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THE OCTET NON-LEPTONIC HAMILTONIAN AND CURRENT ALGEBRA
ON THE LATTICE WITH WILSON FERMIONS⁺)

L. Maiani^{*)}, G. Martinelli

CERN - Geneva

G.C. Rossi

Dipartimento di Fisica, Università di Roma "Tor Vergata"
INFN. Sezione di Roma, Italy

and

M. Testa

INFN. Sezione di Roma, Italy

A B S T R A C T

We discuss in detail the construction of the octet non-leptonic Hamiltonian in lattice QCD with Wilson fermions.

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^{*)} On leave of absence from Dipartimento di Fisica, Università di Roma "La Sapienza", Roma, Italy.

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1. Introduction

The understanding of the non-leptonic, $\Delta S = \pm 1$, weak transitions has proved to be a rather difficult problem in particle physics.

Considerable progress has been made after the discovery that the basic strong interactions are described by an asymptotically free gauge theory⁽¹⁾, and following the construction of a renormalizable gauge theory of the electroweak interactions⁽²⁾. The octet weak hamiltonian was shown⁽³⁾ to be enhanced, and the 27-plet suppressed, by short distance effects arising from gluon exchange at large momentum scales μ , $M_W > \mu > m_c$ (m_c denotes the charmed quark mass). The enhancement thus found is, however, only of a factor 2 to 3, and a considerable effect is still required to account for the experimental octet versus 27-plet amplitude ratio of about 20. Further investigations identified new operators which come into play in the effective hamiltonian at momentum scales lower than m_c , the so-called "penguin" operators⁽⁴⁾, and may account for the remaining part of the effect. However, the coefficients of the penguin operators turn out to be small and their calculation, within perturbative QCD, doubtful. Furthermore, the approximation schemes employed to estimate the hadronic matrix elements (vacuum saturation, bag model and the like⁽⁵⁾) are not so well justified theoretically as to give a convincing proof that the required enhancement is, in fact, present.

Although the overall picture seems rather sound, it is fair to say that a quantitative understanding, based on first-principle calculations is still lacking.

An important and related subject is that of the parameter ϵ' , which describes the direct CP-violation in $K \rightarrow \pi\pi$ amplitudes. Non vanishing ϵ' distinguishes a milliweak origin of the CP-violation, such as predicted by the standard Kobayashi-Maskawa model⁽⁶⁾, from the more exotic superweak interaction⁽⁷⁾. The calculation of ϵ' involves again the matrix elements of penguin operators, and it is affected by similar uncertainties as those discussed above. Even in this case, it is quite difficult to say whether the standard theory agrees or not with the experimental data.

Monte Carlo simulations of QCD on a discrete lattice⁽⁸⁾ offer the possibility of a full-fledged non-perturbative calculation of the matrix elements of the weak hamiltonian. Preliminary results^(9,10) have already been published, with encouraging results. The lattice calculation of $\Delta I = 3/2$ transitions is theoretically well justified⁽¹¹⁾ and we have little to add to the subject. In this paper we focus entirely on the octet, $\Delta I = 1/2$, sector of the non leptonic transitions.

The construction of the octet non-leptonic hamiltonian to be used in lattice QCD suffers from two different but related problems: severe ultraviolet divergences in the limit of vanishing lattice spacing, and bad chiral behaviour. Both problems arise because of the hard chiral symmetry breaking entailed by

the Wilson term in the fermionic action, a term which is necessary to avoid quark species doubling and to control the flavour symmetry^(f). The way to circumvent these problems has been indicated in ref.(11), and will be illustrated in full detail in the present paper (a short account of the main results has been given in ref.(12)). An overall view of the line of reasoning and of the results goes as follows.

The Wilson short distance expansion gives the effective hamiltonian as a combination of local operators of dimension six or lower. We want to compute the Wilson coefficients from QCD improved perturbation theory in the continuum, and the matrix elements of the operators by Monte Carlo simulation on the lattice. To define the latter operators we have to follow a two-step procedure⁽¹¹⁾: first, subtract equal or lower dimension operators from the bare ones, so as to remove all the lattice artifacts induced by the Wilson term and, secondly, apply a multiplicative renormalization, so as to obtain lattice operators which are normalized in the same way as the continuum ones.

Starting from a given (bare) four fermion operator, the structure of the subtractions is determined by dimensional counting and by the symmetries of the problem. It turns out that the subtractions corresponding to higher dimension operators have a finite dependence from the bare QCD coupling, and can therefore

(f) It is not known whether the alternative formulation, with Kogut-Susskind fermions, can be of use in this connection because of the complicated mixing between spin and flavour quantum numbers which is present in the scheme.

be determined in perturbation theory. The same applies to the overall normalization factor (consistently with the perturbative calculation of the Wilson coefficients). However, in contrast to the $\Delta I = 3/2$ case, the octet hamiltonian requires also the subtraction of lower dimension operators, a subtraction which is so divergent that it is necessary to make it in a non perturbative way. It is to determine the coefficients of the latter subtraction that the chiral Ward identities play a crucial role⁽¹²⁾. With them, we shall prove that the properly subtracted operator obeys all the familiar soft-pion relations implied by the continuum current algebra for a V-A operator (i.e. for an operator transforming according to the (8,1) representation of $SU(3)_L \times SU(3)_R$). At this point, we may turn the argument around, and determine the coefficients themselves by enforcing the soft-pion theorems on the lattice. The problem thus reduces to (i) determine which operators have to be subtracted non-perturbatively from a given four fermion operator; (ii) identify suitable matrix elements whose computation on the lattice is able to fix the unknown subtraction coefficients and give us the (finite and well normalized) weak transition amplitudes.

Having to fight against ultraviolet divergences, it is convenient to start with four fermion operators as convergent as possible. This is the case if our lattice is so fine grained as to allow for the propagation of the charmed quark and make the GIM cancellations operative. In this case, we find that the lattice determination of $K \rightarrow 0$ and $K \rightarrow \pi$ matrix elements is sufficient to fix all divergent coefficients, and to determine

the $K \rightarrow \pi\pi$ weak amplitude, via PCAC and to lowest order in light quark masses.

In the opposite case (lattice spacing, a , greater than m_c^{-1}) the presence of one additional subtraction in the parity-conserving hamiltonian makes the knowledge of $K \rightarrow \pi$ matrix elements insufficient. The $K \rightarrow \pi\pi$ matrix elements have to be computed directly, which is possible after the $K \rightarrow 0$ ones are used to remove the single divergence present in the parity-violating channel. Irrespective of the presence or absence of a dynamical charmed quark on the lattice, the determination of ϵ' has to go through the calculation of the $K \rightarrow \pi\pi$ matrix elements of penguin-like operators.

After the subtraction coefficients have been determined from the kaon sector, the same weak octet hamiltonian can be used to compute any other $\Delta S = \pm 1$ transition, such as those between baryonic states.

It must be stressed that the use of the Wilson expansion and all the subsequent manipulations are but a necessary expedient because we are unable to put the full Weinberg-Salam theory on the lattice^(f). If we could do so, we would compute directly the weak transitions for any value of the light quark masses. Having chosen the short-distance approach, and, consequently, being limited to compute the Wilson coefficients with perturbation theory, we have imposed on us certain limitations from the outset.

(f) This is at present impossible because of practical but also because of conceptual reasons. The Wilson term breaks the $SU(2) \times U(1)$ gauge invariance so it is not clear that one can put consistently a chiral gauge theory on the lattice⁽¹³⁾.

As discussed in this paper, the Wilson coefficients can be computed reliably only in the chiral limit, $m_{s,d,u} \rightarrow 0$ and, correspondingly, we find that the subtraction procedure suffers from (finite) ambiguities when non vanishing masses of the light quarks are allowed for. As a consequence of all that, the weak hamiltonian is unambiguously defined in the chiral limit only and we cannot determine the dependence of the weak amplitudes upon light quark masses.

The latter considerations bring us to the question of how far from first-principles is the calculation of the weak amplitudes we propose to do on the lattice. One limitation is the use of perturbation theory at various stages. This we think we can justify up to the same level of rigour as, e.g., in deep inelastic processes, on the basis of asymptotic freedom.

More important is the smoothness assumption needed to recover the physical amplitudes from their chiral symmetric extrapolation. The $K \rightarrow \pi\pi$ amplitude is first order in quark masses⁽¹⁴⁾ and it is related to the chiral limit of the $K \rightarrow \pi$ matrix elements only by chiral perturbation theory.⁽¹⁵⁾ However, for the case where lattice spacing allows charm propagation, one may compute both $K \rightarrow \pi$ and $K \rightarrow \pi\pi$ amplitudes, and obtain an internal check of the smoothness hypothesis. Once the check goes through, we think the lattice calculation does provide a very significant test of the present strong and electroweak interaction theory.

The plan of the paper is the following. We recall in Section 2 the elements of the short-distance expansion of the weak hamil-

tonian, and the relevant operator basis. The aim is basically that of introducing the notation, but some points about the chiral limit of Wilson coefficients and the normalization conditions will be considered in detail for the sake of the subsequent discussion. In section 3 we recall how chiral symmetry works on the lattice, and give a discussion of the chiral Ward identities in the massless quark limit. Ward identities in the massive case are worked out in Section 4 to discuss the ambiguities inherent to the definition of chiral operators, and the connection to soft-pion theorems. In Section 5 the structure of the subtractions to the relevant four fermion operators is presented. Finally, in Section 6, we assemble all the information and spell out the explicit procedure to compute the physical weak transition amplitudes. Technical details are reported in the Appendices.

2. The effective weak hamiltonian and the operator basis in the continuum.

In the standard $SU(3)_{\text{color}} \times SU(2) \times U(1)$ theory, the flavour changing amplitudes between hadronic states h_2 and h_1 , are given by the simple expression⁽¹⁶⁾:

$$\begin{aligned} \langle h_2 | H_{\text{eff}} | h_1 \rangle &= \\ &= \frac{1}{2} g_w^2 \int d^4x \, D(x, M_w) g^{\mu\nu} \langle h_2 | T(J_\mu^{(+)}(x) J_\nu^{(-)}(0)) | h_1 \rangle \end{aligned} \quad (2.1)$$

where D is the W -propagator, and the weak currents are expressed in terms of quark fields according to:

$$J_{\mu}^{(+)} = \bar{u}_L \gamma_{\mu} (\cos \theta d_L + \sin \theta s_L) + \bar{c}_L \gamma_{\mu} (-\sin \theta d_L + \cos \theta s_L) \quad (2.2)$$

θ is the Cabibbo angle and we neglect mixing with further quark generations (a third generation will be considered in Section 6 to discuss the CP-violating ϵ' parameter).

To define the renormalized weak hamiltonian, one has to add to H_{eff} appropriate counterterms

$$(H_{eff})_R = H_{eff} + \delta H_{eff} \quad (2.3)$$

The admissible counterterms to the $\Delta S = \pm 1$ hamiltonian have the form: (16)

$$\delta H_{eff} = \frac{G}{\sqrt{2}} \sin \theta \cos \theta \ c \ O_m \quad (2.4)$$

$$O_m = (m_s + m_d) \bar{s} d + (m_s - m_d) \bar{s} \gamma_5 d + h.c. \quad (2.5)$$

The coefficient c in eq.(2.4) is finite because of the GIM can-

cellation. The structure of $\mathcal{O}_m$ is dictated entirely by the symmetries of the problem. The masses m_s and m_d come in because of chiral symmetry, and the symmetry (antisymmetry) of the scalar (pseudoscalar) term under $m_s \leftrightarrow m_d$ exchange is implied by the fact that H_{eff} is even under the discrete symmetry CPS, where:

$$CPS = (\text{charge conjugation}) \times (\text{parity}) \times (s \leftrightarrow d \text{ exchange}) \quad (2.6)$$

$\mathcal{O}_m$ behaves as a $(3, \bar{3})$ operator under quark field $SU(3)_L \times SU(3)_R$ transformations:

$$\begin{aligned} q_L &\rightarrow U q_L & ; & \quad q_L = \frac{1-\gamma_5}{2} q \\ q_R &\rightarrow V q_R & ; & \quad q_R = \frac{1+\gamma_5}{2} q \\ U^\dagger U &= V^\dagger V = 1 \\ \det U &= \det V = 1 \end{aligned} \quad (2.7)$$

and as an $(8, 1)$ if, together with (2.7) we transform the light quark mass matrix according to:

$$m = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} \rightarrow U^\dagger m V \quad (2.8)$$

In perturbation theory, the coefficient c can be determined by requiring that quarks with definite flavour are mass eigenstates even in presence of the weak interaction. In the real world, where we deal only with confined hadronic states, a substitute must be found. It is natural to replace the condition

on quark matrix elements with one for the matrix elements between pseudoscalar mesons whose masses are directly related to quark masses because of spontaneous chiral symmetry breaking. An important role in finding the appropriate normalization conditions is played by the chiral transformation properties of the current x current product, which are those of an (8,1) multiplet. On general grounds (as is discussed in greater detail in Appendix I) the matrix elements of any such operator in the pseudoscalar meson sector can be parametrized as follows:

$$\begin{aligned}
\langle 0 | H_{eff} | K^0 \rangle &= i \frac{G}{\sqrt{2}} \sin \theta \cos \theta \left[(m_K^2 - m_\pi^2) \delta + \dots \right] \\
\langle \pi^+(q) | H_{eff} | K^+(k) \rangle &= \frac{G}{\sqrt{2}} \sin \theta \cos \theta \left[- \frac{m_K^2}{f_\pi} \delta + (q \cdot k) \gamma + \dots \right] \\
\langle \pi^+ \pi^- | H_{eff} | K^0 \rangle &= i \frac{G}{\sqrt{2}} \sin \theta \cos \theta \left[\frac{m_K^2 - m_\pi^2}{f_\pi} \gamma + \dots \right] \quad (2.9)
\end{aligned}$$

where dots denote higher order terms in external momenta and/or pseudoscalar masses. δ and γ are two independent parameters.

To lowest order in $m_{K,\pi}^2$ and external momenta, the chiral behavior of $\mathcal{O}_m$ implies (see Appendix I):

$$\begin{aligned}
\langle 0 | \mathcal{O}_m | K^0 \rangle &= i (m_K^2 - m_\pi^2) \delta' + \dots \\
\langle \pi^+(q) | \mathcal{O}_m | K^+(k) \rangle &= - \frac{m_K^2}{f_\pi} \delta' + \dots \\
\langle \pi^+ \pi^- | \mathcal{O}_m | K^0 \rangle &= 0 \quad (2.10)
\end{aligned}$$

An admissible and convenient normalization condition we shall adopt in the following, is to require

$$C \delta' + \delta \equiv \delta_R = 0 \quad (2.11)$$

so that the renormalized hamiltonian has a vanishing $K \rightarrow 0$ matrix element and, correspondingly, no constant term in the $K \rightarrow \pi$ matrix element.^(f)

We consider now the short distance expansion^(3,20) of the renormalized current x current product:

$$(H_{eff})_R = \frac{G}{\sqrt{2}} \sin\theta \cos\theta \sum_n C^{(n)}(\mu, M_W) \mathcal{O}^{(n)}(\mu) \quad (2.12)$$

Since the matrix elements of $(H_{eff})_R$ are completely determined, the Wilson coefficients $C^{(n)}$ are specified once the normalization conditions of the composite, local, operators $\mathcal{O}^{(n)}$ are given.

(f) There has been some discussion in the literature, concerning the possible vanishing of δ_R in the chiral limit. As discussed in Appendix I and also in the papers by Bernard et al.⁽¹⁷⁾ and by Le Yaouanc et al.⁽¹⁸⁾ the mere assumptions of (8,1) behaviour and CPS symmetry do not lead to vanishing δ_R , contrary to the claim by Crewther⁽¹⁹⁾. However, since $(H_{eff})_R$ is defined only up to terms of the form given in eq.(2.5), one can always require, as we do, $\delta_R = 0$. At any rate, the actual value of δ_R is irrelevant in that this coefficient does not affect the physical amplitudes (see our discussion at the end of Section 6).

The latter will involve, in general, a normalization scale μ and the μ -independence of the r.h.s. of eq.(2.12) gives the renormalization group equations for the evolution of $C^{(n)}$.

We can classify the operators in the expansion eq.(2.12) according to their chiral properties. Possible, allowed operators transform as (8,1) or, due to the explicit chiral symmetry breaking, as $(3, \bar{3})$ operators multiplied by light quark masses. There are only two operators of the latter type, namely $\mathcal{O}_m$, eq. (2.5), and $\mathcal{O}_\sigma$:

$$\mathcal{O}_\sigma = g_0 \left[(m_s + m_d) \bar{s} \sigma_{\mu\nu} F^{\mu\nu} d + (m_s - m_d) \bar{s} \sigma_{\mu\nu} \tilde{F}^{\mu\nu} d \right] \quad (2.13)$$

where g_0 is the bare color gauge coupling and $F_{\mu\nu}$ the non abelian gluon field strength.

The physical $K \rightarrow \pi\pi$ amplitude to first order in quark masses, is obtained from the chiral limit of the $K \rightarrow \pi$ matrix elements of the operator $\mathcal{O}^{(n)}$, according to:

$$\gamma = \sum_n C^{(n)}(\mu, M_W) \frac{\partial}{\partial(q \cdot k)} \langle \pi^+(q) | \mathcal{O}^{(n)}(\mu) | K^+(k) \rangle \Big|_{\substack{m_{u,d,s} = 0 \\ (q \cdot k) = 0}} \quad (2.14)$$

$$\langle \pi^+ \pi^- | (H_{eff})_R | K^0 \rangle = i \frac{G}{\sqrt{2}} \sin \theta \cos \theta \frac{m_K^2 - m_\pi^2}{f_\pi} \gamma \quad (2.15)$$

To lowest order in chiral perturbation theory, we need to know the Wilson coefficients in the chiral limit only. This is an important point since, in this limit, the Wilson coefficients can be computed reliably with perturbation theory, provided $\mu \gg \Lambda_{\text{QCD}}$. The mass dependent terms in $C^{(n)}$, on the other hand, are affected by finite, non perturbative effects which arise as a feed-back of the non perturbative normalization condition, eq.(2.11).

We now discuss the operator basis relevant to the Wilson expansion in two different cases, corresponding to choosing μ above or below the charmed quark threshold.

i. $M_W \gg \mu > m_c$. Above the charmed quark mass, the operator basis for the octet component of H_{eff} can be restricted^(f) to the operators $\mathcal{O}^{(\pm)}$ and $\mathcal{O}_m$, with:

$$\mathcal{O}^{(-)} = \left[(\bar{s}_L \gamma_\mu d_L) (\bar{u}_L \gamma^\mu u_L) - (\bar{s}_L \gamma_\mu u_L) (\bar{u}_L \gamma^\mu d_L) \right] +$$

$$- (u \rightarrow c) \quad (2.16)$$

$$\mathcal{O}^{(+)} = \frac{1}{5} \left[(\bar{s}_L \gamma_\mu d_L) (\bar{u}_L \gamma^\mu u_L) + (\bar{s}_L \gamma_\mu u_L) (\bar{u}_L \gamma^\mu d_L) + \right.$$

$$+ 2 (\bar{s}_L \gamma_\mu d_L) (\bar{d}_L \gamma^\mu d_L) + 2 (\bar{s}_L \gamma_\mu d_L) (\bar{s}_L \gamma^\mu s_L) \left. \right] +$$

$$- \left[(\bar{s}_L \gamma_\mu d_L) (\bar{c}_L \gamma^\mu c_L) + (\bar{s}_L \gamma_\mu c_L) (\bar{c}_L \gamma^\mu d_L) \right] \quad (2.17)$$

(f) As usual, we neglect operators which appear with Wilson coefficients of order $(m_c^2 - m_u^2)/\mu^2$ such as the operator $\mathcal{O}_\sigma$, eq.(2.13).

To properly define the renormalized operators, one needs to impose two normalization conditions which, in perturbation theory, would specify their two quark and four quark matrix elements. The discussion of the former conditions, in the confined situation, goes exactly as that for Heff. In correspondence to eq.(2.11), we define:

$$\tilde{\mathcal{O}}^{(\pm)} = \mathcal{O}^{(\pm)} + c^{(\pm)} \mathcal{O}_m$$

and require:

$$\langle 0 | \tilde{\mathcal{O}}^{(\pm)} | K^0 \rangle = 0 \quad (2.18)$$

so that:

$$\langle \pi^+(q) | \tilde{\mathcal{O}}^{(\pm)} | K^+(k) \rangle = (q \cdot k) \gamma^{(\pm)} + \dots \quad (2.19)$$

which determines $\tilde{\mathcal{O}}^{(\pm)}$ up to a normalization factor. The latter can be obtained by normalizing $\tilde{\mathcal{O}}^{(\pm)}$ at a large euclidean momentum μ , where perturbation theory is applicable. We define:

$$\mathcal{O}^{(\pm)}(\mu) = Z^{(\pm)}(\mu) \tilde{\mathcal{O}}^{(\pm)}$$

and

$$\langle qq | \mathcal{O}^{(\pm)}(\mu) | qq \rangle_\mu = 1 \quad (2.20)$$

Eqs. (2.18) and (2.20) determine the Wilson coefficients $C^{(\pm)}$. In fact, with the condition, eq. (2.18), one obtains⁽³⁾:

$$(H_{\text{eff}})_R = \frac{G}{\sqrt{2}} \sin \theta \cos \theta \left[C^{(-)}(\mu, M_W) \mathcal{O}^{(-)}(\mu) + \right. \\ \left. + C^{(+)}(\mu, M_W) \mathcal{O}^{(+)}(\mu) + O\left(\frac{m_c^2 - m_u^2}{\mu^2}\right) \right] \quad (2.21)$$

ii. $m_c > \mu > \Lambda_{\text{QCD}}$. The relevant operators have been classified in ref. (4) and we list them here only for completeness. There are four dimension-six operators:

$$\mathcal{O}_1 = (\bar{s}_L \gamma_\mu d_L)(\bar{u}_L \gamma^\mu u_L) - (\bar{s}_L \gamma_\mu u_L)(\bar{u}_L \gamma^\mu d_L) \quad (2.22)$$

$$\mathcal{O}_2 = (\bar{s}_L \gamma_\mu d_L)(\bar{u}_L \gamma^\mu u_L) + (\bar{s}_L \gamma_\mu u_L)(\bar{u}_L \gamma^\mu d_L) + \\ + 2(\bar{s}_L \gamma_\mu d_L)(\bar{d}_L \gamma^\mu d_L) + 2(\bar{s}_L \gamma_\mu d_L)(\bar{s}_L \gamma^\mu s_L) \quad (2.23)$$

$$\mathcal{O}_5 = (\bar{s}_L \gamma_\mu t^A d_L)(\bar{u}_R \gamma^\mu t^A u_R + \bar{d}_R \gamma^\mu t^A d_R + \\ + \bar{s}_R \gamma^\mu t^A s_R) \quad (2.24)$$

$$\mathcal{O}_6 = (\bar{s}_L \gamma_\mu d_L)(\bar{u}_R \gamma^\mu u_R + \bar{d}_R \gamma^\mu d_R + \bar{s}_R \gamma^\mu s_R) \quad (2.25)$$

one dimension-five operator, $\mathcal{O}_\sigma$ given in eq.(2.13), and $\mathcal{O}_m$.

The operators of dimension-five and six require one subtraction proportional to $\mathcal{O}_m$, to regulate quadratic divergences. For any of them we define:

$$\tilde{\mathcal{O}}_i = \mathcal{O}_i + c_i \mathcal{O}_m$$

($i=1,2,5,6,\sigma$) and fix C_i so that^(f):

$$\langle 0 | \tilde{\mathcal{O}}_i | K^0 \rangle = 0 \quad (2.26)$$

Logarithmic divergences correspond to an anomalous dimension matrix which mixes all the above operators. The final, renormalized operators are:

$$\mathcal{O}_i(\mu) = Z_{ij}(\mu) \tilde{\mathcal{O}}_j$$

($i,j=1,2,5,6,\sigma,m$), with Z such that:

$$\begin{aligned} \langle qq | \mathcal{O}_{1,2,5,6}(\mu) | qq \rangle_\mu &= 1 \\ \langle q + \text{gluon} | \mathcal{O}_{1,2,5,6}(\mu) | q \rangle_\mu &= 0 \\ \langle q + \text{gluon} | \mathcal{O}_\sigma(\mu) | q \rangle_\mu &= 1 \\ \langle q | \mathcal{O}_m(\mu) | q \rangle_\mu &= 1 \end{aligned} \quad (2.27)$$

(f) A consequence of this definition is that the matrix elements between pseudoscalar states of $\tilde{\mathcal{O}}_\sigma$ are effectively of order m_s^2 , i.e. m_K^4 , see Appendix I.

As in the previous case, the matrix Z and the corresponding Wilson coefficients can be computed in perturbation theory.

Before closing this Section, it is important to stress that the above discussion, which follows the conventional wisdom, is somewhat formal, in that it assumes that a regularization of the bare theory can be found which respects the chiral symmetry. To our knowledge, no such a regularization exists^(f).

Certainly, the lattice regularization with Wilson fermions does not respect chiral symmetry, and indeed we shall find that the structure of the divergences and the corresponding subtractions are considerably more complicated. It is a non-trivial result of ref.(11), that also in this case we shall recover in the end the operator structure given above, in the limit of vanishing lattice spacing.

3. Chiral operators on the lattice: the massless case

We summarize here the construction of partially conserved axial currents on the lattice, and the definition of operators with given chiral properties.⁽¹¹⁾ In the present Section the latter point is discussed in the limit where quarks have a vanishing physical mass, deferring the discussion of the massive case to the next Section.

^(f) See our comments in the Introduction, for what concerns the use of Kogut-Susskind fermions in this connection.

Quarks are described by the action: (8,20)

$$S_F = \sum_x \left\{ -\frac{1}{2a} \sum_{\mu} \left[\bar{q}(x) (\tau - \gamma_{\mu}) U_{\mu}(x) q(x+\mu) + \right. \right. \\ \left. \left. + \bar{q}(x+\mu) U_{\mu}^{\dagger}(x) (\tau + \gamma_{\mu}) q(x) \right] + \bar{q}(x) \left(M_0 + \frac{4\tau}{a} \right) q(x) \right\} \quad (3.1)$$

The partial conservation equations which correspond to the axial transformations given in eq. (2.7) ($U=V^{\dagger}=\exp(-i\frac{\lambda^a}{2} \epsilon^a)$), are:

$$\langle h_2 | \nabla_{\mu} A_{\mu}^a | h_1 \rangle = \langle h_2 | \bar{q} \left\{ \frac{\lambda^a}{2} \gamma_5, M_0 \right\} q + X^a | h_1 \rangle \quad (3.2)$$

where

$$A_{\mu}^a(x) = \frac{1}{2} \left[\bar{q}(x) U_{\mu}(x) \frac{\lambda^a}{2} \gamma_5 q(x+\mu) + h.c. \right] \quad (3.3)$$

$$\nabla_{\mu} f(x) = \frac{f(x) - f(x-\mu)}{a} \quad (3.4)$$

$$X^a(x) = -\frac{\pi}{2a} \sum_{\mu} \left\{ \left[\bar{q}(x) \frac{\lambda^a}{2} \gamma_5 U_{\mu}(x) q(x+\mu) + \right. \right. \\ \left. \left. + \bar{q}(x+\mu) U_{\mu}^{\dagger}(x) \frac{\lambda^a}{2} \gamma_5 q(x) \right] + [X \rightarrow X-\mu] + \right. \\ \left. - 4 \bar{q}(x) \frac{\lambda^a}{2} \gamma_5 q(x) \right\} \quad (3.5)$$

Eq. (3.2) is anomalous in that the axial current is not conserved in the limit $M_0 \rightarrow 0$, due to the term X^a . In the continuum limit, however, a non-anomalous equation is obtained, for non-singlet currents, after a proper redefinition of quark masses and a

rescaling of the currents^(11,21):

$$\langle h_2 | \nabla_\mu \hat{A}_\mu^a(x) | h_1 \rangle = \langle h_2 | \bar{q}(x) \left\{ \frac{\lambda^a}{2} \gamma_5, m \right\} q(x) | h_1 \rangle \quad (3.6)$$

$$\hat{A}_\mu^a(x) = Z_A A_\mu^a(x) \quad (3.7)$$

$$m = M_0 - \bar{M} \quad (3.8)$$

As shown in ref.(11), the conditions that $\bar{X}^a$:

$$\bar{X}^a \equiv X^a + \bar{q} \left\{ \frac{\lambda^a}{2} \gamma_5, \bar{M} \right\} q + (Z_A - 1) \nabla_\mu A_\mu^a$$

decouples from the physical states:

$$\langle h_2 | \bar{X}^a | h_1 \rangle \underset{a \rightarrow 0}{=} 0$$

and that $\hat{A}_\mu$ obeys the continuum current algebra are sufficient to determine^(f) the normalization factor, Z_A , and the mass subtraction, $\bar{M}$. When this is done, m coincides in the continuum limit with the bare current quark mass (the same we introduced in Section 2, eq.(2.8)). $\bar{M}$ subtracts the linear divergence present in M_0 , leaving in m a residual logarithmic divergence. As in the case of the continuum, the latter divergence matches the

(f) An essential ingredient in the discussion is that Z_A is a function of the bare coupling g_0 and of the Wilson parameter r , only. This was shown to be the case to all orders in perturbation theory, ref.(22).

normalization of the scalar density $\bar{q}q$, so that:

$$\langle h_2 | \bar{q}q | h_1 \rangle = \text{finite} \\ a \rightarrow 0$$

The non perturbative determination of Z_A and m , on the lattice, has been studied numerically in ref.(23).

The existence of a good axial current, conserved in the chiral limit $m \rightarrow 0$, implies that one should be able to construct composite operators with well defined chiral properties. To this aim, one starts from the appropriate lattice Ward identity, which reads:

$$\begin{aligned} \sum_x \nabla_\mu^x \langle h_2 | T(\hat{A}_\mu^a(x) \mathcal{O}(0)) | h_1 \rangle = \\ = \sum_x \langle h_2 | T(\bar{q}(x) \{ \frac{1}{2} \gamma_5, m \} q(x) \mathcal{O}(0)) | h_1 \rangle + \\ + \sum_x \langle h_2 | T(\bar{X}^a(x) \mathcal{O}(0)) | h_1 \rangle + \langle h_2 | i \frac{\delta \mathcal{O}(0)}{\delta \epsilon^a} | h_1 \rangle \end{aligned} \quad (3.9)$$

where the last term in the r.h.s. is the variation of $\mathcal{O}$ under a global axial transformation, eq.(2.7).

Analogously to eq.(3.2), the Ward identity (3.9) is anomalous. Even if $\bar{X}^a$ vanishes on-shell, the product of $\bar{X}^a(x)$ with $\mathcal{O}(0)$ does in general contain localized, contact terms which do not vanish as $a \rightarrow 0$, thus spoiling the naive transformation properties of $\mathcal{O}$. In addition the presence of the Wilson term gives rise to severe (power behaved) divergences in the matrix elements of $\mathcal{O}$ between physical states.

A definite chiral behaviour and the removal of lattice-artifact divergences can be obtained at the same time⁽¹¹⁾. We have to follow two steps. First, one subtracts from the naive operator a combination of operators with equal or lower dimension and all possible naive chiralities, in such a way as to obtain an operator with the correct chiral properties with respect to the "good" axial and vector charges. At this point, not all divergences have been, in general, eliminated, but one is left with an overall multiplicative renormalization which is determined by requiring the lattice operators to be normalized equally to the continuum ones.

In the chiral limit this procedure is straightforward. Let us consider a particular operator $\mathcal{O}_\alpha$, which transforms, under naive axial transformations, according to some representation R (e.g. (8,1)):

$$\frac{\delta \mathcal{O}_\alpha}{\delta \epsilon^a} = i R_{\alpha\beta}^a \mathcal{O}_\beta \quad (3.10)$$

R^a are the axial generators of the particular representation $\mathcal{O}$ belongs to; α and β are flavour labels appropriate to R. The operator which truly transforms according to R is^(f)

^(f) In writing eq.(3.11) we have made the conventional choice to give coefficient one to $\mathcal{O}_\alpha$ and zero to any other possible operator with the same naive chiral transformation properties. This choice is not restrictive: for any operator $\mathcal{O}^{(i)}$ transforming with R, we can similarly construct the corresponding $\tilde{\mathcal{O}}^{(i)}$, and the set of all these new operators is a complete basis.

$$\tilde{\mathcal{O}}_d = \mathcal{O}_d + \sum_{n,\sigma} d_{d\sigma}^{(n)} \mathcal{O}_\sigma^{(n)} \quad (3.11)$$

with $\mathcal{O}^{(n)}$ transforming according to $R^{(n)} \neq R$. The coefficients $d^{(n)}$ are restricted by the condition that they must respect the conserved vector symmetry.

In the chiral limit the integrated axial current Ward identity for $\tilde{\mathcal{O}}$ reads:

$$\begin{aligned} \sum_x \nabla_\mu \langle h_2 | T(\hat{A}_\mu^a(x) \tilde{\mathcal{O}}(0)) | h_1 \rangle &= \\ &= \sum_x \langle h_2 | T(\bar{X}^a(x) \tilde{\mathcal{O}}(0)) | h_1 \rangle + \langle h_2 | i \frac{\delta \tilde{\mathcal{O}}}{\delta \epsilon^a} | h_1 \rangle \end{aligned} \quad (3.12)$$

where:

$$\frac{\delta \tilde{\mathcal{O}}}{\delta \epsilon} = i \left(R \mathcal{O} + \sum_n d^{(n)} R^{(n)} \mathcal{O}^{(n)} \right) \quad (3.13)$$

and we have omitted flavour indices. For the appropriate choice of the coefficients $d^{(n)}$ the contact terms arising in the correlation function of $\bar{X}$ with $\tilde{\mathcal{O}}$ will correct the complicated naive chiral variation, eq.(3.13), in such a way as to reproduce the desired transformation property. The condition for this is:

$$\sum_x \langle h_2 | T(\bar{X}^a(x) \tilde{\mathcal{O}}_a(0)) | h_1 \rangle + \langle h_2 | i \frac{\delta \tilde{\mathcal{O}}_a}{\delta \epsilon^a} | h_1 \rangle =$$

$$= - R_{\alpha\beta}^a \langle h_2 | \tilde{\mathcal{O}}_\beta | h_1 \rangle \quad (3.14)$$

Eqs.(3.14) are a set of linear, inhomogeneous equations in $d^{(n)}$ which has a unique solution. The $\tilde{\mathcal{O}}$ thus constructed is not yet finite, for $a \rightarrow 0$. Eqs.(3.12) and (3.14) show that the only freedom left is to take linear superpositions of operators transforming in the same way under chiral rotations. We thus define

$$\hat{\mathcal{O}} = Z_{LATT} \tilde{\mathcal{O}} \quad (3.15)$$

Z_{LATT} being a matrix which mixes equivalent representations only, and such that:

$$\langle h_2 | Z_{LATT} \tilde{\mathcal{O}} | h_1 \rangle = \text{finite} \quad (3.16)$$

This condition is consistent with the l.h.s. of eq.(3.12). In fact if both $\hat{A}_\mu$ and $Z_{LATT} \tilde{\mathcal{O}}$ have finite matrix elements, possible divergent terms in their product are localized operators whose contribution to the l.h.s. of eq.(3.12) vanishes upon

taking the four-divergence and summing over x .

To complete the program, we must specify more precisely the normalization condition (3.16). In the chiral limit, this requires the introduction of a subtraction point μ , so that:

$$Z_{LATT} = Z_{LATT}(a\mu, g_0) \quad (3.17)$$

$$\hat{\mathcal{J}}(\mu) = Z_{LATT} \tilde{\mathcal{J}} \quad (3.18)$$

$Z_{LATT}(a\mu, g_0)$ is completely determined, including the finite terms, by the requirement that $\hat{\mathcal{J}}(\mu)$, for some value $\mu \ll a^{-1}$, obeys the same normalization condition as the continuum operator $\mathcal{J}(\mu)$. With this choice, $\hat{\mathcal{J}}(\mu)$ does in fact coincide with $\mathcal{J}(\mu)$.

The lattice definition of the weak hamiltonian is then:

$$H_{eff} = \frac{G}{\sqrt{2}} \sin\theta \cos\theta \sum_n C^{(n)}(\mu, M_W) \hat{\mathcal{J}}^{(n)}(\mu) \quad (3.19)$$

where the Wilson coefficients are those computed in the continuum, as discussed in Section 2.

Before closing this Section, we comment on the degree of divergence of the coefficients $d^{(n)}$, eq.(3.11). The existence of the chiral limit requires:

$$d^{(n)} = \frac{1}{(a)^{\Delta_n}} f^{(n)}(g_0, \tau) \quad (3.20)$$

where Δ_n is given by dimensional counting and $f^{(n)}$ depends only upon the bare coupling constant and the Wilson parameter $r^{(f)}$. As a consequence, the coefficients of the dimension-six operators ($\Delta_n = 0$) can be reliably computed in perturbation theory. This is not the case, in general, for power divergent coefficients. In Section 6 we discuss a practical way to determine the latter coefficients in a non-perturbative fashion.

4. Chiral classification of the operators in the massive case

We have seen in Section 2 that the computation of the $K \rightarrow \pi\pi$ weak transition requires the knowledge of the Wilson coefficients and of the $K \rightarrow \pi$ matrix elements in the chiral limit. This limit can be obtained only as an extrapolation from the massive case, where additional divergences can be present in the definition of the relevant lattice operators. Thus, we need also to study the renormalization in the massive quark situation, in order to construct the finite operators whose matrix elements can then be smoothly extrapolated to the chiral limit.

(f) The absence of $\log(\mu a)$ divergences in $d^{(n)}$ guarantees that the μ dependence of $\hat{\mathcal{O}}(\mu)$ arises only from Z_{LATT} , which in turn is necessary for it to have the same scaling behaviour with μ , as the continuum operator.

The integrated Ward identity, for massive quarks, reads:

$$\begin{aligned} \sum_x \langle h_2 | T(\bar{q}(x) \{ \frac{1}{2} \gamma_5, m \} q(x) \tilde{\mathcal{O}}(0)) | h_1 \rangle = \\ = - \sum_x \langle h_2 | T(\bar{X}^a(x) \tilde{\mathcal{O}}(0)) | h_1 \rangle - \langle h_2 | \frac{\delta \tilde{\mathcal{O}}(0)}{\delta \epsilon^a} | h_1 \rangle \end{aligned} \quad (4.1)$$

where $\tilde{\mathcal{O}}$ is given by an expression analogous to eq.(3.11).

In contrast with the massless case, eq.(4.1) determines the operators $\tilde{\mathcal{O}}$ only up to certain ambiguities which, however, will turn out to be harmless. The reason has to do with the additional divergences which can be present in the r.h.s. of eq.(4.1), due to the short distance behaviour of the operator product, and which cannot be removed by renormalizing the matrix elements of $\tilde{\mathcal{O}}$ only.

Such divergences are removed by a further subtraction. As a consequence, the finite Ward identity reads:

$$\begin{aligned} \sum_x \langle h_2 | T(\bar{q}(x) \{ \frac{1}{2} \gamma_5, m \} q(x) Z \tilde{\mathcal{O}}(0)) | h_1 \rangle + C.T. = \\ = - \sum_x \langle h_2 | T(\bar{X}^a(x) Z \tilde{\mathcal{O}}(0)) | h_1 \rangle - Z \langle h_2 | \frac{\delta \tilde{\mathcal{O}}(0)}{\delta \epsilon^a} | h_1 \rangle_+ \end{aligned} \quad (4.2)$$

+ C.T.

C.T. denotes a set of counterterms, at least of order m , which render finite the correlation in the l.h.s. of the equation, and

Z the appropriate normalization matrix^(f) which removes the residual logarithmic divergences from $\tilde{\mathcal{O}}$. Eq.(4.2) is the one to compare with the continuum Ward identity, expressed in terms of a finite T* product of operators:

$$\begin{aligned} \int d^4x \langle h_2 | T^* (\bar{q}(x) \{ \frac{\lambda^a}{2} \gamma_5, m \} q(x) Z \tilde{\mathcal{O}}_a(0) | h_1 \rangle = \\ = R_{\alpha\beta}^a \langle h_2 | Z \tilde{\mathcal{O}}_\beta(0) | h_1 \rangle \end{aligned} \quad (4.3)$$

The condition that the finite operator $Z \tilde{\mathcal{O}}$ transforms correctly according to the representation R gives then the equation:

$$\begin{aligned} \sum_x \langle h_2 | T (\bar{q}(x) \{ \frac{\lambda^a}{2} \gamma_5, m \} q(x) Z \tilde{\mathcal{O}}_a(0) | h_1 \rangle + C. T. = \\ = R_{\alpha\beta}^a \langle h_2 | Z \tilde{\mathcal{O}}_\beta(0) | h_1 \rangle \end{aligned} \quad (4.4)$$

or, equivalently:

$$\begin{aligned} \sum_x \langle h_2 | T (\bar{X}^a(x) Z \tilde{\mathcal{O}}_a(0) | h_1 \rangle + Z \langle h_2 | i \frac{\delta \tilde{\mathcal{O}}_a(0)}{\delta \epsilon^a} | h_1 \rangle + \\ - C. T. = - R_{\alpha\beta}^a \langle h_2 | Z \tilde{\mathcal{O}}_\beta(0) | h_1 \rangle \end{aligned} \quad (4.5)$$

(f) For simplicity of notation, we denote with Z the lattice normalization matrix, called Z_{LATT} in Section 3.

It is important to stress that the previous equations determine uniquely the divergent parts of all quantities involved. We may fix the divergent parts in the coefficients $d^{(n)}$ and in Z by requiring $Z \tilde{\mathcal{O}}$ to have finite matrix elements. Then eq.(4.4) determines the divergent counterterms needed to render finite the correlation function of the pseudoscalar density with $Z \tilde{\mathcal{O}}$. It is also clear, however, that the problem of determining the finite operators which truly behave according to the representation R suffers from finite ambiguities of order m . We can add to $Z \tilde{\mathcal{O}}$ a finite operator and modify accordingly the definition of the T^* product, i.e. the counterterms C.T., in such a way that eq.(4.4) is still satisfied, and vice versa.^(f)

The structure of the ambiguous terms can be determined with a functional method devised to generate the Ward identities for a broken symmetry⁽²⁴⁾. This method is explained in Appendix II, and the result is very simple. Given an operator $Z \tilde{\mathcal{O}}$ which satisfies eq.(4.4), we can add to it any combination of operators, functions of quark field and masses, such that they behave according to the representation R , under the simultaneous transformations given in eqs.(2.7) and (2.8). For $(8,1)$ operators, and to first order in quark masses, we have only two such operators with dimension less or equal to six, namely the finite operators $\mathcal{O}_m(\mu)$ and $\mathcal{O}_\sigma(\mu)$ introduced in Section 2.

We can explore a little further the issue, and show that

^(f) This ambiguity is, in fact, not surprising. It is the usual ambiguity which goes with the definition of a finite T^* product, starting from the unambiguous, regularized but unrenormalized T -product. In this respect, the ambiguity is not specific of the lattice regularization, and it would be present in any continuum regularization as well.

there exists a choice of the counterterms in eq.(4.4) which leads to operators $Z \tilde{\theta}$ whose matrix elements obey the continuum soft pion theorems. We divide the l.h.s. of eq.(4.4) into a pole term contribution, due to diagrams where a pseudoscalar meson is excited from the vacuum by the pseudoscalar density, and a rest. With the definition:

$$\begin{aligned} \langle 0 | \nabla_\mu \hat{A}_\mu^a | \pi^b \rangle &= \langle 0 | \bar{q} \left\{ \frac{\lambda^a}{2} \gamma_5 \right\} q | \pi^b \rangle = \\ &= \frac{f_\pi}{\sqrt{2}} m_\pi^2 \delta^{ab} \end{aligned} \quad (4.6)$$

one has:

$$\begin{aligned} \sum_x \langle h_2 | T(\bar{q}(x) \left\{ \frac{\lambda^a}{2} \gamma_5 \right\} q(x) Z \tilde{\theta}(0)) | h_1 \rangle_{\text{pole}} &= \\ &= \frac{f_\pi}{\sqrt{2}} \langle h_2 + \pi^a(q=0) | Z \tilde{\theta}(0) | h_1 \rangle \end{aligned} \quad (4.7)$$

A possible choice of the counterterms is such that (at $q=0$) they cancel exactly the non-pole terms in the l.h.s. of eq.(4.4), so that the finite T^* product coincides with the pole contribution^(f), eq.(4.7). With this choice, we obtain:

$$\frac{f_\pi}{\sqrt{2}} \langle h_2 + \pi^a(q=0) | Z \tilde{\theta} | h_1 \rangle = R^a \langle h_2 | Z \tilde{\theta} | h_1 \rangle \quad (4.8)$$

(f) This definition coincides with the familiar PCAC statement that $\nabla_\mu \hat{A}_\mu$ is, up to a factor, the interpolating pion field.

which is nothing but the continuum soft-pion theorem for the matrix elements of $Z \tilde{\mathcal{O}}$.

At this point, we turn the argument around. We can forget the Ward identity eq.(4.4), and determine the coefficients $d^{(n)}$ in the expansion eq.(3.11), i.e. the operator $\tilde{\mathcal{O}}$, simply by requiring that its matrix elements on the lattice obey the standard, soft pion theorems. The latter conditions determine $\tilde{\mathcal{O}}$ apart from the overall normalization matrix Z , and from the addition of finite operators like $\mathcal{O}_m(\mu)$ and $\mathcal{O}_\sigma(\mu)$. The latter addition is harmless, since $\mathcal{O}_m(\mu)$ and $\mathcal{O}_\sigma(\mu)$ obey identically the soft-pion theorems (see Appendix I). To complete the analysis, at this point, we have to accomplish two further steps:

- i) identify the operators we need to subtract from the bare operators relevant to the weak hamiltonian;
- ii) give a practical set of rules to determine the coefficients $d^{(n)}$ from the matrix elements on pseudoscalar meson states of the previous operators.

These two points will be considered in the next two Sections.

5. Relation between lattice and continuum weak Hamiltonian

In this Section we give the explicit structure of the operators $\hat{\mathcal{O}}$ which appear in the non leptonic Hamiltonian, including the subtractions needed to correct the lattice artifacts introduced by the Wilson term. We shall assume that the operator structure of the divergent terms is the same as dictated by

perturbation theory, even in the confined situation.

We have to distinguish two cases, i.e.:

$$a^{-1} > \mu > m_c \quad (5.1)$$

and:

$$m_c > a^{-1} > \mu \quad (5.2)$$

where m_c is the charmed quark mass. In the first case, the charmed quark is a dynamical field on the lattice; in the second, charmed quarks have been integrated out in the continuum and m_c appears as an external scale in the Wilson coefficients. Correspondingly, the operator structure of H_{eff} is more complex, including the so-called "penguin" operators⁽⁴⁾.

We discuss the active charm case first. At the bare level, there are only two four fermion operators, $\mathcal{O}^{(\pm)}$, which have already been given in eqs.(2.16), (2.17). The operators to be subtracted from $\mathcal{O}^{(\pm)}$, to obtain multiplicatively renormalizable operators can be classified according to their dimensionality.

i. Dimension-six operators. These operators have been classified in ref.(25). The corresponding coefficients d , in eq.(3.11), are functions of g_0 and r , and can be reliably computed in lattice perturbation theory. Accordingly, we replace $\mathcal{O}^{(\pm)}$ with:

$$\mathcal{O}^{(\pm)} + \delta_6 \mathcal{O}^{(\pm)} \quad (5.3)$$

where $\delta_6 \mathcal{O}^{(\pm)}$ is taken from ref.(25).

It has been verified⁽¹¹⁾ that, to one loop, the four quark

matrix elements of the operators (5.3) obey the correct Ward identity.

ii. Dimension-five operators. In the limit $a \rightarrow 0$, we have two possible operators, namely:

$$g_0 \bar{s} \sigma_{\mu\nu} F^{\mu\nu} d \quad (5.4)$$

$$g_0 \bar{s} \sigma_{\mu\nu} \tilde{F}^{\mu\nu} d \quad (5.5)$$

An example of a lattice operator which corresponds to (5.4) is:

$$\frac{1}{2a^2} \bar{s} \sigma_{\mu\nu} (\square_{\mu\nu} - \square_{\mu\nu}^+) d \quad (5.6)$$

$$\square_{\mu\nu} = \text{Tr} (U_\mu(x) U_\nu(x+\mu) U_\mu^+(x+\nu) U_\nu^+(x))$$

In perturbation theory the mixing with these operators arises from the diagrams in Fig.1. A closer inspection shows that the diagram of Fig.1 (a) has no projection over the anomalous magnetic moment operators, while diagrams like those shown in Figs.1 (b) and (c) give a non vanishing contribution.

The mixing coefficients must respect the CPS symmetry defined in eq.(2.6), as well as be antisymmetric under $u \leftrightarrow c$ exchange. This restricts considerably the form of the dimension-five operators we must add to $\mathcal{O}^{(\pm)}$:

$$\delta_5 \mathcal{O}^{(\pm)} = d_\sigma^{(\pm)} g_0 (\bar{s} \sigma_{\mu\nu} F^{\mu\nu} d) + \quad (5.7)$$

$$+ \tilde{d}_\sigma^{(\pm)} g_0 (\bar{s} \sigma_{\mu\nu} \tilde{F}^{\mu\nu} d)$$

$$d_\sigma^{(\pm)} = A_1^{(\pm)} (m_c - m_u) + O\left[(m_s + m_d) \frac{m_c^2 - m_u^2}{\mu^2}\right] \quad (5.8)$$

$$\tilde{d}_\sigma^{(\pm)} = O\left[(m_s - m_d) \frac{m_c^2 - m_u^2}{\mu^2}\right] \quad (5.9)$$

A linearly divergent contribution to d_σ is ruled out by the GIM mechanism. Terms of the form $\tilde{a}^4 \ln(m)$ are excluded by the existence of the chiral limit. A non-vanishing d_σ is due only to lattice artifacts, as can be seen from the fact that eq.(5.8) does not respect the chiral selection rule, which implies that d_σ should vanish for $m_s, d, u \rightarrow 0$. We have not included in d_σ a logarithmically divergent term of the form $A_0 (m_c - m_u) \ln(a)$ permitted by the chiral limit and by the discrete symmetries of $\mathcal{J}^{(\pm)}$. The presence of such a term would be disastrous. It would give to $\mathcal{J}^{(\pm)}$ a μ -dependence, arising from the lattice, which could not be compensated by the μ -dependence of the Wilson coefficients as determined from the continuum theory. Therefore A_0 must vanish. To the two-loop level this is shown to be the case in Appendix III.

As is the case for dimension-six operators, d_σ can be reliably computed in perturbation theory, from the diagrams in Fig.1 (b) and (c) and all related ones. Such a calculation is lacking at present.

iii. Dimension-three operators. We have two possible bilinear quark operators:

$$\delta_3 \mathcal{O}^{(\pm)} = d_S^{(\pm)} (\bar{s} d) + d_P^{(\pm)} (\bar{s} \gamma_5 d) \quad (5.10)$$

The chiral structure of $\mathcal{J}^{(\pm)}$ implies that d_s and d_p are of order g_0^2 at least (a first non vanishing contribution arises from the diagram in Fig.2). CPS symmetry and $u \leftrightarrow c$ antisymmetry forbid a cubically divergent contribution, and imply the general form:

$$d_S^{(\pm)} = C_0^{(\pm)} \frac{m_c - m_u}{a^2} + \dots \quad (5.11)$$

$$d_P^{(\pm)} = D_0^{(\pm)} \frac{(m_c - m_u)(m_s - m_d)}{a} + \dots$$

where only the most divergent terms have been indicated. The severe ultraviolet divergence make a perturbative evaluation completely illusory, and require this subtraction to be performed in a non-perturbative fashion.

Summing up, in the active charm case, the finite operators $\hat{\mathcal{J}}^{(\pm)}$ are given by:

$$\begin{aligned} \hat{\mathcal{J}}^{(\pm)} = Z_{\text{LAT}}^{(\pm)}(\mu a) & \left[\mathcal{J}^{(\pm)} + \delta_6 \mathcal{O}^{(\pm)} + \delta_5 \mathcal{O}^{(\pm)} + \right. \\ & \left. + d_S^{(\pm)} (\bar{s} d) + d_P^{(\pm)} (\bar{s} \gamma_5 d) \right] \end{aligned} \quad (5.12)$$

The case a^{-1} smaller than m_c can be treated along similar lines with only one important difference, namely that the parity conserving, anomalous magnetic moment operator has to be subtracted with a linearly divergent coefficient. Starting from the bare operators $\mathcal{O}_{1-2}$, $\mathcal{O}_{5-6}$ given in Sect.2, the multiplicatively renormalizable operators, $\tilde{\mathcal{O}}_i$ are given by the expression:

$$\begin{aligned} \tilde{\mathcal{O}}_i = & \mathcal{O}_i + \delta_6 \mathcal{O}_i + d_\sigma^{(i)} g_0 (\bar{s} \sigma_{\mu\nu} F^{\mu\nu} d) + \\ & + \tilde{d}_\sigma^{(i)} g_0 (\bar{s} \sigma_{\mu\nu} \tilde{F}^{\mu\nu} d) + d_S^{(i)} (\bar{s} d) + d_P^{(i)} (\bar{s} \gamma_5 d) \end{aligned} \quad (5.13)$$

$\delta_6 \mathcal{O}_i$ is the appropriate superposition of dimension-six operators on the lattice, which has been given in refs.(10) and (25) to one loop in perturbation theory. Dimensional counting and the CPS rule imply that:

$$\begin{aligned} d_\sigma^{(i)} = & A_0^{(i)} \frac{1}{a} + A_1^{(i)} (m_s + m_d) \ln(\mu a) + \\ & + A_2^{(i)} (m_s + m_d) + O(m_q^2) \end{aligned} \quad (5.14)$$

$$\begin{aligned} \tilde{d}_\sigma^{(i)} = & B_1^{(i)} (m_s - m_d) \ln(\mu a) + B_2^{(i)} (m_s - m_d) + \\ & + O(m_q^2) \end{aligned} \quad (5.15)$$

The lack of a GIM cancellation makes d_σ to be linearly divergent, and the corresponding subtraction has to be done non-perturbatively.

This time, logarithmically divergent terms are allowed, since $\tilde{\mathcal{O}}_i$ mix with the anomalous magnetic moment operators already

in the continuum⁽²⁶⁾. The requirement that the whole μ -dependence of the renormalized operator is compensated by the μ -dependence of the Wilson coefficient implies that:

$$A_1^{(i)} = B_1^{(i)} \quad (5.16)$$

both coefficients being equal to the continuum mixing coefficients. The convention adopted in Sects.3 and 4 was to include in the subtractions to $\mathcal{O}_i$ only terms corresponding to lattice artifacts (see eq.(3.11)). To be consistent with this convention, we should enlarge the operator basis, including the anomalous magnetic moment operator $\mathcal{O}_\sigma$, eq.(2.13), and transfer all the logarithmically divergent terms into the overall normalization matrix $(Z_{LATT})_{ij}$.

Consideration of dimension-three operators goes as before, except that now d_s (d_p) is cubically (quadratically) divergent for $a \rightarrow 0$. Summing up, the renormalized (8,1) operators, in the frozen charm case, are given by the expression:

$$\hat{\mathcal{O}}_i = [Z_{LATT}(\mu a)]_{ij} \tilde{\mathcal{O}}_j \quad (5.17a)$$

$$(i, j = 1, 2, 5, 6, \sigma)$$

$$\begin{aligned} \tilde{\mathcal{O}}_i = & \mathcal{O}_i + \delta_6 \mathcal{O}_i + d_\sigma^{(i)} g_0 (\bar{s} \sigma_{\mu\nu} F^{\mu\nu} d) + \\ & + d_S^{(i)} (\bar{s} d) + d_P^{(i)} (\bar{s} \gamma_5 d) \end{aligned} \quad (5.17b)$$

$$(i = 1, 2, 5, 6)$$

Comparing eqs.(5.17) with (5.12) we see that the frozen charm case requires two non-perturbative subtractions, versus only one,

in the parity conserving sector. The parity violating sector has only one non-perturbative subtraction in both cases.

6. The renormalized weak Hamiltonian on the lattice

We are now in the position to assemble all the elements of the weak hamiltonian.

We consider the active charm case first, i.e. the construction of $\mathcal{O}^{(\pm)}$. Assume that $K \rightarrow 0$ and $K \rightarrow \pi$ matrix elements of the bare operators have been computed by a Monte Carlo simulation. The matrix elements can be parametrized as follows^(f):

$$\langle 0 | \mathcal{O}^{(\pm)} + \delta_6 \mathcal{O}^{(\pm)} + \delta_5 \mathcal{O}^{(\pm)} | K^0 \rangle = i \delta_1^{(\pm)} \quad (6.1)$$

$$\langle \pi^+(q) | \mathcal{O}^{(\pm)} + \delta_6 \mathcal{O}^{(\pm)} + \delta_5 \mathcal{O}^{(\pm)} | K^+(k) \rangle = \delta_2^{(\pm)} + (q \cdot k) \gamma_2^{(\pm)} + \dots \quad (6.2)$$

where dots indicate higher order terms in momentum transfer.

Similarly, one can determine the matrix elements of the scalar and pseudoscalar densities:

$$\langle 0 | \bar{s} \gamma_5 d | K^0 \rangle = i \delta_P \quad (6.3)$$

$$\langle \pi^+(q) | \bar{s} d | K^+(k) \rangle = \delta_S + (q \cdot k) \gamma_S + \dots \quad (6.4)$$

(f) The factor of i in eqs.(6.1) and (6.3) has been inserted by hand, to compare with the corresponding Minkowski space quantities.

For $a \rightarrow 0$, $\delta_{1,2}^{(\pm)}$ and $\gamma_{1,2}^{(\pm)}$ are functions of quark masses with quadratic or linearly divergent coefficients, while $\delta_{p,s}$ and γ_s have a common logarithmically divergent factor. Recall that the multiplicatively renormalizable $\tilde{\mathcal{O}}^{(\pm)}$ are given by the expression:

$$\tilde{\mathcal{O}}^{(\pm)} = \mathcal{O}^{(\pm)} + \delta_6 \mathcal{O}^{(\pm)} + \delta_5 \mathcal{O}^{(\pm)} + d_s^{(\pm)} (\bar{s}d) + d_p^{(\pm)} (\bar{s}\gamma_5 d)$$

In correspondence to the normalization conditions, eqs.(2.18) and (2.19), the coefficients $d_{s,p}$ are given by:

$$d_P^{(\pm)} = -\delta_1^{(\pm)} / \delta_P \quad (6.5)$$

$$d_s^{(\pm)} = -\delta_2^{(\pm)} / \delta_s \quad (6.6)$$

Correspondingly, for the $K \rightarrow \pi\pi$ matrix elements of $\tilde{\mathcal{O}}^{(\pm)}$ and of the weak octet hamiltonian we find:

$$\begin{aligned} \langle \pi^+ \pi^- | \tilde{\mathcal{O}}^{(\pm)} | K^0 \rangle &= \\ &= i Z_{LAT}^{(\pm)} (a_\mu) \frac{m_K^2 - m_\pi^2}{f_\pi} \bar{\gamma}^{(\pm)} \end{aligned} \quad (6.7)$$

$$\begin{aligned} \langle \pi^+ \pi^- | (H_{eff})_R | K^0 \rangle &= \frac{G}{\sqrt{2}} \sin \theta \cos \theta \cdot \\ &\cdot \sum_{i=\pm} C^{(i)}(\mu, M_W) \langle \pi^+ \pi^- | \tilde{\mathcal{O}}^{(i)} | K^0 \rangle \end{aligned} \quad (6.8)$$

with:

$$\bar{\gamma}^{(\pm)} = \lim_{m_{u,d,s} \rightarrow 0} \left(\gamma_2^{(\pm)} - \gamma_s \frac{\delta_2^{(\pm)}}{\delta_s} \right) \quad (6.9)$$

This is our main result, which shows that the computation of $K \rightarrow \pi\pi$ matrix element can be performed, on the lattice, by computing the $K \rightarrow \pi$ matrix elements for small momentum transfer, and using PCAC as a way of implementing the subtractions of the divergent quantities.

The frozen charm case, as seen from eqs.(5.17), requires one more subtraction in the parity conserving sector. As a consequence, a computation of $K \rightarrow \pi$ matrix elements alone is not sufficient to determine d_s , d_σ and $\bar{\gamma}$. In this case, the possible strategy is to determine d_p from the $K \rightarrow 0$ matrix elements, and compute directly the $K \rightarrow \pi\pi$ transitions on the lattice. Once this is done, one can determine d_s and d_σ from the $K \rightarrow \pi$ matrix elements, and obtain again the complete octet non-leptonic hamiltonian.

A second result concerns the CP-violating parameter ϵ' , which describes the intrinsic CP-violation in $\Delta S = \pm 1$, K decay amplitudes⁽⁵⁾. It is a simple exercise to insert the mixing of a third family back in the previous considerations. Above charm threshold, CP-violation in the $\Delta S = \pm 1$ hamiltonian is associated with the operators:

$$\mathcal{O}'_5 = (\bar{s}_L \gamma_\mu t^A d_L) (\bar{q}_R \gamma^\mu t^A q_R) \quad (6.10)$$

$$\mathcal{O}'_6 = (\bar{s}_L \gamma_\mu d_L) (\bar{q}_R \gamma^\mu q_R) \quad (6.11)$$

where $\bar{q} \dots q$ is the SU(4) singlet combination. The divergences of $\mathcal{O}'_{5,6}$ can be analyzed in a way identical to that used for $\mathcal{O}_{5,6}$ in Sect.5, with the conclusion that two subtractions are required for the parity conserving component (corresponding to the operators $g_0(\bar{s} \sigma_{\mu\nu} F^{\mu\nu} d)$ and $(\bar{s}d)$) and one for the parity violating component (corresponding to $(\bar{s} \gamma_5 d)$). The conclusion is that, short of putting a dynamical t-quark on the lattice, the only possibility to compute ϵ' is to fix the subtraction in the parity violating channel from the $K \rightarrow 0$ matrix element, and compute directly the $K \rightarrow \pi\pi$ matrix elements of $\mathcal{O}'_{5,6}$ thus renormalized.

Few additional comments may be useful.

- i) The difference between frozen and dynamical charm can be traced back to the hard chiral breaking induced by the Wilson term. This makes $d_s(\pm)$ and $d_p(\pm)$ to be unrelated to each other and d_σ to be non-vanishing. Were it not for the Wilson term, the $K \rightarrow 0$ subtraction, eq.(6.3), would be sufficient to determine the weak hamiltonian in all cases.
- ii) In the continuum theory, the subtraction of $\bar{s} \gamma_5 d$ is not needed to compute the $K \rightarrow \pi\pi$ transition, the latter operator being a total four divergence^(17,27) because of the equations of motion. It is not so on the lattice, where:

$$\langle h_2 | \bar{s} \gamma_5 d | h_1 \rangle = \frac{1}{m_s + m_d} \langle h_2 | \nabla_\mu \hat{A}_\mu - \bar{X} | h_1 \rangle \quad (6.12)$$

Even though the matrix element $\langle h_2 | \bar{X} | h_1 \rangle$ vanishes with the

lattice spacing, it need not do so when multiplied by a divergent coefficient. In fact:

$$d_P \langle \pi^+ \pi^- | \bar{s} \gamma_5 d | K^0 \rangle \neq 0$$

since d_P is linearly (quadratically) divergent in the dynamical (frozen) charm case.

- iii) To specify completely the weak hamiltonian we have used the condition $\delta_R=0$, see e.g., eqs.(6.5) and (6.6). This is not necessary: any finite value of δ_R would simply add irrelevant terms to the final result, which vanish when the chiral limit in eq.(6.9) is taken. Indeed, as discussed in the Introduction and later in Sections 3 and 4, only the chiral limit of the weak hamiltonian is unambiguously defined. We must use chiral perturbation theory to relate our calculation to the physical $K \rightarrow \pi \pi$ amplitude.
- iv) Once the weak hamiltonian is determined from the pseudoscalar meson matrix elements, one can use it to compute any other weak transition in the chiral limit, notably the non-leptonic decay amplitudes of strange baryons.

Appendix I.-Effective Lagrangians and chiral perturbation theory.

We derive here the form of the matrix elements of chiral operators between pseudoscalar mesons, in the framework of the so-called chiral perturbation theory. A compact way of generating all matrix elements is provided by the language of the non-linear effective lagrangians^(15,17,28). The basic tool is the SU(3) matrix which realizes a non-linear representation of the chiral symmetry in the Goldstone boson sector:

$$\Sigma(x) = \exp\left(i \frac{2}{f_\pi} \pi(x)\right) \quad (\text{A.I.1})$$

$$\pi(x) = \pi^a(x) \frac{\lambda^a}{\sqrt{2}} \quad (\text{A.I.2})$$

π^a ($a=1,\dots,8$) represent the (real) octet components of the Goldstone pseudoscalar bosons. Under a chiral transformation Σ transforms according to:

$$\Sigma \rightarrow U \Sigma V^\dagger \quad (\text{A.I.3})$$

where U and V are constant SU(3) matrices. The effective, strong interaction lagrangian, at most quadratic in the momenta is:

$$\mathcal{L}_S = \frac{f_\pi^2}{8} \text{Tr}(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger) + v \text{Tr}(\Sigma m^\dagger + m \Sigma^\dagger) \quad (\text{A.I.4})$$

where $f_\pi \cong 135$ MeV is the π -decay constant, m the quark mass matrix which is related to the mass of the pseudoscalar mesons according to, e.g.:

$$m_{\pi^+}^2 = \frac{4v}{f_\pi^2} (m_u + m_d) \quad (\text{A.I.5})$$

The invariance of the lagrangian (A.I.4) under the transformations (A.I.3) is broken by the mass term. However, $\mathcal{L}_s$ is invariant if we transform:

$$m \rightarrow U m V^\dagger \quad (\text{A.I.6})$$

together with Σ . This invariance is in fact equivalent to the full set of the global broken symmetry Ward identities, (as discussed in Appendix II).

The construction of the matrix elements of an operator $\mathcal{O}$, transforming according to a given representation R of $SU(3)_L \times SU(3)_R$, goes as follows. The one-particle irreducible matrix elements are obtained by constructing the most general tensor transforming like R , out of Σ and m . The complete matrix elements are then obtained from the tree diagrams generated by the irreducible vertices and by the effective lagrangian (A.I.4).

Within chiral perturbation theory, we restrict ourselves to invariants which contain at most two powers of the external momenta or one power of m , consistently with eq.(A.I.5). We are interested in the matrix elements of CP-even operators transforming in two different ways.

- i) (8,1). Any such operator is characterized by a 3x3 matrix, Λ , real and symmetric for CP even operators (e.g. $\Lambda = \lambda^6$ for $\Delta S = \pm 1$ transitions). The irreducible vertices take the form:

$$\begin{aligned} \Gamma_{(8,1)}(\Sigma, m) = & \alpha_1 \text{Tr}(\Lambda \partial_\mu \Sigma \partial_\mu \Sigma^\dagger) + \\ & + \alpha_2 \frac{4\nu}{f_\pi^2} \text{Tr}[\Lambda (m \Sigma^\dagger + \Sigma m^\dagger)] \quad (\text{A.I.7}) \\ \Lambda \xrightarrow{\text{CP}} \Lambda^\pi \end{aligned}$$

Specializing to $\Lambda = \lambda^6$, eq.(A.I.7) gives the following matrix elements:

$$\langle 0 | \mathcal{O}_{(8,1)} | K^0 \rangle = i 2 \alpha_2 \frac{m_K^2 - m_\pi^2}{f_\pi} \quad (\text{A.I.8})$$

$$\langle \pi^+(q) | \mathcal{O}_{(8,1)} | K^+(k) \rangle = \frac{4\alpha_1}{f_\pi^2} (q \cdot k) - 2 \alpha_2 \frac{m_K^2}{f_\pi^2} \quad (\text{A.I.9})$$

$$\langle \pi^+ \pi^- | \mathcal{O}_{(8,1)} | K^0 \rangle = i 4 \alpha_1 \frac{m_K^2 - m_\pi^2}{f_\pi^3} \quad (\text{A.I.10})$$

External states are on the mass-shell, the $K^0 \rightarrow \pi^+ \pi^-$ amplitude is computed at vanishing momentum transfer and pole diagrams have been included when needed. For $\Delta S = \pm 1$, the mass dependent coupling in (A.I.7) is a total four-

Appendix II.-Ward identities for a softly broken symmetry⁽²⁴⁾

We start from the generating functional:

$$e^{W(J, \sigma, \pi, \dots)} = \int \mathcal{D}q \mathcal{D}\bar{q} \dots \exp \left[-S + \int J \hat{\mathcal{O}} + \dots \right] \quad (\text{A.II.1})$$

S is the QCD action, which we split into a mass-independent term, S_0 , and generalized mass terms:

$$S = S_0 + \sigma^a \bar{q} \frac{\lambda^a}{2} q + i \pi^a \bar{q} \frac{\lambda^a}{2} \gamma_5 q \quad (\text{A.II.2})$$

We have introduced fictitious scalar and pseudoscalar fields, which, in the end will be set equal to:

$$\sigma^a \frac{\lambda^a}{2} = m \quad (\text{A.II.3})$$

$$\pi^a = 0 \quad (\text{A.II.4})$$

m is the physical (bare) quark mass, to be identified with $M_0 - \bar{M}$, eq.(3.8) in the lattice case. $\hat{\mathcal{O}}$ is an operator which transforms according to the representation R and dots represent additional source terms, needed to generate the external states.

The axial current Ward identities follow from the invariance of W under a simultaneous transformation of J (according to $\bar{R}$) and of σ and π (according to the $(3, \bar{3})$ representation). The

invariance of W implies:

$$\begin{aligned}
 & - \int dx \frac{\delta W}{\delta \sigma^b(x)} T_2 \left(\lambda^b \left\{ \frac{\lambda^a}{2}, \pi(x) \right\} \right) + \int dx \frac{\delta W}{\delta \pi^b(x)} T_2 \left(\lambda^b \left\{ \frac{\lambda^a}{2}, \sigma(x) \right\} \right) + \\
 & + i \int dx \frac{\delta W}{\delta J_a(x)} \left(R_{a\beta}^a \right)^* J_\beta(x) = 0
 \end{aligned} \tag{A.II.5}$$

Taking a further derivative with respect to J , one obtains:

$$\left. - \int dx \frac{\delta^2 W}{\delta \pi^b(x) \delta J_a(y)} T_2 \left(\lambda^b \left\{ \frac{\lambda^a}{2}, \sigma(x) \right\} \right) \right|_{\substack{J=\pi=0 \\ \sigma=m}} = \tag{A.II.6}$$

$$= -i R_{a\beta}^a \frac{\delta W}{\delta J_\beta(y)} \bigg|_{\substack{J=\pi=0 \\ \sigma=m}}$$

This equation coincides with the Ward identity given in eq.(4.3) of the text, under the identification

$$\langle Z \tilde{\theta}_\alpha(0) \rangle = \frac{\delta W}{\delta J_\alpha(0)} \Big|_{\substack{J=\pi=0 \\ \sigma=m}} \quad (\text{A.II.7})$$

$$\begin{aligned} i \int dx \langle T^* (\bar{q}(x) \{ \frac{\lambda^a}{2} \gamma_5, m \} q(x) Z \tilde{\theta}_\alpha(0)) \rangle &= \\ = \int dx \frac{\delta^2 W}{\delta \pi^b(x) \delta J_\alpha(0)} T_2 \left(\lambda^b \{ \frac{\lambda^a}{2}, \sigma(x) \} \right) \Big|_{\substack{J=\pi=0 \\ \sigma=m}} & \quad (\text{A.II.8}) \end{aligned}$$

We can determine now all possible changes in $\hat{\theta} = Z \tilde{\theta}$ which can be compensated by a redefinition of the Green's function of $\hat{\theta}$ with the mass insertion, in such a way that the structure of the Ward identity remains unchanged.

We consider the replacement:

$$\int dx J(x) \hat{\theta}(x) \rightarrow \int dx J(x) [\hat{\theta}(x) + \theta'(\sigma, \pi, q, \bar{q}, \dots)] \quad (\text{A.II.9})$$

where θ' is a local function of all fields. Provided that θ' transforms in the same way as $\hat{\theta}$ under a variation of all fields, σ and π included, the substitution (A.II.9) gives a new, equally invariant generating functional, $W'(J, \sigma, \pi, \dots)$. As a consequence, the new Ward identity keeps the same form as before,

with the replacements:

$$\langle h_2 | \hat{\mathcal{O}} | h_1 \rangle \rightarrow \langle h_2 | (\hat{\mathcal{O}} + \mathcal{O}') | h_1 \rangle \quad (\text{A.II.10})$$

$$\begin{aligned} & \int dx \langle h_2 | T^* (\bar{q}(x) \{ \frac{\lambda^a}{2} \delta_{5,m} \} q(x) \hat{\mathcal{O}}(0)) | h_1 \rangle \rightarrow \\ & \rightarrow \int dx \langle h_2 | T^* [\bar{q}(x) \{ \frac{\lambda^a}{2} \delta_{5,m} \} q(x) (\hat{\mathcal{O}}(0) + \mathcal{O}'(0))] | h_1 \rangle + \\ & + \int dx \langle h_2 | \frac{\delta \mathcal{O}'(0)}{\delta \pi^b(x)} T_2 (\lambda^b \{ \frac{\lambda^a}{2}, \sigma(x) \}) | h_1 \rangle \Big|_{\substack{J=\pi=0 \\ \sigma=m}} \equiv (\text{A.II.11}) \\ & \equiv \int dx \langle h_2 | T^* [\bar{q}(x) \{ \frac{\lambda^a}{2} \delta_{5,m} \} q(x) (\hat{\mathcal{O}}(0) + \mathcal{O}'(0))] | h_1 \rangle \end{aligned}$$

In conclusion, the Ward identity determines the operator $\hat{\mathcal{O}}$ only up to ambiguities of the form given in eq.(A.II.9), namely operators which transform according to R if we vary simultaneously quark fields and the mass matrix according to eqs.(2.7) and (2.8) of the text.

In the case of interest, where $\hat{\mathcal{O}}$ belongs to an (8,1), one sees immediately that the ambiguous terms are combinations of the two operators:

$$\mathcal{O}_m^a = \bar{q}_L \frac{\lambda^a}{2} m q_R + \bar{q}_R m^\dagger \frac{\lambda^a}{2} q_L$$

$$\mathcal{O}_\sigma^a = g_0 \left[\bar{q}_L \frac{\lambda^a}{2} m \sigma_{\mu\nu} F^{\mu\nu} q_R + \bar{q}_R m^\dagger \frac{\lambda^a}{2} \sigma_{\mu\nu} F^{\mu\nu} q_L \right]$$

to first order in m.

Appendix III.-Logarithmic divergences in the mixing of $\mathcal{J}^{(\pm)}$ with the anomalous magnetic moment operator

We consider the two-loop diagram of Fig. 3, where the four quark vertex represents the insertion of $\mathcal{J}^{(\pm)}$ (eqs.(2.16), (2.17) of text). Solid, internal lines denote the lattice quark propagators ($a = \text{lattice spacing} = 1$):

$$\begin{aligned} S(k) &= \Delta^{-1}(k) \left[i \sum_{\mu} \gamma_{\mu} \sin k_{\mu} + m + W(k) \right] \\ \Delta(k) &= \sum_{\mu} \sin^2 k_{\mu} + (m + W(k))^2 \\ W(k) &= 2\tau \sum_{\mu} \sin^2(k_{\mu}/2) \end{aligned} \quad (\text{A.III.1})$$

Wavy lines denote gluons and the lattice quark-gluon vertex is:

$$V_{\mu}(p \rightarrow p') = -ig_0 t^A \left[\gamma_{\mu} \cos \left(\frac{p+p'}{2} \right)_{\mu} - i\tau \sin \left(\frac{p+p'}{2} \right)_{\mu} \right] \quad (\text{A.III.2})$$

$w(k)$ and the second term in the bracket of (A.III.2) are the chiral symmetry breaking parts induced by the Wilson term. A charmed or an up quark run in the loop, and the two contributions have to be subtracted from each other. Fig.3 gives one of the possible two-loop diagrams whereby $\mathcal{J}^{(\pm)}$ is mixed with the operator:

$$g_0 \bar{\psi} \sigma_{\mu\nu} F^{\mu\nu} \psi \quad (\text{A.III.3})$$

The extension of the following analysis to the other diagrams necessary to compute the mixing coefficient is straightforward.

We want to show that the diagram in Fig.3 does not give rise to a term of the form:

$$A_0 (m_c - m_u) \ln(\mu a) \bar{q}_0 \bar{s} \sigma_{\mu\nu} F^{\mu\nu} d \quad (\text{A.III.4})$$

with a coefficient A_0 independent of the external m_s and m_d masses (μ is the subtraction point, $\mu > m_c$). If we denote by A_c the projection over the operator (A.III.3) of the diagram with only the charmed quark running in the loop, this amounts to saying that:

$$\left. \frac{\partial A_c}{\partial m_c} \right|_{m_c=0} = \text{infrared finite} \quad (\text{A.III.5})$$

for vanishing external momenta and masses. The argument goes in two steps.

i) We consider first the subintegration over q . To obtain the part of A_c which is independent of the external masses we must allow for a violation of chirality along the strange quark line, i.e. we must take either the Wilson term $w(q+p')$ in the numerator of the strange quark propagator, or the anomalous part of the gluon $-\bar{s}$ - s vertex. In either case, the product (propagator) $\times V_\mu$ for the strange quark line is infrared finite. Setting external

momenta and masses to zero, and keeping m_c to first order, the $q \rightarrow 0$ behaviour of the diagram Fig.3 is:

$$m_c \int \frac{d^4 q}{q^2} \frac{1}{(k-q)^2} \quad (\text{A.III.6})$$

in the worst case in which the m_c insertion is done in the line with momentum $k'=k-q+p-p'$. Projecting over the operator (A.III.3) increases by one unit the degree of divergence, since we have to extract one power of p or p' . In any case, we find from (A.III.6) a convergent subintegration, for $k \neq 0$.

ii) Infrared divergence may thus arise only from the final k integration and it is determined by the overall superficial degree of infrared divergence. It is crucial to note that a single m_c insertion in the fermion loop gives a non-vanishing contribution only if it is accompanied by a chiral breaking term into another c -quark propagator or into one of the gluon vertices. Thus, unlike the continuum case, the m_c insertion does not increase the degree of superficial infrared divergence. Setting again all external momenta and masses to zero, and to first order in m_c , one finds that the diagram behaves in the infrared according to:

$$m_c \int \frac{d^4 q}{q^2} \frac{d^4 k}{k^2} \frac{1}{(k-q)^2} \quad (\text{A.III.7})$$

A_c has one more unit in the degree of divergence, due to the projection over the operator (A.III.3), but it is nonetheless infrared convergent.

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Figure Captions

- Fig.1 - Typical diagrams which induce the mixing of a four-fermion operator with the anomalous magnetic moment operators, $\bar{s}\sigma_{\mu\nu}F^{\mu\nu}d$ and $\bar{s}\sigma_{\mu\nu}\tilde{F}^{\mu\nu}d$.
- Fig.2 - Typical diagram for the mixing of a four fermion operator with $\bar{s}d$ and $\bar{s}\gamma_5 d$.
- Fig.3 - Kinematics of the diagram in Fig. 1(b).

divergence, when the equations of motion derived by (A.I.4) are used. This is the reason why no d_2 term is present in (A.I.10).

- ii) $(3, \bar{3})$. By CP invariance, any such operator is characterized by a real (not necessarily symmetric) 3×3 matrix, $\textcircled{H}$. Irreducible vertices are generated by:

$$\begin{aligned} \Gamma_{(3, \bar{3})}(\Sigma, m) = & \beta_1 \text{Tr}(\textcircled{H} \Sigma^+) + \beta_2 \text{Tr}(\textcircled{H} \square \Sigma^+) + \\ & + \frac{4v}{f_\pi^2} \left[\beta_3 \text{Tr}(\textcircled{H} \Sigma^+ m \Sigma^+) + \beta_4 \text{Tr}(\textcircled{H} \Sigma^+) \text{Tr}(m \Sigma^+) + \right. \\ & \left. + \beta_5 \text{Tr}(\textcircled{H} \Sigma^+) \text{Tr}(m^+ \Sigma) \right] + (\text{CP-conjugate}) \end{aligned} \quad (\text{A.I.11})$$

For $\Delta S = \pm 1$ operators,

$$\begin{aligned} \textcircled{H} &= \frac{1}{2} (c_s \lambda^6 + i c_p \lambda^7) \\ \textcircled{H} &\xrightarrow{\text{CP}} \textcircled{H}^* \end{aligned} \quad (\text{A.I.12})$$

the two components differ in their behaviour under $s \leftrightarrow d$ exchange. Invariance under this operation requires c_p (c_s) to be antisymmetric (symmetric) under exchange of s and d quark masses, as indeed, is found in the explicit formulae of Sections 2 and 5. From eq.(A.I.11) one derives:

$$\langle 0 | \mathcal{O}_{(3,\bar{3})} | K^0 \rangle = i \frac{2C_P}{f_\pi} \left[\beta_1 + m_K^2 (-\beta_2 + \beta_3 + \beta_4 + \beta_5) + \frac{m_\pi^2}{2} (\beta_4 + \beta_5) \right] \quad (\text{A.I.13})$$

$$\begin{aligned} \langle \pi^+(q) | \mathcal{O}_{(3,\bar{3})} | K^+(k) \rangle = & -2 \frac{C_S}{f_\pi^2} \left[\beta_1 + m_K^2 (-\beta_2 + \beta_3 + \beta_4 + \beta_5) + \right. \\ & \left. + \frac{m_\pi^2}{2} (-2\beta_2 + 2\beta_3 + \beta_4 + \beta_5) + 2(q \cdot k) \beta_2 \right] \end{aligned} \quad (\text{A.I.14})$$

$$\langle \pi^+ \pi^- | \mathcal{O}_{(3,\bar{3})} | K^0 \rangle = -i \frac{4C_P}{f_\pi^3} m_\pi^2 (\beta_3 + \beta_4 + \beta_5) \quad (\text{A.I.15})$$

The finite, ambiguous terms that have been found in Section 4, are linear combinations of the operators $\mathcal{O}_m(\mu)$ and $\mathcal{O}_\sigma(\mu)$, eqs.(2.5) and (2.13) of text. They correspond to $(3,\bar{3})$ operators with coefficients C_P and C_S related to quark masses according to:

$$\begin{aligned} C_S &= C (m_s + m_d) \\ C_P &= C (m_s - m_d) \end{aligned} \quad (\text{A.I.16})$$

with C a finite coefficient. To first order in quark masses, we have to keep only the β_1 term in eqs.(A.I.13) to (A.I.15) and we see that the addition of such terms is equivalent to a redefinition of α_2 in eqs.(A.I.8) to (A.I.10). The addition of such terms to an $(8,1)$ operator does not affect the coefficient of $(q \cdot k)$ in the $K \rightarrow \pi$ matrix element, nor the relation between this coefficient and the $K \rightarrow \pi\pi$ matrix element. For definiteness, the renormalized operator $\mathcal{O}_\sigma(\mu)$ is defined in the text *so* as to obey the condition (2.26). In this case, β_1 vanishes and all the matrix elements of $\mathcal{O}_\sigma(\mu)$ between pseudoscalars are of order m_s^2 .

a)

b)

c)

fig.1

fig.2

fig.3