Shift Theorem Involving the Exponential of a Sum of Non-Commuting Operators in Path Integrals

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Abstract

We show that for expressions of the form of an exponential of the sum of two non-commuting operators inside path integration, it is possible to shift one of the non-commuting operators from the exponential to other functions when the domain of integration of the argument of that function is the entire real axis. In particular we prove that,

\[
\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \ f_1(y) < x | e^{-[(a(y)x+h \frac{d}{dy})^2 + b(\frac{d}{dy}) + c(y)]} | x > f_2(y) \\
= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \ f_1(y) < x - \frac{h}{a(y)} \frac{d}{dy} | e^{-[a^2(y)x^2 + b(\frac{d}{dy}) + c(y)]} | x - \frac{h}{a(y)} \frac{d}{dy} > f_2(y).
\]

Here \(h\) is a constant and \(f, a, b, c\) are functions of single variable chosen that the integration over \(x\) is well defined. In this expression \(a(y), c(y)\) do not commute with the derivative operator \(\frac{d}{dy}\) in the exponential. This shift theorem should be useful in evaluating Path Integrals which arise when using the background field formalism. The crucial ingredient is that the integration over \(x\) be the entire real axis.

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I. INTRODUCTION

In background field methods in quantum field theory one often encounters the exponential of the sum of non-commuting operators inside the path integration. A simple example of this type occurs in charged scalar field theory in the presence of a background field $A$.

The vacuum to vacuum transition amplitude in the presence of background field $A$ is given by

$$<0|0>|A> = \int [d\phi^* d\phi] e^{i\int d^4x \phi^* M[A] \phi} = Det^{-1}[M[A]]/Det^{-1}[M[0]] = e^{iS^{(1)}}$$

where for scalar field theory

$$M[A] = (\hat{p} - eA)^2 - m^2, \quad \text{with}\quad \hat{p}_\mu = \frac{1}{i} \frac{\partial}{\partial x^\mu}.$$  \hspace{1cm} (2)

The one loop effective action is

$$S^{(1)} = i Tr \ln[(\hat{p} - eA)^2 - m^2] - i Tr \ln[\hat{p}^2 - m^2],$$

where the $Tr$ is given by

$$Tr\mathcal{O} = \int d^4x <x|\mathcal{O}|x>.$$  \hspace{1cm} (4)

Since it is convenient to work with the trace of the exponential we replace the logarithm by

$$\ln \frac{a}{b} = \int_0^\infty ds \frac{ds}{s} [e^{-is(b-ic)} - e^{-is(a-ic)}].$$  \hspace{1cm} (5)

Hence we get from eq. (3)

$$S^{(1)} = -i \int_0^\infty ds \frac{ds}{s} Tr[e^{-is(\hat{p} - eA)^2 - m^2 - i\epsilon}] - e^{-is(\hat{p}^2 - m^2 - i\epsilon)}].$$  \hspace{1cm} (6)

Consider the case where one has a time dependent electric field $E(t)$ in the (for convenience) $z$ (or 3) direction. We choose the Axial gauge $A_3 = 0$ so that

$$A_0 = -E(t)z.$$  \hspace{1cm} (7)

Using eq. (7) in eq. (6) we obtain

$$S^{(1)} = -i \int_0^\infty ds \int_{-\infty}^{+\infty} dt <t| \int_{-\infty}^{+\infty} dx <x| \int_{-\infty}^{+\infty} dy <y| \int_{-\infty}^{+\infty} dz <z| [e^{-is(\hat{p_0} + eE(t)z)^2 - \hat{p}_z^2 - \hat{p}_3^2 - m^2 - i\epsilon}] |z > |y > |x > |t >.$$  \hspace{1cm} (8)
Inserting complete set of $|p_T>$ states $\int d^2p_T|p_T><p_T| = 1$ we find (we use the normalization $<q|p> = \frac{1}{\sqrt{2\pi e^{i\hbar p}}}$)

$$S^{(1)} = \frac{-i}{(2\pi)^2} \int_0^\infty \frac{ds}{s} \int d^2x_T \int d^2p_T e^{is(p^2_T+m^2+i\epsilon)} [\int_{-\infty}^{+\infty} dt < t | \int_{-\infty}^{+\infty} dz < z | e^{-is([-\frac{1}{2}\hbar E(t)z^2-\vec{p}^2]])} z > | t > - \int dt \int dz \frac{1}{4\pi s}].$$

(9)

In the “$x$” representation the operators $\hat{p}_0 = \frac{1}{i\hbar} \frac{\partial}{\partial t}$ and $E(t)$ do not commute with each other. In order to evaluate this type of Path Integral it is quite useful to be able to shift the derivative operator from the exponential to pre-factor and post-factor functions that occur when we insert complete sets of states in order to evaluate the Path Integral.

In particular we would like to show that the following theorem is true: (in what follows we suppress the integration over the variable “$y$”)

$$\int_{-\infty}^{+\infty} dx f_1(y) < x | e^{-(a(y)x+h\frac{\partial}{\partial y})^2 +b(\frac{\partial}{\partial y}) +c(y)} | x > f_2(y)$$

$$= \int_{-\infty}^{+\infty} dx f_1(y) < x - \frac{h}{a(y)} \frac{d}{dy} | e^{-a^2(y)x^2 + b(\frac{\partial}{\partial y}) +c(y)} | x - \frac{h}{a(y)} \frac{d}{dy} > f_2(y).$$

(10)

Here $h$ is a constant and $f, a, b, c$ are functions of single variable, such that the integration over $x$ is well defined. This leads to special case

$$W = \int_{-\infty}^{+\infty} dx f_1(y) < x | e^{-(a(y)x+h\frac{\partial}{\partial y})^2} | x > f_2(y)$$

$$= \int_{-\infty}^{+\infty} dx f_1(y) < x - \frac{h}{a(y)} \frac{d}{dy} | e^{-a^2(y)x^2} | x - \frac{h}{a(y)} \frac{d}{dy} > f_2(y).$$

(11)

Let us evaluate the left and right hand side of the above equation separately. For the left hand side we find

$$W = \int_{-\infty}^{+\infty} dx f_1(y) < x | e^{-(a(y)x+h\frac{\partial}{\partial y})^2} | x > f_2(y)$$

$$= \int_{-\infty}^{+\infty} dx f_1(y) < x | e^{-(a(y)x+h\frac{\partial}{\partial y})^2} f_2(y)$$

$$= \frac{1}{2\pi} \int dp \int_{-\infty}^{+\infty} dx f_1(y) e^{-(a(y)x+h\frac{\partial}{\partial y})^2} f_2(y)$$

(12)

where we have used $<x|p> = \frac{1}{\sqrt{2\pi e^{i\hbar p}}}$ after inserting complete set of $|p>$ states ($\int dp |p><p| = 1$). Although eq.(12) is formally infinite, the derivative of $W$ with respect to $p$ is finite and given by
\[
\frac{dW}{dp} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \ dx \ f_1(y) \ e^{-(a(y)x + h \frac{d}{dy})^2} \ f_2(y). \tag{13}
\]

Now evaluating the right hand side of eq. (11) we find (after inserting complete set of \(x_1\) states)

\[
W = \int_{-\infty}^{+\infty} dx \ \int_{-\infty}^{+\infty} dx_1 \ f_1(y) \ < x - \frac{h}{a(y)} \frac{d}{dy} | x_1 > e^{-a^2(y)x_1^2} \ \delta(x - \frac{h}{a(y)} \frac{d}{dy} - x_1) \ f_2(y) \\
= \int_{-\infty}^{+\infty} dx_2 \ \int_{-\infty}^{+\infty} dx_1 \ f_1(y) \ < x_2 | x_1 > e^{-a^2(y)x_1^2} \ \delta(x_2 - x_1) \ f_2(y) \\
= \frac{1}{2\pi} \int dp \ \int_{-\infty}^{+\infty} dx \ f_1(y) \ e^{-a^2(y)x^2} \ f_2(y) \tag{14}
\]

where we have used \(x = x_2 - \frac{h}{a(y)} \frac{d}{dy}\) and the \(x_2\) integration range remains from \(-\infty\) to \(+\infty\).

From the above equation the \(x\) integral expression is given by

\[
\frac{dW}{dp} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \ f_1(y) \ e^{-a^2(y)x^2} \ f_2(y). \tag{15}
\]

By equating eqs. (13) and (15) we find (as a result of the shift theorem) that the following integrals are equivalent.

\[
I(y) = \int_{-\infty}^{+\infty} dx \ e^{-(a(y)x + h \frac{d}{dy})^2} \ f(y) = \int_{-\infty}^{+\infty} dx \ e^{-a^2(y)x^2} \ f(y). \tag{16}
\]

This form of the shift theorem does not suffer from the formally infinite constant

\[
\langle x | x \rangle = \delta(0) = \frac{1}{2\pi} \int dp. \tag{17}
\]

We will explicitly verify the validity of eq. (16) by direct expansion of the exponential later in this paper.

This last identity allows one to immediately perform the \(x\) integration to obtain:

\[
I(y) = \int_{-\infty}^{+\infty} dx \ e^{-(a(y)x + h \frac{d}{dy})^2} \ f(y) = \frac{\sqrt{\pi}}{a(y)} f(y). \tag{18}
\]

As a result of this theorem being true, one can the rewrite the one loop effective action in the alternative form,

\[
S^{(1)} = \frac{i}{(2\pi)^2} \int_{0}^{\infty} \frac{ds}{s} \int d^2x_T \int d^2p_T e^{is(\tilde{p}_T^2 + m^2 + i\epsilon)} \\
\int dt \int dz \frac{1}{4\pi s} \int_{-\infty}^{+\infty} dt < t | \int_{-\infty}^{+\infty} dz < z + \frac{i}{E(t) \ dt} | e^{-i\frac{E^2}{2}(t)z^2 - \tilde{p}_T^2} | z > + \frac{i}{E(t) \ dt} | t > \tag{19}
\]
where the $z$ integration must be performed from $-\infty$ to $+\infty$ for the shift theorem to be applicable. The shift in $z$ variable in the above equation is not valid if the $z$ integration limit is finite.

This latter form can be used to evaluate the effective action which we will do in a subsequent paper.

This above theorem eq. (10) can be generalized to involve matrices as follows

$$I_{ij}(y) = \int_{-\infty}^{+\infty} dx \left[ f_{1j}(y) < x | e^{-[(A(y)x + h\sum_{n=1}^{k} \frac{d^n}{dy^n})^2 + B(\frac{d}{dx}) + C(y)]} | x > f_{2j}(y) \right]^{ij}$$

$$= \int_{-\infty}^{+\infty} dx [ f_{1j}(y) < x - \frac{h}{A(y)} \sum_{n=1}^{k} \frac{d^n}{dy^n} ] e^{-[A^2(y)x^2 + B(\frac{d}{dx}) + C(y)]} | x - \frac{h}{A(y)} \sum_{n=1}^{k} \frac{d^n}{dy^n} > f_{2j}(y) ]^{ij}$$

(20)

where $k$ is any arbitrary number (which can be up to $\infty$). Here $h$ is a constant and $f_{1j}(y)$, $f_{2j}(y)$, $A^{ij}(y)$, $C^{ij}(y)$, are $(i, j)$ dimension matrices which do not commute with $\frac{d}{dy}$; $G_{1j}^{ij}(x)$, $G_{2j}^{ij}(x)$ are functions of $x$ and $\delta^{ij} B(\frac{d}{dx})$ is a function of $\frac{d}{dx}$, chosen that the integration over $x$ is well defined.

This shift by derivative technique will be very useful when one studies particle production from arbitrary background fields via Schwinger-like mechanisms in QED and QCD [1, 2]. Quark and gluon production from arbitrary classical chromo fields is expected to be an important ingredient in the production and equilibration of the quark-gluon plasma found at the RHIC and LHC [3, 4].

This paper is organized as follows. In section II we provide general proofs of eqs. (10), and (20) by using similarity transformation techniques. These theorems can be verified by explicit expansion of the exponential at hand. As an example we verify explicitly in section III. eq. (16) using a theorem we prove in the Appendix. We present our conclusions in section IV.

II. SIMILARITY TRANSFORMATION APPROACH FOR PROVING THE “SHIFT THEOREM”

In this section we provide a general proof of eqs. (10) and (20) by using similarity transformations.
A. Shift Theorem Involving Non-Commuting Operators in the Exponential

Consider the following similarity transformations acting on $x$:

$$x \pm \frac{h}{a(y)} \frac{d}{dy} = e^{\pm \frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} x e^{\mp \frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}},$$  \hspace{1cm} (21)

Since $e^{\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}}$ commutes with $b\left(\frac{d}{dx}\right)$ we find

$$(a(y)x + h \frac{d}{dy})^2 + b\left(\frac{d}{dx}\right) = e^{\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} \left[(e^{-\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} a(y) e^{\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} x^2 + b\left(\frac{d}{dx}\right) \right] e^{-\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}}. \hspace{1cm} (22)

Hence

$$I(y) = \int_{-\infty}^{+\infty} dx f_1(y) < x \left| e^{-[a(y)x + h \frac{d}{dy}]^2 + b\left(\frac{d}{dx}\right) + c(y)} \right| x > f_2(y)$$

$$= \int_{-\infty}^{+\infty} dx f_1(y) < x \left| e^{-\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} a(y) e^{\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} (x - \frac{h}{a(y)} \frac{d}{dy})^2 + b\left(\frac{d}{dx}\right) + e^{-\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} c(y) e^{\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} \right| x - \frac{h}{a(y)} \frac{d}{dy} > f_2(y). \hspace{1cm} (23)

Next we change the $x$ integration variable to $x'$ where

$$x = x' - \frac{h}{a(y)} \frac{d}{dy}. \hspace{1cm} (24)

With the above change in integration variable the integration limits for $x'$ remain $\pm \infty$. Under this change of integration variable one also has $dx = dx'$. With these changes we find from the equation (23)

$$I(y) = \int_{-\infty}^{+\infty} dx f_1(y) < x - \frac{h}{a(y)} \frac{d}{dy} \left| e^{-\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} a(y) e^{\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} (x - \frac{h}{a(y)} \frac{d}{dy})^2 + b\left(\frac{d}{dx}\right) + e^{-\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} c(y) e^{\frac{h}{a(y)} \frac{d}{dy} \frac{d}{dx}} \right| x - \frac{h}{a(y)} \frac{d}{dy} > f_2(y). \hspace{1cm} (25)

Using eq. (21) for the similarity transformation of $(x - \frac{h}{a(y)} \frac{d}{dy})$ we find

$$I(y) = \int_{-\infty}^{+\infty} dx f_1(y) < x \left| e^{-[a(y)x + h \frac{d}{dy}]^2 + b\left(\frac{d}{dx}\right) + c(y)} \right| x > f_2(y)$$

$$= \int_{-\infty}^{+\infty} dx f_1(y) < x - \frac{h}{a(y)} \frac{d}{dy} \left| e^{-[a(y)x + h \frac{d}{dy}]^2 + b\left(\frac{d}{dx}\right) + c(y)} \right|$$

$$|x - \frac{h}{a(y)} \frac{d}{dy} > f_2(y). \hspace{1cm} (26)

This proves eq. (10), and consequently the special case (16).
B. Shift Theorem Involving Matrices and Non-Commuting Operators in the Exponential

The above proofs can be easily extended to involve matrices. This will be useful for evaluating path integrals in SU(3) gauge theory in the presence of background field. We will prove eq. (20) in this section by using similarity transformation involving matrices.

We next consider the similarity transformation on the matrices $x\delta^{ij}$ as follows

$$\delta^{ij}x \pm \left[-\frac{h}{A(y)}\left[\sum_{n=1}^{k} \frac{d^n}{dy^n}\right]\right]^{ij} = \left[e^{\pm \frac{h}{A(y)}\left[\sum_{n=1}^{k} \frac{d^n}{dy^n}\right]}xe^{\pm \frac{h}{A(y)}\left[\sum_{n=1}^{k} \frac{d^n}{dy^n}\right]}\right]^{ij}. \quad (27)$$

where $A^{ij}(y)$ is y dependent matrix.

Since $\left[e^{\frac{h}{A(y)}\left[\sum_{n=1}^{k} \frac{d^n}{dy^n}\right]}\right]^{ij}$ commutes with $\delta^{ij}B\left(\frac{d}{dx}\right)$ we find

$$\left[\left(A(y)x + h\left[\sum_{n=1}^{k} \frac{d^n}{dy^n}\right]\right)^2\right]^{ij} + \delta^{ij}B\left(\frac{d}{dx}\right) =
\left[e^{\frac{h}{A(y)}\left[\sum_{n=1}^{k} \frac{d^n}{dy^n}\right]}\right]^{ij}\left[(e^{-\frac{h}{A(y)}\left[\sum_{n=1}^{k} \frac{d^n}{dy^n}\right]}A(y)e^{\frac{h}{A(y)}\left[\sum_{n=1}^{k} \frac{d^n}{dy^n}\right]}x)^2 + B\left(\frac{d}{dx}\right)\right]^{mn}
\left[e^{-\frac{h}{A(y)}\left[\sum_{n=1}^{k} \frac{d^n}{dy^n}\right]}\right]^{ij}. \quad (28)$$

Repeating the same logic as used previously, we obtain

$$I^{ij}(y) = \int_{-\infty}^{+\infty} dx [f_1(y) < x | e^{-[A(y)x + h\left[\sum_{n=1}^{k} \frac{d^n}{dy^n}\right]]^2 + B\left(\frac{d}{dx}\right) + C(y)\left| x > f_2(y)\right]^{ij}}
\int_{-\infty}^{+\infty} dx [f_1(y) < x - \frac{h}{A(y)}\left[\sum_{n=1}^{k} \frac{d^n}{dy^n}\right]e^{-[A^2(y)x^2 + B\left(\frac{d}{dx}\right) + C(y)]\left| x - \frac{h}{A(y)}\left[\sum_{n=1}^{k} \frac{d^n}{dy^n}\right] > f_2(y)\right]^{ij}}
\quad (29)$$

which proves eq. (20).

This completes the general proof of eqs. (10) and (20) by using similarity transformations. Since this "proof" is rather formal and relies on similarity transformations that are not very familiar, we will now explicitly demonstrate the validity of the shift theorem in the case of eq. (16), by splitting the exponential of the two non commuting operators into a single exponential times a power series and performing the integrations term by term in the expansion.
III. EXPLICIT VERIFICATION OF A SPECIAL CASE

In this section we would like to verify by explicit calculation the special case

\[
\int_{-\infty}^{+\infty} dx \ e^{-(a(y)x + \frac{h^2}{2y})^2} f(y) = \int_{-\infty}^{+\infty} dx \ e^{-a^2(y)x^2} f(y),
\]  

(30)

To do this we will use a theorem which we prove in the appendix (see eq. (A21)) for two non-commuting operators \(A, B\)

\[
e^{-A+B} = e^{-A} \left[ 1 + \sum_{n=1}^{\infty} (-1)^n \prod_{i=1}^{n} \left[ \int_{0}^{x_{i-1}} dx_i \ e^{x_iA} B e^{-x_iA} \right] \right].
\]  

(31)

Using eq. (31) in (30) we find

\[
e^{-a(y)x + \frac{h^2}{2y}} f(y) = e^{-A} \left[ 1 - \int_{0}^{1} dx_1 e^{x_1A} B e^{-x_1A} + \int_{0}^{1} dx_1 e^{x_1A} B e^{-x_1A} \int_{0}^{x_1} dx_2 e^{x_2A} B e^{-x_2A} \right.
\]

\[
- \int_{0}^{1} dx_1 e^{x_1A} B e^{-x_1A} \int_{0}^{x_1} dx_2 e^{x_2A} B e^{-x_2A} \int_{0}^{x_2} dx_3 e^{x_3A} B e^{-x_3A} + \ldots \right] f(y)
\]  

(32)

where

\[
(a(y)x + \frac{h}{dy})^2 = A + B,
\]

\[
A = a^2(y)x^2,
\]

\[
B = 2x a(y) \frac{d}{dy} + x \frac{da(y)}{dy} + h^2 \frac{d^2}{dy^2}.
\]  

(33)

Integrating \(x\) from \(-\infty\) to \(+\infty\) in eq. (32) we write

\[
\int_{-\infty}^{+\infty} dx \ e^{-(a(y)x + \frac{h^2}{2y})^2} f(y) = \int_{-\infty}^{+\infty} dx \ e^{-A} \left[ 1 - I_1 + I_2 - I_3 + I_4 + \ldots \right]
\]  

(34)

where

\[
\int_{-\infty}^{+\infty} dx \ e^{-A} I_n = \int_{-\infty}^{+\infty} dx \ e^{-A} \left[ \prod_{i=1}^{n} \left[ \int_{0}^{x_{i-1}} dx_i \ e^{x_iA} B e^{-x_iA} \right] \right] f(y).
\]  

(35)

with \(x_0 = 1\) and \(n=1,2,3,\ldots\) etc. Using the expressions for \(A\) and \(B\) from eq. (33) and performing the \(x\) and \(x_i\)'s integrations explicitly in eq (35) we find

\[
-I_1 = - \frac{f[y][a'[y]^2]}{2a[y]^2} + \frac{a'[y]f'[y]}{a[y]} + \frac{1}{2} \frac{f[y][a''[y]}{a[y]} - f''[y]
\]  

(36)
\[ I_2 = \frac{5f[y]a'[y]^4}{24a[y]^4} - \frac{a'[y]^3 f'[y] - f[y]a'[y]^2 a''[y]}{a[y]^3} + \frac{1}{a[y]^2} \left( \frac{1}{2} f[y]a'[y]^2 + 2a'[y]f'[y]a''[y] + \frac{3}{8} f[y]a''[y]^2 + \frac{4}{3} a'[y]^2 f''[y] + \frac{7}{12} f[y]a'[y]a''[y] \right) + \frac{1}{a[y]} (a'[y]f''[y] - \frac{1}{2} f[y]a''[y] - \frac{7}{6} a''[y]f''[y] - \frac{2}{3} f'[y]a'(3)[y] - a'[y]f'(3)[y] - \frac{1}{6} f[y]a'(4)[y]) + (f''[y] + \frac{1}{2} f'(4)[y]) \]  

(37)


(38)

and


(-10640 f[y]a'[y]^4 - 553280 a'[y]3 f'[y]a''[y]^2 - 434672 f[y]a'[y]^2 a''[y]^2 -
In the above equations \( a[y] = a(y), f[y] = f(y), a''[y] = h \frac{d^2a(y)}{dy^2}, f'[y] = h \frac{df(y)}{dy}, a''[y] = h^2 \frac{d^2a(y)}{dy^2}, f''[y] = h^2 \frac{d^2f(y)}{dy^2}, a^{(n)}[y] = h^n \frac{d^n a(y)}{dy^n}, f^{(n)}[y] = h^n \frac{d^n f(y)}{dy^n}, a^{(n)}[y]^k = h^{n+k}(\frac{d^n a(y)}{dy^n})^k \) and \( f^{(n)}[y]^k = h^{n+k}(\frac{d^n f(y)}{dy^n})^k \). It can be easily seen that all the terms having odd powers in \( \frac{d}{dy} \) are zero because of odd integrations in \( x \) (see the expression for \( B \) in eq. (33)).

Now let us look at the above terms more carefully. The terms which contain two powers
in $\frac{d}{dy}$ are only in $I_1$ and $I_2$. We find from eqs. (36) and (37)

$$I_2 - I_1 = \frac{5f[y]a'[y]^4}{24a[y]^4} - \frac{a'[y]^2(a'[y]f'[y] + f[y]a''[y])}{a[y]^3} + \frac{1}{a[y]}\left(\frac{3}{8}f[y]a''[y]^2 + \frac{4}{3}a'[y]^2f''[y] + a'[y](2f'[y]a''[y] + \frac{7}{12}f[y]a^{(3)}[y])\right) + \frac{1}{6a[y]}(-7a''[y]f''[y] - 4f'[y]a^{(3)}[y] - 6a'[y]f^{(3)}[y] - f[y]a^{(4)}[y]) + \frac{1}{2}f^{(4)}[y]$$

(40)

which does not contain any terms having two powers in $\frac{d}{dy}$. All the terms having two powers in $\frac{d}{dy}$ in $I_1$ exactly canceled with all the terms having two powers in $\frac{d}{dy}$ in $I_2$. Similarly let us look more closely at all the terms with four powers in $\frac{d}{dy}$. These are contained in $I_2$, $I_3$ and $I_4$ only. From eqs. (36), (37), (38) and (39) we find

$$-I_1 + I_2 - I_3 + I_4 = -\frac{30203(f[y]a'[y]^{(8)})}{1920a[y]^8} + \frac{a'[y]^6(2675a'[y]f'[y] + 8734f[y]a''[y])}{120a[y]^7} - \frac{1}{6720a[y]^6}(a'[y]^4(80a'[y](6699f'[y]a''[y] + 1118a'[y]f''[y]) + f[y](134120a'[y]^2 + 659013a''[y]^2 + 177750a'[y]a^{(3)}[y]))), + \frac{1}{336a[y]^5}(a'[y]^2(12997f[y]a''[y]^3 + 6a'[y]a''[y](4076f'[y]a''[y] + 2839f[y]a^{(3)}[y]) + 126a'[y]^3(62f'[y] + 9f^{(3)}[y]) + 3a'[y]^2(3617a''[y]f''[y] + 2556f'[y]a^{(3)}[y] + f[y](6034a''[y] + 663a^{(4)}[y]))) + \frac{1}{13440a[y]^4}(-28261f[y]a''[y]^4 - 4a'[y]a''[y]^2(46496f'[y]a''[y] + 53199f[y]a^{(3)}[y]) - 4a'[y]^2(16a''[y](2913a''[y]f''[y] + 4988f'[y]a^{(3)}[y]) + f[y](102158a''[y]^2 + 16201a^{(3)}[y]^2 + 22440a''[y]a^{(4)}[y])) - 168a'[y]^4(852f''[y] - 53f^{(4)}[y]) - 8a'[y]^3(20188f[y]a^{(3)}[y] + 6708f''[y]a^{(3)}[y] + 2520a''[y]f^{(3)}[y] + 240f'[y](287a''[y] + 18a^{(4)}[y]) + 987f[y]a^{(5)}[y])) + \frac{1}{1680a[y]^3}(a''[y]a''[y](19a''[y]f''[y] + 2464f'[y]a^{(3)}[y]) + 2f[y](1428a''[y]^2 + 775a^{(3)}[y]^2 + 471a''[y]a^{(4)}[y])) + 2a'[y](-a''[y](2335f''[y]a^{(3)}[y] + 2877a''[y]f^{(3)}[y]) + f'[y](6888a''[y]^2 + 206a^{(3)}[y]^2 - 328a''[y]a^{(4)}[y]) + f[y](368a^{(3)}[y]a^{(4)}[y] + a''[y](4053a^{(3)}[y] + 33a^{(5)}[y]))) + 1680a''[y]^3(f^{(3)}[y] - f^{(5)}[y]) + a'[y]^2(-6132a^{(3)}[y]f^{(3)}[y] + 658f[y]a^{(4)}[y] - 3432f''[y]a^{(4)}[y] + 84a''[y](122f''[y] - 83f^{(4)}[y]) + 6f'[y](868a^{(3)}[y] - 187a^{(5)}[y]) - 165f[y]a^{(6)}[y])) + \frac{1}{240a[y]^2}(99f[y]a^{(3)}[y]^2 + 244f''[y]a^{(3)}[y]^2 + 184f'[y]a^{(3)}[y]a^{(4)}[y] + 16f[y]a^{(4)}[y]^2 + a''[y]^2(560f''[y] + 373f^{(4)}[y]) + 30f[y]a^{(3)}[y]a^{(5)}[y] + a''[y](720a^{(3)}[y]f^{(3)}[y] + 186f[y]a^{(4)}[y] + 434f''[y]a^{(4)}[y] + 10f'[y](68a^{(3)}[y] + 15a^{(5)}[y]) + 160a'[y](7f^{(3)}[y] + 3f^{(5)}[y]) + 3f[y]a^{(6)}[y]) + 120a'[y]^2(3f^{(4)}[y] + f^{(6)}[y]) + 2a'[y](258f^{(3)}[y]a^{(4)}[y] + 307a^{(3)}[y]f^{(4)}[y] + 64f[y]a^{(5)}[y] + 6f''[y](90a^{(3)}[y] + 23a^{(5)}[y]) + f'[y](288a^{(4)}[y] + 43a^{(6)}[y]) + 6f[y]a^{(7)}[y]) +$
\[
\frac{1}{120a[y]}(-320a^{(3)}[y]f^{(3)}[y] - 234f''[y]a^{(4)}[y] - 260a''[y]f^{(4)}[y] - 86a^{(4)}[y]f^{(4)}[y]
+ \frac{1}{24}(8f^{(6)}[y] + f^{(8)}[y])
\]

which does not contain any terms having two and four powers in \(\frac{d}{dy}\). All the terms having four powers in \(\frac{d}{dy}\) in \(I_3\) exactly canceled with all the terms having four powers in \(\frac{d}{dy}\) in \(I_2 + I_4\). This process can be repeated up to arbitrary powers of \(\frac{d}{dy}\) and after adding all the terms in the infinite series, we find each power \(n = k + l\) found by expanding in B and organizing our series in derivatives of the form \(a^{(k)}(y)f^{(n-k)}(y), k \leq n\) then the coefficients of all the terms for \(n=1,2,3,...\) vanishes. Hence we find that

\[
\int_{-\infty}^{+\infty} dx \ e^{-(a(y)x + \frac{dx}{dy})^2} f(y) = \int_{-\infty}^{+\infty} dx \ e^{-A} [1 - I_1 + I_2 - I_3 + I_4 + ...] = \int_{-\infty}^{+\infty} dx \ e^{-a^2(y)x^2} f(y),
\]

This explicitly verifies eq. (16).

**IV. CONCLUSIONS**

To conclude, we have shown that, remarkably, inside of integrals over the entire real line one can shift the non-commuting derivative operator (not depending on the integration variable) which occurs in exponentials just as if it were a constant. In particular we have shown that

\[
\int_{-\infty}^{+\infty} dx \ f_1(y) < x| e^{-[a(y)x + \frac{dx}{dy})^2 + b(\frac{d}{dy}) + c(y)]} |x > f_2(y) = \\
\int_{-\infty}^{+\infty} dx \ f_1(y) < x - \frac{h}{a(y) dy} | e^{-[a^2(y)x^2 + b(\frac{d}{dy}) + c(y)]} |x - \frac{h}{a(y) dy} > f_2(y),
\]

as well as its extension to Matrix functions, where \(f, a, b, c\) are chosen that the \(x\) integration is well defined. This shift theorem should prove useful in the evaluation of Path Integrals that occur when utilizing the background field method. A discussion of pair production from time dependent fields which will utilize this theorem will be the subject of a subsequent paper.
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APPENDIX A: SERIES EXPANSION FOR THE EXPONENTIAL OF SUM OF TWO NON-COMMUTING OPERATORS

In this section we derive

\[
\exp\left[-(A+B)\right] = \left[1 + \sum_{n=1}^{\infty} (-1)^n \prod_{i=1}^{n} \left( \int_{0}^{x_{i-1}} dx_i \cdot e^{x_i A} B e^{-x_i A} \right) \right] (A1)
\]

where \(x_0 = 1\). We will prove eq. (A1) by two separate methods. First we will use a simple procedure which is equivalent to the interaction picture technique in quantum mechanics (see for example the textbook of Ashok Das [6]) and then we will use another procedure based on the properties of Beta functions.

1. Iterative method using the Interaction Picture Technique

Let us write

\[
\exp[-x(A + B)] = \exp[-xA] S(x)
\]

and then determine \(S(x)\). In the above equation \(A\) and \(B\) are two non-commuting operators and \(x\) is a parameter which commutes with \(A\) and \(B\). From the above equation we find

\[
S(x) = \exp[xA] \exp[-x(A + B)].
\]

Differentiating \(S(x)\) with respect to \(x\) we find

\[
\frac{dS(x)}{dx} = \exp[xA] (A) \exp[-x(A + B)] + \exp[xA] (-A - B) \exp[-x(A + B)]
= \exp[xA] (-B) \exp[-xA] \exp[xA] \exp[-x(A + B)] = -\exp[xA] B \exp[-xA] S(x).
\]

(A4)
Hence the above equation
\[
\frac{dS(x)}{dx} = - \exp[xA] B \exp[-xA] S(x)
\] (A5)
can be iteratively solved to yield
\[
S(x) = S(x = 0) - \int_0^x dx_1 e^{x_1A} B e^{-x_1A} S(x_1) = 1 - \int_0^x dx_1 e^{x_1A} B e^{-x_1A} S(x_1)
\] (A6)
where we have used \( S(x = 0) = 1 \) from eq. (A3). This process can be continued iteratively to yield
\[
S(x) = [1 + \sum_{n=1}^{\infty} (-1)^n \prod_{i=1}^n \left[ \int_0^{x_{i-1}} dx_i \ e^{x_iA} B e^{-x_iA} \right] ]
\] (A7)
where \( x_0 = x \). Using \( x = 1 \) in eqs. (A7) and (A2) we find
\[
\exp[-(A + B)] = \exp[-A] [1 + \sum_{n=1}^{\infty} (-1)^n \prod_{i=1}^n \left[ \int_0^{x_{i-1}} dx_i \ e^{x_iA} B e^{-x_iA} \right] ]
\] (A8)
with \( x_0 = 1 \). This proves eq. (A1).

2. Direct evaluation of the Power Series using the integral Representation for the Beta Function

In this section we will re-derive eq. (A1) by completely expanding the exponential in a power series in \( Bjj \) and then re-exponentiating using the integral representation for the Beta function. For this purpose we write the exponential of sum of two non-commuting operator \( A \) and \( B \) as
\[
e^{-(A+B)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (A + B)^n = \sum_{k=0}^{\infty} O(B^k)
\] (A9)
where \( O(B^k) \) means sum of all the terms of order \( B^k \).

Sum of all the terms linear in \( B \) are given by
\[
O(B) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{m=0}^{n-1} (A)^m B (A)^{n-m-1} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n-m+1}}{(n+m+1)!} (A)^m B (A)^n.
\] (A10)

Using the following definition of the Beta function
\[
\frac{1}{(n + m + j + 1)!} = \frac{1}{(m)! (n + j)!} \int_0^1 dx \ (1 - x)^m x^{n+j}
\] (A11)
we find
\[ O(B) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+m+1}}{(n)! (m)!} \int_0^1 dx ((1-x)A)^m B (xA)^n \]
\[ = -e^{-A} \int_0^1 dx \quad e^{xA} B \quad e^{-xA}. \]

(A12)

This linear order in $B$ result is often cited in the literature \[5\]. Here we obtain all the terms of the series in $B$ explicitly by summing the series. We evaluate the terms which has two powers in $B$. For this purpose we write the exponential as
\[ e^{-A+B} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (A+B)^n \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (A+B)^m (A+B)(A+B)^p (A+B)(A+B)^{n-m-p-2}. \] (A13)

To obtain the terms with two powers in $B$ in the above equation we put $A=0$ in $(A+B)$ and $B=0$ in the power terms $(A+B)^k$. The result of this process is
\[ \mathcal{O}(B^2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{n+m+p+2}}{n! m! (n+m+p+1)!} A^m B A^p B A^n. \] (A14)

Using eq. (A11) for the beta function definition we find
\[ \mathcal{O}(B^2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{n+m+p+2}}{m! (n+p+1)!} \int_0^1 dx (1-x)^m x^{n+p+1} A^m B A^p B A^n \]
\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{n+m+p+2}}{m! p! n!} \int_0^1 dx (1-x)^m x^{n+p+1} \int_0^1 dy (1-y)^p y^n A^m B A^p B A^n \]
\[ = e^{-A} \int_0^1 dx \quad \int_0^1 dy \quad e^{xA} B \quad e^{-xA} e^{yA} B \quad e^{-yxA}. \] (A15)

Hence we find up to second power of $B$
\[ e^{-(A+B)} = e^{-A} [1 - \int_0^1 dx_1 \quad e^{x_1 A} B \quad e^{-x_1 A} + \int_0^1 dx_1 \quad e^{x_1 A} B \quad e^{-x_1 A} \int_0^{x_1} e^{x_2 A} B \quad e^{-x_2 A} + \mathcal{O}(B^3)..... \]. (A16)

Now let us evaluate all the terms which has three powers of $B$. For this purpose we write the exponential as
\[ e^{-(A+B)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (A+B)^m (A+B)(A+B)^p (A+B)(A+B)^q (A+B)(A+B)^{n-m-p-q-3}. \] (A17)
To obtain the three powers of $B$ from the above equation we put $A=0$ in $(A+B)$ and $B=0$ in the power terms $(A+B)^k$. We obtain

\[ O(B^3) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{m=0}^{n-3} \sum_{p=0}^{n-3-m} \sum_{q=0}^{n-3-m-p} A^m B^p B^q B^n \]

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(-1)^{n+m+p+q+3}}{(n+m+p+q+3)!} A^m B^p B^q B^n. \quad (A18) \]

Using eq. (A11) for the beta function definition we find

\[ O(B^3) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(-1)^{n+m+p+q+3}}{m!p!q!n!} \int_0^1 dx (1-x)^{m+n+p+q+2} \int_0^1 dy (1-y)^p y^{n+q+1} \]

\[ \int_0^1 dw (1-w)^q w^n A^m B^p B^q B^n = -e^{-A} \int_0^1 dx e^{xA} B e^{-xA} \int_0^x dy e^{yA} B e^{-yA} \]

\[ \int_0^y dw e^{wA} B e^{-wA}. \quad (A19) \]

Hence the series expansion of the exponential in $B$ up to terms of order $B^3$ is given by:

\[ e^{-(A+B)} = e^{-A} \left[ 1 - \int_0^1 dx_1 e^{x_1 A} B e^{-x_1 A} + \int_0^1 dx_1 e^{x_1 A} e^{-x_1 A} \int_0^{x_1} dx_2 e^{x_2 A} B e^{-x_2 A} - \right. \]

\[ \left. \int_0^1 dx_1 e^{x_1 A} B e^{-x_1 A} \int_0^{x_1} dx_2 e^{x_2 A} B e^{-x_2 A} \int_0^{x_2} dx_3 e^{x_3 A} B e^{-x_3 A} + \mathcal{O}(B^4) \ldots \right]. \quad (A20) \]

Generalizing this pattern we obtain

\[ e^{-(A+B)} = e^{-A} \left[ 1 + \sum_{n=1}^{\infty} (-1)^n \prod_{i=1}^{n} \left[ \int_0^{x_{i-1}} dx_i e^{x_i A} B e^{-x_i A} \right] \right] \quad (A21) \]

which agrees with eq. (A1).

---


[5] We thank George Sterman for pointing this to us.