. Witten, Nucl. Phys. B186 (1981) 412
- [2] E. Witten, Fermion Quantum Numbers in Kaluza-Klein Theory, Princeton Preprint (1983)
 N.S. Manton, Nucl. Phys. B193 (1981) 502
 G. Chapline and N.S. Manton, Nucl. Phys. B184 (1981) 391
 G. Chapline and R. Slansky, Nucl. Phys. B209 (1982) 461
- [3] A.N. Schellekens, Nucl. Phys. B248 (1984) 706
- [4] E. Witten, Phys. Lett. 149B (1984) 351
 P.H. Frampton, H. van Dam and K. Yamamoto, Phys. Rev. Lett. 54 (1985) 1114
 P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B258 (1985) 46
 K. Pilch and A.N. Schellekens, Fermion Spectra from Superstrings, Stony Brook preprint, ITP-SB-85-14
- [5] F.A. Bais and P. Batenburg, Nucl. Phys. B253 (1985) 162
- [6] N.S. Manton, Nucl. Phys. B158 (1979) 141
 P. Forgács and N.S. Manton, Comm. Math. Phys. 72 (1980) 15
- [7] J.P. Bourguignon, H.B. Lawson and J. Simons, Proc. Nat. Acad. Sci. 76 (1979) 1550
 J.P. Bourguignon and H.B. Lawson, Comm. Math. Phys. 79 (1981) 189
- [8] T. Eguchi, P.B. Gilkey and A.J. Hanson, Phys. Rep. 66 (1980) 213

- [9] F.A. Bais, K.J. Barnes, P. Forgács and G. Zoupanos, Nucl. Phys. B, to be published
- [10] E.B. Bogomolny, Sov. J. Nucl. Phys. 24 (1976) 449
- [11] H.B. Nielsen and P. Olesen, Nucl. Physics B61 (1973) 45
- [12] M.A. Lohe, Phys. Lett 70B (1977) 325
- [13] E. Corrigan, C. Devchand, D.B. Fairlie and J. Nuyts, Nucl. Phys. B214 (1983) 452
- [14] R.S. Ward, Nucl. Phys. B236 (1984) 381
- [15] Y. Brihaye, C. Devchand and J. Nuyts, Phys. Rev. D32 (1985) 990
- [16] a. H. Georgi and S.L. Glashow, Phys. Rev. D6 (1972) 429
b. D.J. Gross and R. Jackiw, Phys. Rev. D6 (1972) 477
- [17] S. Okubo, Phys. Rev. D16 (1977) 3528
- [18] J. Banks and H. Georgi, Phys. Rev. D14 (1976) 1159
- [19] P.H. Frampton and T.W. Kephart, Phys. Rev. D28 (1983) 1010
- [20] P.H. Frampton, Phys. Lett. 122B (1983) 351
J. Preskill, P.H. Frampton and H. van Dam, Phys. Lett. 124B (1983) 209
L. Caneschi, G.R. Farrar, A. Schwimmer, Phys. Lett. 138B (1984) 386
T.W. Kephart, Phys. Lett. 151B (1985) 267
- [21] J. Patera and R.T. Sharp, J. Math. Phys. 22 (1981) 2352, section IV
- [22] R. Jackiw, Cocycles in Mathematics and Physics, MIT preprint, CTP 1230, Dec. 1984, to be published in the E.S. Fradkin Festschrift
- [23] See for example,
R.O. Wells, Differential Analysis on Complex Manifolds, Springer Verlag, New York (1980)
S.I. Goldberg, Curvature and Homology, Dover (1982)
S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Acad. Press., New York (1978)
- [24] A. Trautman, Int. J. Theor. Physics 16 (1977) 561
J. Nowakowski and A. Trautman, J. Math. Phys. 19 (1977) 561
J. Harnad, J. Tafel and S. Shnider, J. Math. Phys. 21 (1980) 2236
- [25] Z. Horváth and L. Palla, Nucl. Phys. B142 (1978) 327

- [26] B. Grossman, T.W. Kephart and J.D. Stasheff, Comm. Math. Phys. 96 (1984) 431, Comm. Math. Phys. 100 (1985) 311

S. Fubini and H. Nicolai, the Octonionic Instanton, preprint CERN-TH, 4115/85
- [27] N. Steenrod, Topology of fibre bundles, Princeton Math. Series No. 14, (1950)
- [28] P.A.M. Dirac, Proc. Roy. Soc. A133 (1931) 60
- [29] A.A. Belavin, A.M. Polyakov, A. Schwartz and Y. Tyupkin, Phys. Lett. 59B (1975) 85
- [30] M.F. Atiyah, N.J. Hitchin and I.M. Singer, Proc. Roy. Soc. London A362 (1978) 425
- [31] C.W. Bernard, N.H. Christ, A.H. Guth and E.J. Weinberg, Phys. Rev. D16 (1977) 2967
- [32] L.S. Brown, R.D. Carlitz and C. Lee, Phys. Rev. D16 (1977) 417
- [33] A.N. Schellekens, Nucl. Phys. B250 (1985) 252
- [34] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Yu.I. Manin, Phys. Lett. 65A (1978) 185
- [35] E. Corrigan, P. Goddard and A. Kent, Comm. Math. Phys. 100 (1985) 1
- [36] G. 't Hooft, Phys. Rev. D14 (1976) 3432
- [37] See for example, S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, vol.I, Interscience, New York (1969)
- [38] F.A. Bais and P. Batenburg, in preparation.
- [39] D.H. Tchrakian, J. Math. Phys. 21 (1980) 166
- [40] D.H. Tchrakian, Phys. Lett. 150B (1985) 360