Braneworlds, Conformal Fields and the Gravitons

Rui Neves*

Centro de Electrónica, Optoelectrónica e Telecomunicações (CEOT)
Faculdade de Ciências e Tecnologia
Universidade do Algarve
Campus de Gambelas, 8005-139 Faro, Portugal

Abstract

We investigate the dynamics of Randall-Sundrum AdS$_5$ braneworlds with 5-dimensional conformal matter fields. In the scenario with a compact fifth dimension the class of conformal fields with weight -4 is associated with exact 5-dimensional warped geometries which are stable under radion field perturbations and describe on the brane the dynamics of inhomogeneous dust, generalized dark radiation and homogeneous polytropic dark energy. We analyse the graviton mode fluctuations around this class of background solutions and determine their mass eigenvalues and wavefunctions from a Sturm-Liouville problem. We show that the localization of gravity is not sharp enough for large mass hierarchies to be generated. We also discuss the physical bounds imposed by experiments in particle physics, in astrophysics and in precise measurements of the low energy gravitational interaction.

1 Introduction

In the AdS$_5$ Randall-Sundrum (RS) scenario [1, 2] our visible 4-dimensional (4D) Universe is a 3-brane world embedded in a $Z_2$ symmetric 5D anti-de Sitter (AdS) space. In the RS1 model [1] the fifth dimension is compact and there are two 3-brane boundaries. In this setting gravity is exponentially localized near the hidden positive tension brane and decays towards the observable negative tension brane. The hierarchy problem is then reformulated as an exponential hierarchy between the weak and Planck scales [1]. In the RS2 model [2] the AdS$_5$ orbifold has an infinite fifth dimension and a single visible positive tension brane to which the gravitational field is exponentially bound.

*E-mail: rneves@ualg.pt. Also at Centro Multidisciplinar de Astrofísica - CENTRA, Instituto Superior Técnico, Avenida Rovisco Pais, 1049-001 Lisboa and Departamento de Física, Faculdade de Ciências e Tecnologia, Universidade do Algarve, Campus de Gambelas, 8005-139 Faro, Portugal.
In the visible brane the low energy theory of gravity is 4D general relativity and the cosmology may be Friedmann-Robertson-Walker [1]-[10]. In the RS1 model this requires the stabilization of the radion mode with for example a 5D scalar field [3, 6, 9, 10]. Using the Gauss-Codazzi formulation [11, 12] many other braneworld solutions have been discovered although a number of them have not yet been associated with exact 5D spacetimes [13]-[16].

In this paper we continue the research about the dynamics of a spherically symmetric RS 3-brane when conformal matter fields propagate in the bulk [17]-[20] (see also [21]). Some time ago [17, 18] we have discovered a new class of exact 5D dynamical warped solutions which is associated with conformal fields characterized by an energy-momentum tensor of weight -4. These solutions were shown to describe on the brane the dynamics of inhomogeneous dust, generalized dark radiation and homogeneous polytropic matter [17, 18]. The latter in particular refers to a perfect dark energy fluid and describes the accelerated expansion of our Universe. The radion may be stabilized by a sector of the conformal bulk fields of weight -4 while another sector generates the dynamics on the brane. The stabilization requires a bulk fluid sector with a constant negative 5D pressure and involves new warp functions [20].

If the theory of gravity on the brane deviates from that of Einstein the existence of such dynamical geometries requires the presence of non-conformal matter fields confined to the brane [22]. In this work we analyse the graviton field perturbations around this class of background geometries and determine their mass eigenvalues and wavefunctions from a Sturm-Liouville problem. We show that gravity is not sufficiently localized near the positive tension branes to be able to generate large mass hierarchies. We also discuss the physical bounds imposed by experiments in particle physics, in astrophysics and in precise measurements of Newton’s law of gravity.

2 5D Einstein equations and conformal fields

The most general non-factorizable dynamical metric consistent with the $Z_2$ symmetry in the fifth dimension and with 4D spherical symmetry on the brane is given by

$$d\tilde{s}_5^2 = \Omega^2 \left( dz^2 - e^{2A} dt^2 + e^{2B} dr^2 + R^2 d\Omega_2^2 \right),$$

where $(t, r, \theta, \phi, z)$ are the coordinates mapping the AdS$_5$ orbifold. In this set $z$ is related to the cartesian coordinate $y$ by $z = le^{y/l}$, $y > 0$, with $l$ the AdS radius. The functions $\Omega = \Omega(t, r, z)$, $A = A(t, r, z)$, $B = B(t, r, z)$ and $R = R(t, r, z)$ are $Z_2$ symmetric. $R(t, r, z)$ represents the physical radius of the 2-spheres and $\Omega$ is the warp factor characterizing a global conformal transformation on the metric.

In the RS1 model the classical dynamics is defined by the 5D Einstein equations,

$$\tilde{G}_{\mu}^{\nu} = -\kappa_5^2 \left\{ A_B \delta_{\mu}^{\nu} + \frac{1}{\sqrt{g_{55}}} \left[ \lambda \delta \left( z - z_0 \right) + \lambda' \delta \left( z - z_0' \right) \right] \left( \delta_{\mu}^{\nu} - \delta_{5}^{\nu} \delta_{5}^{\mu} \right) - \tilde{T}_{\mu}^{\nu} \right\},$$

where $(t, r, \theta, \phi, z)$ are the coordinates mapping the AdS$_5$ orbifold. In this set $z$ is related to the cartesian coordinate $y$ by $z = le^{y/l}$, $y > 0$, with $l$ the AdS radius. The functions $\Omega = \Omega(t, r, z)$, $A = A(t, r, z)$, $B = B(t, r, z)$ and $R = R(t, r, z)$ are $Z_2$ symmetric. $R(t, r, z)$ represents the physical radius of the 2-spheres and $\Omega$ is the warp factor characterizing a global conformal transformation on the metric.
where $\Lambda_B$ is the negative bulk cosmological constant and $\kappa_5^2 = 8\pi/M_5^3$ with $M_5$ the fundamental 5D mass scale. $\lambda, \lambda'$ are the brane tensions and $\bar{T}_\mu$ is the stress-energy tensor of the matter fields which in 5D is conserved,

$$\bar{\nabla}_\mu \bar{T}_\mu^\nu = 0. \quad (3)$$

For a general 5D metric $\bar{g}_{\mu\nu}$ [2] and [3] form a complex system of differential equations. To solve it let us introduce some simplifying assumptions [20]. First consider that the bulk matter is described by conformal field s with weight $s$. Under the conformal transformation $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ the stress-energy tensor satisfies $\bar{T}_\mu^\nu = \Omega^{s+2} T_\mu^\nu$. Then separate the conformal tensor $\bar{T}_\mu^\nu$ in two sectors $\bar{T}_\mu^\nu$ and $\bar{U}_\mu^\nu$ with the same weight $s$, $\bar{T}_\mu^\nu = \bar{T}_\mu^\nu + \bar{U}_\mu^\nu$ where $\bar{T}_\mu^\nu = \Omega^{s+2} T_\mu^\nu$ and $\bar{U}_\mu^\nu = \Omega^{s+2} U_\mu^\nu$, and take $s = -4$. Finally consider $A = A(t, r), B = B(t, r), R = R(t, r)$ and $\Omega = \Omega(z)$. Then [2] leads to [20]

$$G_a^b = \kappa_5^2 T_a^b, \quad (4)$$

$$G_5^r = \kappa_5^2 T_5^r, \quad (5)$$

$$6\Omega^{-2}(\partial_z \Omega)^2 + \kappa_5^2 \Omega^2 \Lambda_B = \kappa_5^2 U_5^r, \quad (6)$$

$$\left\{3\Omega^{-1}(\Omega + \kappa_5^2 \Omega^2 \left[\Lambda_B + \Omega^{-1} \left[\lambda \delta(z - z_0) + \lambda' \delta(z - z')\right]\right])\right\} \delta_a^b = \kappa_5^2 U_a^b, \quad (7)$$

where the latin indices represent the 4D coordinates $t, r, \theta$ and $\phi$. On the other hand from [3] we also obtain [20]

$$\nabla_a T_a^0 = 0 \quad (8)$$

and the equations of state $2T_5^r = T_a^a, \quad 2U_5^r = U_a^a$. $U_\mu^\nu$ turns out to be a constant diagonal tensor, $U_\mu^\nu = \text{diag}(\bar{\rho}, \bar{p}_t, \bar{p}_T, \bar{p}_5)$, with the density $\bar{\rho}$ and pressures $\bar{p}_t, \bar{p}_T, \bar{p}_5$ satisfying $\bar{p}_5 = -2\bar{\rho} = 2\bar{p}_t = 2\bar{p}_T$. Note that $\nabla_a U_a^0 = 0$ is an identity. If $T_\mu^\nu = \text{diag}(\rho, p_t, p_T, p_5)$ where $\rho, p_t, p_T$ and $p_5$ are, respectively, the density and pressures then its equation of state is re-written as

$$\rho - p_t - 2p_T + 2p_5 = 0. \quad (9)$$

## 3 Exact 5D warped solutions


$$\Omega(y) = e^{-|y|/l} \left(1 + p_5^5 e^{2|y|/l}\right), \quad \lambda = \lambda_{RS} \frac{1 - p_5^5}{1 + p_5^5}, \quad \lambda' = -\lambda_{RS} \frac{1 - p_5^5 \exp(2\pi r_c/l)}{1 + p_5^5 \exp(2\pi r_c/l)}, \quad (10)$$

where $p_5^5 = p_5/(4\Lambda_B), \lambda_{RS} = 6/(l\kappa_5^2)$ and $r_c$ is the RS compactification scale.

To determine the dynamics on the brane we solve [4] and [5] when $T_\mu^\nu$ satisfies [8] and [10] [20]. Note that as long as $p_5$ balances $\rho, p_t$ and $p_T$ according to [5] and
the 4D equation of state is not constrained. Three examples corresponding to inhomogeneous dust, generalized dark radiation and homogeneous polytropic matter were considered in [17] and [18]. The latter describes the dynamics on the brane of dark energy in the form of a polytropic fluid. The 5D dark energy polytropic solutions are [18]

\[
\tilde{S}^2 = \Omega^2 \left[ -dt^2 + S^2 \left( \frac{dr^2}{1 - k r^2} + r^2 d\Omega^2 \right) \right] + dy^2,
\]

where \( S \) satisfies \( S \dot{S}^2 = \kappa_5^2 (\eta S^3 - 3a) \). \( \alpha \) and \( \eta \) characterize different polytropic phases. For \( -1 \leq \alpha < 0 \) the fluid is in its generalized Chaplygin phase.

### 4 Radion stability

Provided the equation of state of the conformal fluid is independent of the radion perturbation, these solutions are stable for a range of the model parameters if \( p_5^B > 0 \) [20]. Using \( x = y/r_c \) the radion mass is given in terms of the dimensionless radion mass parameter \( M \) (see figure 1)

\[
m_\gamma = \frac{1}{r_c} \sqrt{\frac{4|M|}{3 \int_0^\pi dx \Omega^2}}, \quad M = \lambda r_c \kappa_5^2 \Omega^4(0) + \lambda' r_c \kappa_5^2 \Omega^4(\pi) - \frac{6r_c^2}{l^2} \int_0^\pi dx \Omega^4.
\]

![Figure 1: Plot of radion mass parameter M for l/r_c = 5.](image)

For \( p_5^B > 0 \) the stability of the AdS5 braneworlds also depends on the dimensionless ratio \( l/r_c \). For \( l/r_c < 1.589 \cdots \) all solutions turn out to be unstable. Stable universes begin to appear at \( l/r_c = 1.589 \cdots, p_5^B = 0.138 \cdots \). For \( l/r_c > 1.589 \cdots \) we find stable solutions for an interval of \( p_5^B \) (see in figure 1 the example of \( l/r_c = 5 \)) which increases with \( l/r_c \). For large enough but finite \( l/r_c \) the stability interval approaches the limit \( 0.267 \cdots, 3.731 \cdots \). Naturally, \( M \to 0 \) if \( l/r_c \to \infty \).

The particle physics bound for the radion mass is \( m_\gamma > 35 - 120 \text{ GeV} \) [23]. For fixed \( l/r_c \) the order of magnitude bound \( m_\gamma > 100 \text{ GeV} \) requires a small fifth
dimension, \( r_c < 1 \times 10^{-15} \) mm. Since the 5D mass scale is related to the 4D Planck mass \( M_5^2 = 8\pi/\kappa_5^2 \sim 1 \times 10^{19} \) GeV by \( M_5^2 r_c^2 \int dx \Omega^2 = M_4^2 \) we conclude that \( M_5 > 2 \times 10^{10} \) TeV. Consequently this radion is invisible in astrophysical processes which require at least \( m_\gamma > 100 \) MeV [23]. It is also invisible in precision Newton’s law measurements which imply \( m_\gamma > 6.25 \times 10^{-4} \) eV [23].

If the scale \( M_5 \) is high enough then the radion may be light. The direct bound from particle physics is \( M_5 > 1 - 2 \) TeV [23]. From astrophysics it may be as high as \( M_5 > 1 - 200 \) TeV [23]. Now \( m_\gamma > 6.25 \times 10^{-4} \) eV corresponds to a large fifth dimension with \( r_c < 160 \mu \text{m} \). Then the 5D scale is indeed high, \( M_5 > 4 \times 10^5 \) TeV.

These results are a consequence of the absence of a large mass hierarchy, \( m = \Omega(\pi r_c)m_0 \), \( \Omega(\pi r_c) \sim 1 - 100 \), and as order of magnitude estimates they hold for all the \( p_B^5 \) stability range.

### 5 The gravitons

To analyse the graviton field perturbations around the background \( \tilde{g}_{\mu\nu} \) let us write the classical 5D Einstein equations as

\[
\tilde{G}_{\mu\nu} = -\kappa_5^2 \tilde{\mathcal{M}}_{\mu\nu},
\]

where the energy-momentum tensor \( \tilde{\mathcal{M}}_{\mu\nu} \) is

\[
\tilde{\mathcal{M}}_{\mu\nu} = \frac{1}{\sqrt{g_{\mu\nu}}} \left[ \lambda \delta (y - y_0) + \lambda' \delta (y - y'_0) \right] \left( \tilde{g}_{\mu\nu} - \delta^{\mu}_\nu \tilde{g}_{\nu} + \tilde{g}_{\mu\nu} \delta^{\mu}_\nu + \tilde{g}_{\nu} \delta^{\mu}_\nu \right)
+ \Lambda B \tilde{g}_{\mu\nu} - \tilde{T}_{\mu\nu}.
\]

Consider the graviton perturbation \( h_{\mu\nu} \) such that \( \tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \). Using \( \hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \) we find

\[
\tilde{G}_{\mu\nu} = \bar{G}_{\mu\nu} - \frac{1}{2} \left( \nabla^2 \hat{h}_{\mu\nu} + \bar{g}_{\mu\nu} \nabla^\alpha \nabla^\beta \hat{h}_{\alpha\beta} \right) + \nabla_{(\mu} \nabla^{\alpha} \hat{h}_{\alpha\nu)} + \bar{R}_{\mu\nu}^\beta \hat{h}_{\alpha\beta}
+ \bar{G}_{(\mu} \hat{h}_{\nu)\beta} + \frac{1}{6} \bar{R} \hat{g}_{\mu\nu}.
\]

On the other hand \( \tilde{\mathcal{M}}_{\mu\nu} = \bar{\mathcal{M}}_{\mu\nu} + \delta \mathcal{M}_{\mu\nu} \). Since \( \bar{G}_{\mu\nu} = -\kappa_5^2 \bar{\mathcal{M}}_{\mu\nu} \) we obtain

\[
-\frac{1}{2} \left( \nabla^2 \hat{h}_{\mu\nu} + \bar{g}_{\mu\nu} \nabla^\alpha \nabla^\beta \hat{h}_{\alpha\beta} \right) + \nabla_{(\mu} \nabla^{\alpha} \hat{h}_{\alpha\nu)} + \bar{R}_{\mu\nu}^\beta \hat{h}_{\alpha\beta} =
- \bar{G}_{(\mu} \hat{h}_{\nu)\beta} - \frac{1}{6} \bar{R} \hat{g}_{\mu\nu} - \kappa_5^2 \delta \mathcal{M}_{\mu\nu}.
\]

Assume a flat brane background \( \tilde{g}_{ab} = \Omega^2(y)\eta_{ab}, \tilde{g}_{55} = 1, \tilde{g}_{5a} = \tilde{g}_{a5} = 0, \forall a \). Consider also the graviton perturbation \( \tilde{g}_{ab} = \bar{g}_{ab} + h_{ab}, \tilde{g}_{55} = \bar{g}_{55}, \tilde{g}_{5a} = \bar{g}_{5a} \) and the RS gauge \( \partial^h h_{ab} = h = 0 \). Then

\[
-\frac{1}{2} \nabla^2 h_{\mu\nu} + \bar{R}_{\mu\nu}^\beta \hat{h}_{\alpha\beta} + \bar{G}_{(\mu} \hat{h}_{\nu)\beta} = -\kappa_5^2 \delta \mathcal{M}_{\mu\nu}.
\]
Expanding we find

\[
\left[-\frac{1}{2} \left(\Omega^{-2}\Box_4 + \partial_y^2\right) + 4 \frac{\partial_y^2 \Omega}{\Omega} + 4 \left(\frac{\partial_y \Omega}{\Omega}\right)^2\right] h_{ab} = -\kappa_5^2 \delta M_{ab} \tag{18}
\]

The conformal 5D energy-momentum tensor \( \tilde{T}_{\mu
u} \) is

\[
\tilde{T}_{ab} = \frac{2\Lambda_B p^5_B}{\Omega^2} \tilde{g}_{ab}, \quad \tilde{T}_{a5} = \tilde{T}_{b5} = 0, \quad \tilde{T}_{55} = \frac{4\Lambda_B p^5_B}{\Omega^2} \tilde{g}_{55} \tag{19}
\]

Then the energy-momentum tensor perturbation is

\[
\delta M_{ab} = \Lambda_B \left(1 - \frac{2p^5_B}{\Omega^2}\right) h_{ab} + \left[\lambda \delta (y - y_0) + \lambda' \delta (y - y'_0)\right] h_{ab} \tag{20}
\]

Using the warp equations (6), (7) in the \( y \) coordinate and writing the graviton wave function as \( h_{ab} = e^{ip \cdot x} \psi_{ab}(y) \) where \( p^2 = -m^2 \) we obtain the following Sturm-Liouville problem

\[
-\frac{1}{2} \partial_y^2 \psi_{ab} + \frac{2}{l^2} \left[1 - 2p^5_B e^{2|y|/l} \left(1 + p^5_B e^{2|y|/l}\right)^{-2}\right] \psi_{ab} - \frac{2}{l} \frac{1 - p^5_B}{1 + p^5_B} \delta(y) \psi_{ab} \\
- \frac{2}{l} \frac{1 - p^5_B e^{2\pi r_c/l}}{1 + p^5_B e^{2\pi r_c/l}} \delta(y - \pi r_c) \psi_{ab} = m^2 e^{2|y|/l} \left(1 + p^5_B e^{2|y|/l}\right)^{-2} \psi_{ab} \tag{21}
\]

The eigenvalues \( m^2 \) are positive or zero leading to \( m \geq 0 \). The set is infinite but discrete, \( m_i, i = 0, 1, \cdots + \infty \). There is just one massless graviton, \( m_0 = 0 \). It has a positive wave function \( \psi_0 \) in the fifth dimension. This wave function is localized near the positive tension branes. The massive gravitons have oscillating wave functions \( \psi_i, i = 1, \cdots + \infty \). Their masses and mass splittings decrease when \( r_c \) increases. For the solutions with a stabilized radion there is no hierarchy between graviton wave functions (see figure [2]) so that gravity is not strongly localized.

**Figure 2:** Plot of graviton wave functions \( \psi \) for \( l/r_c = 5 \) and \( p^5_B = 0.25 \). Shown are the massless graviton and the first four massive gravitons. The mass scale is set by \( r_c = 100 \text{m} = 1 \times 10^3 \text{eV}^{-1} \).
Like the radion the massive gravitons may be above the TeV scale. Their mass splittings have the same order of magnitude. The radion and the less massive graviton have similar masses. The massive gravitons may also be light. Newton’s potential is

\[ V(r) = -\frac{G_{4mm'}}{r} \left( 1 + \frac{|\psi_1|^2}{|\psi_0|^2} e^{-m_1 r} + \cdots \right) \]  

(22)

Since \(|\psi_1|^2/|\psi_0|^2 \sim 1\) we must have \(r_c < 160\mu m\) and \(m_1 > 1 \times 10^{-3}\)eV.

6 Conclusions

In this paper we have analysed exact 5D solutions describing the dynamics of AdS_5 braneworlds when conformal fields of weight -4 exist in the bulk. We have considered solutions for which gravity is localized near the brane and the dynamics on the brane is for example that of inhomogeneous dust, generalized dark radiation and homogeneous polytropic dark energy. We have seen that the radion may be stabilized using only the conformal bulk fields of weight -4 which generate the dynamics on the brane. This requires invariance of their equation of state under the radion perturbation, a stabilizing sector with a constant negative 5D pressure and new warp functions. We have also discussed graviton perturbations and determined their mass eigenvalues and wave functions from a Sturm-Liouville problem. Besides a massless graviton localized on the positive tension branes we have seen that this scenario involves an infinite discrete set of increasingly massive gravitons. We have shown that the new stabilizing warp functions are unable to generate a large mass hierarchy and that gravity is not strongly localized near the branes. Possibilities to overcome this problem are the introduction of supersymmetry [24] or of a non-conformal 5D scalar field [3] to stabilize the radion field. This analysis is left for future research. Finally we have also shown that to satisfy the current observational constraints [23] the radion and the less massive graviton may either be heavier than \(\sim 100\text{GeV}\) corresponding to a small fifth dimension \(\sim 1 \times 10^{-15}\)mm or light with mass above \(\sim 1 \times 10^{-3}\)eV corresponding to a compactification scale of the order of 100\(\mu m\).

Acknowledgements

We would like to thank Rui Guerra for his continual support. We also thank the financial support of Fundação para a Ciência e a Tecnologia - FCT and Fundo Social Europeu - FSE under the contract SFRH/BPD/7182/2001 (III Quadro Comunitário de Apoio), of Centro de Electrónica, Optoelectrónica e Telecomunicações - CEOT, of Centro Multidisciplinar de Astrofísica-CENTRA and Conselho de Reitores das Universidades Portuguesas - CRUP with project Acção Integrada Luso-Espanhola E-126/04.
References


Dadhich N and Ghosh S G 2001 *Phys. Lett.* B *518* 1
Germani C and Maartens R 2001 *Phys. Rev.* D *64* 124010
Govender M and Dadhich N 2002 *Phys. Lett.* B *538* 233