The Relation between
Cosmic Temperature and Scale Factor

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Abstract

In this short paper, we drive the explicit relation between the temperature $T$ of classical ideal gas in the universe and the scale factor $a(t)$ of the Friedman-Robertson-Walker metric via kinetic and statistical calculation. This formula is suitable for both the ultra-relativistic and the non-relativistic cases.

In [1], the author point out that, the cooling mechanism of the expanding universe is one of the most interesting problem of the students. In the standard cosmological text books [2] and the pedagogical articles [3, 4, 5, 6], the variation of the temperature of the cosmic fluid during the expansion of the universe has been discussed in several ways. It has been shown that the momentum of particles $p$ in the cosmic fluid decrease with the inverse of scale factor of the expansion $a$. However some of these approaches such as dealing the motion of particles with the special theory of relativity seems does not satisfy the students’ curiosity.

Using the distribution function of particles for the non-relativistic particles, the temperature of cosmic fluid scales with the mean energy of the particles as $T \simeq E[7]$. In [1] the author solved the drifting speed of the particle, i.e. the so-called peculiar

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velocity $v_{pec} = a \frac{dr}{dt}$ by calculating the geodesic, and get $p \propto a^{-1}$. Then concluded that the cosmic temperature $T \propto a^{-1}$ for the ultra-relativistic gases but $T \propto a^{-2}$ for the non-relativistic particles.

One of the present authors once also encountered this problem in research[8]. Using the method similar to [1, 7], we can actually get the analytic expression of the cosmic temperature $T$ with respect to the scalar factor $a$. To solve the function $T(a)$, at First we introduce a useful lemma to solve the drifting speed of the particles

**Lemma.** If the interval of an orthogonal subspace have the following form,

$$ds^2 = A(t)dt^2 + \bar{g}_{\mu\nu}(t)dx^\mu dx^\nu,$$

where $A$ and $\bar{g}_{\mu\nu}$ only depend on the coordinate $t$, then the geodesic in this subspace can be solved by

$$\frac{dx^\mu}{ds} = \bar{g}^{\mu\nu}C_\nu, \quad \frac{dt}{ds} = \frac{1}{A}(1 - \bar{g}^{\mu\nu}C_\mu C_\nu),$$

where $C_\mu$ are constants, and $\bar{g}^{\mu\nu}\bar{g}_{\alpha\beta} = \delta^\mu_\alpha$.

In the Friedman-Robertson-Walker metric, the interval with respect to $(t, r)$ is given by $ds^2 = dt^2 - a(t)^2 dr^2$. So the geodesic equation reads

$$\frac{d}{ds} r = C a^2, \quad \frac{d}{ds} t = \frac{1}{a}\sqrt{a^2 + C^2},$$

where $C$ is a constant only depends on the initial data. By (3) we get the drifting speed of a particle

$$v \equiv \frac{adr}{dt} = \frac{C}{\sqrt{a^2 + C^2}}, \quad C = \frac{v_0}{\sqrt{1 - v_0^2}}a(t_0).$$

So the momentum of the particles with proper mass $m_n$ satisfies

$$p = \frac{m_n v}{\sqrt{1 - v^2}}, \quad p(t)a(t) = p(t_0)a(t_0).$$

Although Eq.(5) is derived in subspacetime $(t, r)$, but it is suitable for all particles due to the symmetry of the Friedman-Robertson-Walker metric. The relation between momentum $p$ and the kinetic energy $K$ reads $p^2 = K(K + 2m)$.

Denoting the temperature of the universe by $T$, according to the principle of equipartition of energy, we have the statistical distribution of the kinetic energy $K$ as

$$f(K) dK = \sqrt{\frac{4K}{\pi(kT)^3}} \exp\left(-\frac{K}{kT}\right) dK.$$
One can easily calculate the average kinetic energy

\[ \bar{K} = \int_0^\infty K f(K) dK = \frac{3}{2} kT. \]

Eq.(6) is suitable for both the ultra-relativistic and non-relativistic speed.

For the drifting movement of a particle with proper mass \( m_n \), by (5) we have \( p_n^2 = \frac{C_n}{a^2} \), where \( C_n \) are constants only depending on the initial data. Then on the one hand, for all particles we have the mean square momentum directly

\[ \bar{p}^2 = \frac{C_0}{a^2}, \]  

(7)

where \( C_0 \) only depends on the initial data.

On the other hand, according to statistical principle we have

\[ \bar{p}^2 = \sum_n \int_0^\infty \frac{N_n}{N} p_n^2 f(K_n) dK_n \]
\[ = \sum_n \int_0^\infty \frac{N_n}{N} K_n (K_n + 2m_n) f(K_n) dK_n \]
\[ = \sum_n kT \frac{N_n}{N} \left( \frac{15}{4} kT + 3m_n \right), \]  

(8)

where \( N_n \) is the number of particles with mass \( m_n \) in a unit volume, and \( N = \sum_n N_n \) is the total number of particles in the volume. Comparing (8) with (7), we get the equation of \( T(a) \) as follows

\[ kT \left( kT + \frac{4}{5} \bar{m} \right) = \frac{C}{a^2}, \quad \bar{m} = \sum_n \frac{N_n}{N} m_n, \]  

(9)

where \( \bar{m} \) is the average mass of all particles, and \( C \) is a constant only depending on initial data.

Solving Eq.(9), we finally get the following temperature function

\[ \frac{1}{2} kT = \frac{\bar{m} b^2}{5a(a + \sqrt{a^2 + b^2})}, \]  

(10)

where \( b \) is a constant determined by the initial data \( a_0 \) and \( T_0 \)

\[ \frac{b}{a_0} = \sqrt{\frac{5kT_0}{\bar{m}} \left( 1 + \frac{5kT_0}{4\bar{m}} \right)}, \]  

(11)

References


