The planned generation of lasers and heavy ion colliders renews the hope to see electron-positron pair creation in strong classical fields (so called spontaneous pair creation). This adiabatic relativistic effect has however not been described in a unified manner. We discuss here the theory of adiabatic pair creation yielding the momentum distribution of scattered pairs in overcritical fields. Our conclusion about the possibility of adiabatic pair creation is different from earlier predictions for laser fields.

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I. INTRODUCTION

The creation of an electron positron pair in an almost stationary very strong external electromagnetic field (a potential well) is often referred to as spontaneous pair creation ([1],[2],[3]). This adiabatic phenomenon emerges straightforwardly from the Dirac sea interpretation of negative energy states: An adiabatically increasing field (a potential well of changing deepness) lifts a particle from the sea to the positive energy subspace (by the adiabatic theorem) where it hopefully scatters and when the potential is switched off one has one free electron and one unoccupied state—a hole ([4],[5]) (see figure below). A better terminology is thus adiabatic pair creation (APC).

The figure shows the bound state energy curve of a bound state emerging from the Dirac sea (vacuum), changing with time $s$ due to the change of the potential well. If the potential reaches a high enough value (critical value) the bound state enters the upper continuum and scatters and if it has enough time to escape from the range of the potential, before the potential decreases under its critical value a pair will be created. Before we discuss experimental possibilities we give first the treatment of APC, thereby improving earlier results [1].

Consider the one particle Dirac equation with external electromagnetic field. On microscopic timescales $\tau = \frac{mc^2}{\hbar} t = \frac{c}{\lambda} t$ the equation reads

$$i \frac{\partial \psi}{\partial \tau} = -i \frac{\hbar}{mc} \sum_{l=1}^{3} \alpha_l \partial_l \psi + A_{\varepsilon\tau}(x) \psi + \beta \psi = (D^{0} + A_{\varepsilon\tau}(x)) \psi$$

$$\equiv (D^{0} + A_{\varepsilon\tau}(x)) \psi = D_{\varepsilon \tau} \psi$$

(1)

where $\varepsilon$ is the adiabatic parameter, representing the slow time variation of the external potential and $Amc^2$ gives the potential in the units $eV$ (we discuss later physical values for $\varepsilon$).

We consider the Dirac equation on the macroscopic time scale $s = \varepsilon \tau$:

$$i \frac{\partial \psi_s}{\partial s} = \frac{1}{\varepsilon} D_s \psi_s .$$

(2)

The spectrum of the Dirac operator without external field is $(-\infty, -1] \cap [1, \infty)$. The adiabatic theorem ensures (for $\varepsilon$ small) that the gap can only be crossed by bound states $\Phi_s$ of the Dirac operator $D_s \Phi_s = E_s \Phi_s$ for which $E_s$ is a curve crossing the gap (see figure). If there is no such curve the probability of pair creation is exponentially small in $1/\varepsilon$.

We assume now, that such a curve exists and consider the bound state $\Phi_0$ (assumed to be non degenerate) at the crossing.

We expand it in generalized eigenfunctions which of course depend also on the "parameter" $s$. We shall need the eigenfunctions for times $\sigma$ close to the critical time $\sigma = 0$.

Consider the eigenvalue equation

$$D_s \varphi = E \varphi$$

(3)
for fixed $\sigma \in \mathbb{R}$. The continuous subspace is spanned by
generalized eigenfunctions $\varphi^j(\mathbf{k}, \sigma, x)$, $j = 1, 2, 3, 4$, with
energy $E = \pm E_0 = \pm \sqrt{k^2 + 1}$. For ease of notation we will
drop the spin index $j$ in what follows.

The generalized eigenfunctions also solve the Lippmann-Schwinger equation
\begin{equation}
\varphi(\sigma, \mathbf{k}, x) = \varphi_0(\mathbf{k}, x) \\
+ \int G_k^+(x-x')A_\sigma(x')\varphi(\sigma, \mathbf{k}, x')d^3x',
\end{equation}
with $\varphi_0(\mathbf{k}, x) = (\xi(\mathbf{k}))e^{i\mathbf{k}\cdot \mathbf{x}}$, the
generalized eigenfunctions of the free Dirac operator $D^0$, i.e. $G_k^+$ is the kernel
of $(E_k - D^0)^{-1} = \lim_{\delta \to 0} (E_k - D^0 + i\delta)^{-1}$.

Introducing the operator $T_\sigma$
\begin{equation}
T_\sigma f = \int G_k^+(x-x')A_\sigma(x')f(x')d^3x',
\end{equation}
(11) becomes
\begin{equation}
(1 - T_\sigma)\varphi(\sigma, \mathbf{k}, \cdot) = \varphi_0(\mathbf{k}, \cdot).
\end{equation}

Note that
\begin{equation}
(1 - T_0^0)\Phi_0 = 0.
\end{equation}

We estimate the propagation of a wave function generated by
the static Dirac Operator $D_\sigma = D^0 + A_\sigma(x)$, where $\sigma > 0$ should be thought
of as near the critical value (the relevant regime turns out to be of order $\sigma = O(e^{1/\alpha})$).

Since the generalized eigenfunctions for $(\sigma, \mathbf{k}) \approx (0,0)$
are close to the bound state $\Phi_0$ it is reasonable to write in
leading order:
\begin{equation}
\varphi(\sigma, \mathbf{k}, x) \approx \eta_\sigma(\mathbf{k})\Phi_0(x).
\end{equation}

Since they solve (11), the first summand of (11) must become
negligible with respect to $\eta_\sigma(\mathbf{k})\Phi_0$, which is part
of the second summand. Hence $\eta_\sigma(\mathbf{k})$ must diverge
for $(\sigma, \mathbf{k}) \to (0,0)$. For the outgoing asymptote of the state
$\Phi_0$ (generalized Fourier transform) evolved with $D_\sigma$ near
criticality we have with (8) that
\begin{equation}
\hat{\Phi}_\text{out}(\sigma, \mathbf{k}) := \int (2\pi)^{-\frac{3}{2}}\Phi_0(x)\tilde{\eta}(\sigma, \mathbf{k}, x)d^3x
\approx (2\pi)^{-\frac{3}{2}}\hat{\eta}_\sigma(\mathbf{k}).
\end{equation}

Now, for $(\sigma, k)$ close to but different from $(0,0)$, $\eta_\sigma(\mathbf{k}) \sim
\hat{\Phi}_\text{out}(\sigma, \mathbf{k})$ will be peaked around a value $k(\sigma)$ with width
$\Delta(\sigma)$ (determined below) defined by
\begin{equation}
\eta_\sigma(k(\sigma) \pm \Delta(\sigma)) \approx \eta_\sigma(k(\sigma))/\sqrt{2}.
\end{equation}

We may use the width for the rough estimate
\begin{equation}
|\partial_k \hat{\Phi}_\text{out}(\sigma, \mathbf{k})| < \Delta(\sigma)^{-1}\hat{\Phi}_\text{out}(\sigma, k\sigma),
\end{equation}
where the right hand side should be multiplied by some
appropriate constant which we—since it is
not substantial—take to be one. Using (8), (9),
$d^3k = k^2d\Omega dk$ and partial integration (observing
$-ie\sigma \partial_k e^{-i(\frac{k^2}{2} + i\sigma)}\theta = e^{-i(\frac{k^2}{2} + \sigma)}\theta$) we get
\begin{equation}
U_\sigma(s,0)\Phi_0 = e^{-isD_\sigma}\Phi_0 \approx \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{-i(\frac{k^2}{2} + \sigma)}\hat{\Phi}_\text{out}\hat{\Phi}_\text{out}d^3k
\approx \frac{-i\sigma}{s} \int e^{-i(\frac{k^2}{2} + \sigma)}\partial_k (|\hat{\Phi}_\text{out}|^2\Phi_0(x)k d\Omega) dk
\end{equation}
By (11), (9) and (10), assuming that $\Delta(\sigma) \ll k(\sigma)$
\begin{equation}
|\partial_k (|\hat{\Phi}_\text{out}|^2k) \partial k d\Omega dk \approx |\hat{\Phi}_\text{out}|^2 \left( \frac{2}{\Delta(\sigma)} + \frac{1}{k^2} \right)d^3k
\approx |\hat{\Phi}_\text{out}|^2 \left( \frac{2}{\Delta(\sigma)k(\sigma)} \right)d^3k.
\end{equation}
Hence
\begin{equation}
|U_\sigma(s,0)\Phi_0(x)| \leq \frac{2\varepsilon|\hat{\Phi}_\text{out}(x)|}{\Delta(\sigma)k(\sigma)} \int |\hat{\Phi}_\text{out}|^2 d^3k.
\end{equation}
Since $\Phi_0$ is normalized we get for the decay time $s_d$, defined
by $|\langle U(s_d,0)\Phi_0, \Phi_0 \rangle| \approx 1/2$
\begin{equation}
s_d \approx 4\varepsilon(k(\sigma)\Delta(\sigma))^{-1}.
\end{equation}

The important information we must provide is thus $\eta_\sigma(k)$ for $\sigma \approx 0$. In view of (6) and (8) we have that
\begin{equation}
(1 - T_\sigma^k)\eta_\sigma(\mathbf{k})\Phi_0 \approx \varphi_0(\mathbf{k}, \cdot).
\end{equation}

We can estimate $\eta_\sigma(k)$ by considering the scalar product of (13) with $A_0\Phi_0$:
\begin{equation}
\eta_\sigma(k)\langle (1 - T_\sigma^k)\Phi_0, A_0\Phi_0 \rangle \approx \langle \varphi_0(\mathbf{k}, \cdot), A_0\Phi_0 \rangle.
\end{equation}

One finds that $\langle \varphi_0(\mathbf{k}, \cdot), A_0\Phi_0 \rangle = Ck + O(k^2)$ with an appropriate $C \neq 0$. Thus
\begin{equation}
\eta_\sigma(k) \approx Ck((1 - T_\sigma^k)\Phi_0, A_0\Phi_0))^{-1}.
\end{equation}
Expanding $T_\sigma$ in orders of $k$ around $k = 0$ until fourth order yields (the first order term turns out to be zero on
general grounds (17, 18))
\begin{equation}
\eta_\sigma(k) \approx \frac{-Ck}{C_0\sigma - (C_2 + \Theta(\sigma))k^2 - i[(C_3 + \Theta(\sigma))k^2]}.
\end{equation}

For $C_0 \sigma \approx C_2 k^2$ the denominator behaves like $C_3 k^3$, other-
wise it behaves like $C_0 \sigma - C_2 k^2$. Hence by (19)
\begin{equation}
|\hat{\Phi}_\text{out}(\sigma, k)|^2 \approx Ck(k^2 - (C_0 \sigma - |C_2| k^2)^2 + |C_3|^2) k^6)^{-1}
\end{equation}
This result (17) differs from the results given in the liter-
ature (see e.g. formula (7) in (10)). The right hand side of
(14) obviously diverges for $(\sigma, \mathbf{k}) \to (0,0)$. For fixed
$0 \neq \sigma \approx 0$ the divergent behavior is strongest close to the
resonance at $(C_0 \sigma - |C_2| k^2)^2 = 0$
\begin{equation}
k(\sigma) = \sqrt{\sigma C_0|C_2|}. \end{equation}
In view of (10) $\Delta(\sigma)$ can be roughly estimated by setting the right hand side of (14) equal to $1/2$ of its maximal size, i.e.

$$C_0 \sigma - |C_2| (k(\sigma) + \Delta(\sigma))^2 \approx |C_3| k^3(\sigma)$$

hence

$$\Delta(\sigma) \approx k(\sigma)^2 |C_2|(2|C_2|)^{-1}.$$  

(16)

For a rough estimate of the decay time $s_d$ we set $\sigma = s_d$ and use (12), (15) and (16). This yields

$$s_d^2 = \frac{8\varepsilon |C_2|^2}{|C_0|^2 |C_3|}.$$  

(17)

We turn now to the experimental verifications. The experimental verification of APC has been sought in heavy ion collisions (HIC) (but without success so far [2], [8]). Here the adiabatic time scale on which the field increases is directly determined by the relative speed with which the heavy ions approach each other and one computes that $\varepsilon$ is of order $10^{-1}$ [1]. Using (17) $s_d(\text{HIC}) \propto \varepsilon^{-2}$. The rigorous estimate taking into account the time dependence of the external field, yields $s_d \propto \varepsilon^{-2}$ ([2], [16]). Hence if the field stays overcritical for times much larger than $\varepsilon^{-2}$ the probability of pair creation is one, in the adiabatic case this is well satisfied. Thus HIC-APC is theoretically proven. An interesting prediction is the shape of the momentum distribution of the created positron. It would be nice if the shape would be simply the resonance [14] as suggested e.g. in [21]. But that requires a somewhat different situation than what one has in HIC. It would require an overcritical static field (adiabatic is not enough) of a life time much larger than the decay time $s_d$. Then in fact the resonance [14] would stay more or less intact (see also [25]). In an adiabatically changing field the resonance [14] changes however with the field and it is highly unclear how.

Another experimental situation with adiabatically changing fields is provided by lasers. For laser fields (wavelength $\lambda$) $\varepsilon = \lambda/\lambda_c \ll 1$. It is in fact hoped, that APC can be seen in a new generation of lasers which are able to create in focus a very well localized overcritical classical field. But the adiabatic scenario requires a bound state curve bridging the gap. Unfortunately such curves do not exist for the Dirac equation without electric potential, i.e. they do not exist for laser fields. For this it is important to note that only the four-vector-potential enters in the Dirac equation. Descriptions in terms of $E^-$ or $B^-$ fields may therefore be dangerously misleading. This is similar to the role of the $A$ and $B$ field in the Aharonov Bohm effect.

We give now a simple argument, why for vector potentials with $A_0 = 0$ it is unlikely that a bound state curve of the Dirac equation which crosses the spectral value zero exists. Starting with $(\Delta = \sum_{j=1}^3 A_j \alpha_j)$

$$(D_0 + \Delta)\Phi = (E - A_0)\Phi$$

squaring yields

$$(H + m^2 + \mu |B|)\phi_j = E^2 \phi_j$$

where $H$ is the Schrödinger operator with vector potential $A$ in Coulomb gauge. Assuming $\nabla A$ being constant (constant magnetic field) we go to the eigenspaces of this matrix setting $\varphi_j = \langle s_j, \Phi \rangle$. Then, if $A_0 = 0$, we have that

$$(H + m^2 + \mu |B|)\varphi_j = E^2 \varphi_j.$$  

From this we see that the energy curve of the Schrödinger operator $H + m^2 + \mu |B|$ must be quadratically zero if there is a bound state curve of the corresponding Dirac operator crossing zero. This however is a very special behavior of eigenvalues and thus unlikely. On the other hand if $A_0 \neq 0$ this argument breaks down and one cannot conclude that crossing curves are unlikely. In fact a recent paper [15] works out that in the latter case crossing curves do exist.

On this point the literature on laser pair creation is unclear since it is based on an old result by Schwinger [12], who predicted a Klein’s paradox effect for a constant strong electric field. Since then works only focus on the electric field, modeled in such a way that it produces an electric potential $\lambda$ see also [2], [14]. If it were the case that one could model the laser field by an electric potential than one could conclude that pair creation would be observable with optical lasers at a moderate rate, actually twice per period, see also [27].

Going back to our argument and [15] there is the possibility for experimental APC by combining lasers and heavy ions fields, i.e. shooting heavy ions into the focus of the laser. In [15] a lower bound for the critical val of a (spatially constant) magnetic field is given, namely $\frac{\lambda}{\lambda_c^2} \ll \frac{m^2}{\epsilon c^2}$. We recall that $\frac{m^2}{\epsilon c^2}$ is the critical field strength which satisfies $\mu B = mc^2$. For uranium this yields that the field has to be two times overcritical. (Note that the field for a given time is practically constant over the range of the heavy ion potential.) Such fields are expected to be reached by a new generation of lasers, called XFEL [12].

One further remark is in order. The moral of APC is to have a potential well the deepness of which changes adiabatically in time over an energy range of at least $2mc^2$. One can think to create such a field by electrostatic arrangements like charged plates (condenser) which are in space widely separated (adiabatically changing in space). Our derived time scales do not any more apply in this case because now the constants involve the length scale of the separation. It is nevertheless the case that pair creation will occur with a moderate rate for electric field strengths much smaller than the one predicted by Schwinger [12]. It is however unclear how such pairs can be measured.