Noncommutative field approach in gravity, Lorentz violation and torsion generation


1Instituto de Física, Universidade de São Paulo
Caixa Postal 66318, 05315-970, São Paulo, SP, Brazil
2Departamento de Física, Universidade Federal da Paraíba
Caixa Postal 5008, 58051-970, João Pessoa, Paraíba, Brazil

We develop the noncommutative fields approach for the linearized Einstein gravity. As a result an additive Lorentz-breaking torsion term, proportional to the noncommutativity parameter, is shown to be generated in the Lagrangian. The same term is shown to be generated by the Lorentz-breaking coupling of the gravity field to a spinor field. Its presence implies in nontrivial modification of the dispersion relations which allows us to conclude that the CPT symmetry in the modified theory is broken.

One of the important implications of some formulations of the space-time noncommutativity is the Lorentz violation (see e.g., [2]) and the consequent modifications of dispersion relations [3, 4]. Recently, it was suggested a new mechanism to generate the breaking of the Lorentz invariance, that is the so called noncommutative fields approach [3]. Its essence consists in non-canonical deformations of the commutation relations between the canonical variables of a physical theory, implying in the arise of new additive terms in the classical action. Unlike the popular approaches of the noncommutative field theory (such as Moyal product [1], coherent states [5] and models with dynamical noncommutativity parameter [6]), in this case the space-time coordinates continue to be commutative. As a result of applying the noncommutative fields approach, the Hamiltonian of the theory (and, as a consequence, the Lagrangian after reformulation of the theory in terms of the Lagrange variables) is modified by new additive Lorentz-breaking terms. In the papers [7, 8] this method was applied to the electrodynamics and the Yang-Mills theories. The next problem, which we will address in this paper, consists in the application of this method to the (linearized) Einstein gravity. We will study the generation of Lorentz-breaking terms and investigate their physical implications. We would remind that possibility of generation of Lorentz-breaking terms for the gravity by perturbative corrections was proved in [9]; see also [10] for an alternative approach to

*Electronic address: alysson.mgomes.ajsilva@fma.if.usp.br
†Electronic address: jroberto.passos.petrov@fisica.ufpb.br
this problem. We show that the gravitational waves display a kind of the birefringence phenomenon
with modify dispersion relations.

The starting point of our study is the Einstein-Hilbert action in the weak field approximation,
also known as the Fierz-Pauli action (see f.e. [11]):

\[ S_{\text{free}} = \int d^4x \left( \frac{1}{2} \left[ \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} - \partial_\lambda h \partial_\mu h^{\mu\lambda} \right] + \frac{1}{4} \left[ \partial_\mu h \partial^\mu h - \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} \right] \right). \] (1)

Here the \( h_{\mu\nu} \) is a second rank symmetric tensor characterizing (weak) metric fluctuation
\( h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \) where \( g_{\mu\nu} \) is a metric tensor of the curved space, and \( \eta_{\mu\nu} = \text{diag}(- + + +) \) is the metric
tensor of the flat space), \( h = \eta^{\mu\nu} h_{\mu\nu} \) is the trace of \( h_{\mu\nu} \). This theory is invariant under the gauge
transformations [11]:

\[ \delta h_{\mu\nu} = \frac{1}{2} \left( \partial_\rho \xi_\nu + \partial_\nu \xi_\rho \right). \] (2)

The equations of motion in this theory look like [12]

\[ -\frac{1}{2} \left( \partial_\lambda \partial_\mu h_{\lambda\nu} + \partial_\lambda \partial_\nu h_{\lambda\mu} \right) + \frac{1}{2} \eta_{\mu\nu} \partial_\lambda \partial_\rho h^{\lambda\rho} + \frac{1}{2} \partial_\mu \partial_\nu h + \frac{1}{2} \Box h_{\mu\nu} - \eta_{\mu\nu} \frac{1}{2} \Box h = 0. \] (3)

To quantize the theory we would first construct the classical Hamiltonian. This procedure was
presented in [13] (see [14, 15] for alternative approaches to the problem). Here we give a short
description. First, the indices of the Lagrangian can be split into time and space ones. After some
rearrangements, the Lagrangian corresponding to \( \) takes the form

\[ L = -\frac{1}{4} h_{ij} \dot{h}_{ij} - \frac{1}{2} \partial_k h_{ij} \partial_k h_{00} + \frac{1}{4} \partial_i h_{ij} \partial_i h_{kk} + \]
\[ + \frac{1}{4} h_{ij} \dot{h}_{ij} + \frac{1}{2} \partial_i h_{0j} \partial_i h_{0j} - \frac{1}{4} \partial_i h_{jk} \partial_i h_{jk} + \]
\[ + \dot{h}_{ij} \partial_j h_{00} - \frac{1}{2} \partial_i h_{kk} \partial_i h_{ij} + \frac{1}{2} \partial_i h_{00} \partial_j h_{ij} - \]
\[ - \dot{h}_{ik} \partial_i h_{0k} - \frac{1}{2} \partial_i h_{0i} \partial_j h_{0j} + \frac{1}{2} \partial_i h_{jk} \partial_j h_{ik}. \] (4)

Here the Latin indices stand for the pure space coordinates and \( \dot{f} \equiv \partial_0 f \). We see that the La-
grangian does not depend on the velocities corresponding to \( h_{00}, h_{0i} \), i.e. the conjugated canonical
momenta are equal to zero: \( p^0_\mu = \frac{\partial L}{\partial \partial_0 h^{\mu}} = 0 \). So we can write down the primary constraints:

\[ \Phi_\mu^{(1)} = p_0^\mu \simeq 0, \] (5)

which evidently commute with each other. The other momenta are given by

\[ p_{ij} = \frac{\partial L}{\partial \dot{h}_{ij}} = -\frac{1}{4} \dot{h}_{kk} \delta_{ij} + \frac{1}{2} \dot{h}_{ij} + \partial_k h_{0k} \delta_{ij} - \frac{1}{2} (\partial_i h_{0j} + \partial_j h_{0i}). \] (6)
Under the gauge transformations (2), they are transformed as
\[ \delta p_{ij} = \frac{1}{2} (\delta_{ij} \partial_k \partial_k - \partial_i \partial_j) \xi_0. \] (7)
The velocities are expressed from the equation (6) as
\[ \dot{h}_{ij} = 2p_{ij} - p_{kk} \delta_{ij} + (\partial_i h_{0j} + \partial_j h_{0i}), \] (8)
and the canonical Hamiltonian density is given by,
\[ H = p^{\mu\nu} \dot{h}_{\mu\nu}(p) - L = p_{ij} p_{ij} - \frac{1}{2} p_{kk} p_{ll} + \frac{1}{2} (\partial_i h_{kk} \partial_j h_{ij} - \partial_i h_{jk} \partial_j h_{ik}) + \frac{1}{4} (\partial_i h_{jk} \partial_j h_{ik} - \partial_i h_{jj} \partial_j h_{kk}) - 2h_{0j} \partial_i p_{ij} - \frac{1}{2} h_{00} (\partial_i \partial_i h_{kk} - \partial_i \partial_j h_{ij}). \] (9)
Conservation of the primary constraints (5) implies in
\[ \{ \Phi^{(1)}_\mu, H \} \simeq 0, \] (10)
where \{\} are the Poisson brackets, which gives the secondary constraints:
\[ R_j \equiv \Phi^{(2)}_j = \partial_i p_{ij} \simeq 0; \quad R_0 \equiv \Phi^{(2)}_0 = \partial_i \partial_i h_{kk} - \partial_i \partial_j h_{ij} \simeq 0. \] (11)
The \( h_{00}, h_{0i} \) are the Lagrange multipliers associated to these constraints. We note that the condition of conservation of the secondary constraints imply in only a new, tertiary constraint,
\[ \Phi^{(3)} = \{ \Phi^{(2)}_0, H \} = \partial_i \partial_j p_{ij} \simeq 0, \] (12)
whereas \( \{ \Phi^{(2)}_j, H \} \equiv 0 \). The constraint \( \Phi^{(3)} \) closes the Dirac algorithm since its Poisson bracket with the Hamiltonian is equal to zero (note that all the constraints in the theory are of first class).
We note, however, that this constraint is really a linear combination of the \( \Phi^{(2)}_j \) constraints, so its adding to the Hamiltonian with an arbitrary scalar multiplier \( \lambda \) implies only in the replacement \( h_{0j} \rightarrow h_{0j} + \partial_j \lambda \) in (9)\( \). In fact, only the Lagrange multipliers associated to the constraints \( \Phi^{(2)}_j \) are modified under such a replacement but not the constraints themselves. Therefore we will not consider the tertiary constraint henceforth.

The constraints \( \Phi^{(2)}_0 \equiv R_0 \) and \( \Phi^{(2)}_i \equiv R_i \) generate the gauge symmetry. For the quantization we must convert the metric \( h_{ij} \) and momenta \( p_{ij} \) into operators whose commutation relations will be obtained from the classical Poisson brackets algebra,
\[ \{ p_{ij}(\vec{x}), p_{kl}(\vec{y}) \} = 0, \]
\[ \{ h_{ij}(\vec{x}), h_{kl}(\vec{y}) \} = 0, \]
\[ \{ h_{ij}(\vec{x}), p_{kl}(\vec{y}) \} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}) \delta(\vec{x} - \vec{y}). \] (13)
In this case the constraints \( R_k, R_0 \) generate in the purely spacial sector (which is the physical sector) the gauge transformations:

\[
\begin{align*}
\delta h_{ij} &= \{ h_{ij}, \Delta \xi \} = \frac{1}{2}(\partial_i \xi_j + \partial_j \xi_i); \\
\delta p_{ij} &= \{ p_{ij}, \Delta \xi \} = \frac{1}{2}(\delta_{ij}\partial_k \partial_k - \partial_i \partial_j)\xi_0
\end{align*}
\]

(14)

where

\[
\Delta \xi = - \int d^3x R_k(x)\xi_k(x) + \frac{1}{2} \int d^3x R_0(x)\xi_0(x)
\]

(15)

is the generator of gauge transformations which is equal to

\[
\Delta \xi = \frac{1}{2} \int d^3x p_{ij}(x) \left[ \partial_i \xi_j(x) + \partial_j \xi_i(x) \right] - \frac{1}{2} \int d^3x h_{ij}(x) \left[ \delta_{ij}\partial_k \partial_k - \partial_i \partial_j \right] \xi_0(x)
\]

(16)

and \( \xi_i(x) \) is an arbitrary vector function of coordinates.

Following the method described in [7, 8], to introduce noncommutativity we deform this Poisson bracket algebra as

\[
\begin{align*}
\{ p_{ij}(\vec{x}), p_{kl}(\vec{y}) \} &= \theta_{ijkl}\delta(\vec{x} - \vec{y}), \\
\{ h_{ij}(\vec{x}), h_{kl}(\vec{y}) \} &= 0, \\
\{ h_{ij}(\vec{x}), p_{kl}(\vec{y}) \} &= \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})\delta(\vec{x} - \vec{y}),
\end{align*}
\]

(17)

where \( \theta_{ijkl} \) is a \( c \)-number symbol possessing the following symmetry: \( \theta_{1234} = \theta_{2134} = \theta_{1243} = -\theta_{3412} \). This symmetry is satisfied if

\[
\theta_{ijkl} = \theta_{ik}\delta_{jl} + \theta_{il}\delta_{jk} + \theta_{jl}\delta_{ik} + \theta_{jk}\delta_{il},
\]

(18)

where \( \theta_{ij} \) is a constant antisymmetric matrix. We introduce a modified generator of the gauge transformation; instead of the \( \Phi^{(2)}_k = \partial ip_{ik} \), we choose

\[
R_k = \partial_i p_{ik} - \theta_{klmn} \partial_l h_{nm} = \partial_i p_{ik} - 2(\theta_{kn}\partial_l h_{ln} + \theta_{ln}\partial_l h_{kn}).
\]

(19)

The operator \( \Delta \xi \) providing gauge transformations \( (14) \) in this case takes the form

\[
\begin{align*}
\Delta \xi &= \frac{1}{2} \int d^3x \left\{ (\partial_i \xi_k(x) + \partial_k \xi_i(x)) \left[ p_{ik}(x) + \theta_{rsik} h_{rs}(x) \right] - \xi_0(x) (\delta_{rs}\partial_l \partial_l - \partial_r \partial_s) h_{rs}(x) \right\} \\
&\equiv \frac{1}{2} \int d^3x \left\{ (\partial_i \xi_k(x) + \partial_k \xi_i(x)) \left[ p_{ik}(x) + 2(\theta_{ri} h_{rk}(x) + \theta_{rk} h_{ri}(x)) \right] - \xi_0(x) (\delta_{rs}\partial_l \partial_l - \partial_r \partial_s) h_{rs}(x) \right\}.
\end{align*}
\]

(20)
Modification of the $\Delta \xi$ (or, as is equivalent, of $R_k$) by an additive term proportional to $\theta_{rsik}$ implies in the modification of the Hamiltonian density:

$$H = p_{ij}p_{ij} - \frac{1}{2} p_{kk} p_{ll} + \frac{1}{2} (\partial_i h_{kk} \partial_j h_{ij} - \partial_i h_{jk} \partial_j h_{ik}) + \frac{1}{4} (\partial_i h_{jk} \partial_i h_{jk} - \partial_i h_{jj} \partial_i h_{kk}) -$$

$$- 2h_{0j} (\partial_i p_{ij} - \theta_{jlmn} \partial_l h_{nm}) - \frac{1}{2} h_{00} (\partial_i h_{kk} - \partial_i \partial_j h_{ij}),$$

(21)

which has been augmented by the term

$$\Delta H = 2h_{0j} \theta_{jlmn} \partial_l h_{nm} \equiv 4h_{0k} (\theta_{km} \partial_l h_{lm} + \theta_{lm} \partial_l h_{km}).$$

(22)

The velocities can be easily found to be

$$\dot{h}_{ij} = \{h_{ij}, H\} = 2p_{ij} - p_{ll} \delta_{ij} + (\partial_i h_{0j} + \partial_j h_{0i}),$$

(23)

reproducing Eq. (8) from which the $p_{ij}$ are expressed through velocities just like in Eq. (6).

The canonical conjugate momentum $\hat{\pi}_{ij}$ for $h_{ij}$ (which should satisfy the commutation relation $\{\hat{\pi}_{ij}, \hat{\pi}_{kl}\} = 0$) is

$$\hat{\pi}_{ij} = p_{ij} + \frac{1}{2} \theta_{kl} h_{kl} = p_{ij} + (\theta_{ri} h_{rj} + \theta_{rj} h_{ri}).$$

(24)

The Lagrangian restored by the rule

$$L = \hat{\pi}_{ij} \dot{h}_{ij} - H,$$

(25)

differs from the initial Lagrangian by additive terms. By defining $\theta_{ij} = -\epsilon_{0ijk} \theta^k$, we have

$$L_{new} = \frac{1}{2} (\partial_\lambda h_{\mu\nu} \partial^\mu h^{\lambda\nu} - \partial_\lambda h_\mu h^{\mu\lambda}) + \frac{1}{4} (\partial_\mu h \partial^\nu h - \partial_\lambda h_\mu \partial^\lambda h^{\mu\nu}) -$$

$$- 4h_{0k} (\theta_{km} \partial_l h_{lm} + \theta_{lm} \partial_l h_{km}) + \dot{h}_{ij} (\theta_{ki} h_{kj} + \theta_{kj} h_{ki}),$$

(26)

differing from the initial Lagrangian by additive terms. By defining $\theta_{ij} = -\epsilon_{0ijk} \theta^k$, we have

$$L_{new} = \frac{1}{2} (\partial_\lambda h_{\mu\nu} \partial^\mu h^{\lambda\nu} - \partial_\lambda h_\mu h^{\mu\lambda}) + \frac{1}{4} (\partial_\mu h \partial^\nu h - \partial_\lambda h_\mu \partial^\lambda h^{\mu\nu}) +$$

$$+ 2\epsilon^{\mu\nu\lambda\kappa} \theta_{mu} \partial_\lambda h^{\mu}_\kappa + 2\epsilon_{ijk} \theta_k h_{0i} \left(2\partial_i h_{ij} + 2\partial_j h_{00} - \dot{h}_{0j}\right).$$

(27)

However, the last term vanishes if $h_{00} = h_{0j} = 0$. Thus the last term does not affect the physical dynamics and only modifies the constraints, whose algebra is not changed. Therefore, we can disregard this term by choosing $h_{00} = h_{0j} = 0$ and restrict ourselves to the Lagrangian

$$L_{new} = \frac{1}{2} (\partial_\lambda h_{\mu\nu} \partial^\mu h^{\lambda\nu} - \partial_\lambda h_\mu h^{\mu\lambda}) + \frac{1}{4} (\partial_\mu h \partial^\nu h - \partial_\lambda h_\mu \partial^\lambda h^{\mu\nu}) + \Delta L,$$

(28)
where

$$\Delta L = -2 \epsilon^{\lambda\mu\rho\theta} \rho h_{\nu\sigma} \partial_{\lambda} h_{\mu}^{\sigma}. \quad (29)$$

We have shown that the Lagrangian has been modified by a new term which breaks the Lorentz invariance, and reproduces one of Lorentz-violating terms presented in [16]. Nevertheless, this $\Delta L$ term has a restricted gauge invariance. Indeed, the gauge transformations $\delta$ imply in the following variation of $\Delta L$:

$$\delta \Delta L = 2 \epsilon^{\lambda\mu\rho\theta} \rho \xi_{\nu} \partial_{\lambda} \partial_{\sigma} h_{\mu}^{\sigma}, \quad (30)$$

which vanishes in the harmonic gauge $\partial_{\sigma} h_{\sigma\mu} = \frac{1}{2} \partial^{\mu} h$. \quad (31)

So, $\Delta L$ is not modified under gauge transformations compatible with the harmonic gauge condition, that is, for gauge parameters $\xi^{\mu}$ satisfying the d’Alembert equation $\Box \xi^{\mu} = 0$. Hence we conclude that the new term $\Delta L$ possesses a gauge invariance with a restricted gauge parameter.

Alternatively, we note that if the metric fluctuation satisfies the Hilbert condition [19]:

$$\partial_{\sigma} h_{\sigma\mu} = 0, \quad (32)$$

the variation $\delta \Delta L$ vanishes. Thus, we found that for specific metric configurations which possess some residual gauge symmetry, the $\theta$ dependent contribution to the Lagrangian is gauge invariant.

Moreover, the term (29) has a nontrivial geometric content. Indeed, the torsion is expressed in terms of the vielbein $e_{\mu}^{a}$ and spinor connection as [16]

$$T_{\lambda\mu\nu} = e_{\lambda}^{a}(\partial_{\mu} e_{\nu a} + \omega_{\mu ab} e_{\nu}^{b}) - (\mu \leftrightarrow \nu). \quad (33)$$

The vielbein and the connection, expanded around the flat background [21] are given respectively by

$$e_{\mu}^{a} = \delta_{\mu}^{a} + \frac{1}{2} h_{\mu}^{a}, \quad (34)$$

and

$$\omega_{\mu ab} = -\frac{1}{2} \partial_{a} h_{\mu b} + \frac{1}{2} \partial_{b} h_{\mu a}. \quad (35)$$

Replacing these expressions into (33), we get

$$T_{\lambda\mu\nu} = \frac{1}{4} \left[ h_{\nu}^{b} (\partial_{b} h_{\mu\lambda} - \partial_{\lambda} h_{\mu b}) - h_{\mu}^{b} (\partial_{b} h_{\nu\lambda} - \partial_{\lambda} h_{\nu b}) \right]. \quad (36)$$
Here, as it is always done in the weak field approximation, only the lowest order terms (those of second order) in the metric fluctuation $h_{\mu\nu}$ are taken into account. Therefore the contraction of this term with $4\epsilon^{\lambda\mu\nu\rho}\theta_\rho$ with subsequent integration by parts gives

$$4 \int d^4x \epsilon^{\lambda\mu\nu\rho}\theta_\rho T_{\lambda\mu\nu} = -2 \int d^4x \epsilon^{\lambda\mu\nu\rho}\theta_\rho h_{\nu\sigma} \partial_\lambda h_\sigma,$$

which exactly reproduces the space-time integral of $\Delta L$. We conclude that

$$\Delta L = 4 \int d^4x \epsilon^{\lambda\mu\nu\rho}\theta_\rho T_{\lambda\mu\nu},$$

i.e. the noncommutative fields method allows us to generate a torsion term in the action. As a by-product we find that in the three-dimensional Einstein gravity the noncommutative field approach does not violate Lorentz symmetry (in that case, instead of $\epsilon^{\lambda\mu\nu\rho}\theta_\rho$, the factor $\epsilon^{\lambda\mu\nu}$ will arise, with $\theta$ a scalar noncommutativity parameter). In three dimensions, the torsion is structurally similar to the “electrodynamical” Chern-Simons term (which have arisen in the three-dimensional case [18]).

The equations of motion in the theory [28] look like

$$\frac{1}{2} \left( \partial^\nu \partial_\rho h_{\lambda\nu} + \partial^\lambda \partial_\nu h_{\mu\lambda} \right) + \frac{1}{2} \eta_{\mu\nu} \partial_\lambda \partial_\rho h^\lambda_\rho + \frac{1}{2} \partial_\mu \partial_\nu h - \frac{1}{2} h_{\mu\nu} - \eta_{\mu\nu} \frac{1}{2} \Box h + 4 \epsilon_{\alpha\beta\lambda\mu} \partial^\beta \partial^\lambda h^\alpha_\rho = 0.$$

It is interesting to study the dispersion relations for the modified theory [28], i.e. the relations between energy and spacial momentum. To do it, we can, following [9], split the components of the metric fluctuation as follows:

$$h^{00} = n, \quad h^{0i} = \tilde{n}^i + \partial^i n_L, \quad h^{ij} = \left( \delta^{ij} - \frac{\partial^i \partial^j}{\Box} \right) \phi + \frac{\partial^i \partial^j}{\Box^2} \chi + \partial^i \xi^j + \partial^j \xi^i + \tilde{h}^{ij},$$

where $\tilde{n}^i$, $\tilde{\xi}^i$ and $\tilde{h}^{ij}$ are transversal, $\partial_i \tilde{n}^i = \partial_i \tilde{\xi}^i = \partial_i \tilde{h}^{ij} = 0$. We introduce also $\tilde{\sigma}^i = \tilde{n}^i + \tilde{\xi}^i$ and $\Lambda = \Box^2(n + 2\tilde{n}_L) + \chi + \Box \phi$. As a result, the Lagrangian [28] takes the form

$$L = \frac{1}{4} \tilde{h}^{ij} \Box \tilde{h}_{ij} + \frac{1}{2} \phi \Box \phi - \frac{1}{2} (\partial^i \tilde{\sigma}^j)^2 - \phi \Lambda \ \ - 2\tilde{h}^{ij} \theta_{ki} \hat{h}_{kj} - 4\theta_{ik} \hat{h}_{ij} \partial_j \tilde{\xi}^i + 4(\chi - \phi) \theta_{ki} \partial^i \xi^k + 2\theta_{ki} \xi^k \Box^2 \tilde{\xi}^i + 4\theta_{km} \tilde{n}_k \partial_m (\phi - \chi) - 4\theta_{km} \tilde{n}_k \Box^2 \tilde{\xi}^i - 4\theta_{km} \tilde{n}_l \partial^m \partial^k (\partial_j \tilde{\xi}^i + \tilde{h}_{im})$$

We see that, like in [9], the only propagating degrees of freedom are $\tilde{h}_{ij}$. To study wave propagation, we choose that only $\theta_{12} = -\theta_{21} = \frac{1}{4} \theta$ are different from zero, with the wave propagates in $x_3$. 
direction which coincides with the direction of the noncommutativity vector $\vec{\theta} = \{\theta_i\}$. In this case, the equation of motion turns out to be

$$\Box \tilde{h}_{ij} - 4(\dot{\tilde{h}}_{ik}\theta_{kj} + \dot{\tilde{h}}_{jk}\theta_{ki}) = 0. \quad (42)$$

Due to symmetry and tracelessness, the $\tilde{h}_{ij}$ has two components: $\tilde{h}_{11} = -\tilde{h}_{22} = T$, $\tilde{h}_{21} = \tilde{h}_{12} = S$. So, we have two independent equations:

$$\Box Z + 2i\theta \dot{Z} = 0$$
$$\Box \bar{Z} - 2i\theta \dot{\bar{Z}} = 0, \quad (43)$$

where $Z = T + iS$, $\bar{Z} = T - iS$. The corresponding dispersion relations are given respectively by:

$$E = -\theta \pm \sqrt{p^2 + \theta^2},$$
$$E = \theta \pm \sqrt{p^2 + \theta^2}. \quad (44)$$

Thus we found that the dispersion relations are modified. The theory describes two types of particles-antiparticles, with the dispersion relation for one type being $E = \pm(\sqrt{\theta^2 + p^2} + \theta)$, and for the other being $E = \pm(\sqrt{\theta^2 + p^2} - \theta)$, with $p = |\vec{p}|$. Thus, the propagation of gravitational waves in the noncommutative media displays a kind of birefringence phenomenon. We find two types of waves with the velocities of propagation different from each other and from the speed of light, similarly to the propagation of the electromagnetic waves in the noncommutative space [9]. It must be stressed that these two different polarizations (and velocities) correspond to two "circular" polarizations with respect to the $\vec{\theta}$. The group velocity $\dot{c} = \frac{dE}{dp} = \frac{p}{\sqrt{p^2 + \theta^2}}$ is always less than the speed of light. On the other hand, the phase velocity $\dot{\bar{c}} = \frac{E}{p}$ can be superluminal for one of the polarizations. Also, the symmetry of positive- and negative-energy solutions shows that C-symmetry is preserved. At the same time, the presence of an odd-index Lorentz-violating tensor is known to break the CPT-symmetry [20]. Thus, we conclude that the additive term (29) generates CPT breaking.

Let us make some remarks about conservation laws in the modified gravity theory [28]. Since there exists a preferable direction in the space-time, the angular momentum is no more conserved. However, the energy-momentum tensor is still conserved since the space-time continues to be homogeneous. Following the Noether theorem, we can write the energy-momentum tensor as

$$\Theta^b_a = -\delta^b_a \left[ \frac{1}{2}(\partial_\lambda h_{\mu\nu} \partial^{\nu} h^{\lambda\mu} - \partial_\lambda h \partial_\mu h^{\lambda\nu}) + \frac{1}{4} (\partial_\mu h \partial^{\mu} h - \partial_\lambda h_{\mu\nu} \partial^{\lambda} h^{\mu\nu}) \right] +$$

$$+ \partial_\nu h^{\nu\lambda} \partial_a h_{\lambda} - \partial_a h \partial_\nu h^{\nu\lambda} -$$

$$- 2\delta^b_a \epsilon_{\mu\nu\lambda\rho} \partial^\rho h^{\mu\sigma} \partial^{\lambda} h_{\sigma} + 2\epsilon^{\mu\nu\rho} h_{\mu} \partial_a h_{\nu\lambda} h^{\lambda}_\rho. \quad (45)$$
Under the Hilbert condition (32), the covariantized form of this tensor (up to a surface term) is reduced to

\[ \Theta^{ab} = -\frac{1}{4} \eta^{ab} \left[ \partial_\mu h \partial_\nu h - \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} \right] - 2\eta^{ab} \epsilon_{\mu\nu\lambda\rho} \partial^\rho h_{\mu\nu} + 2\epsilon_{\mu\nu\rho\sigma} \partial^\rho h^{\nu\lambda} h_{\rho\lambda}. \] (46)

Thus, the energy-momentum tensor is modified by an additive $\theta$-dependent term although it remains conserved. Moreover, it is easy to check that $\Theta^{0i} \neq \Theta^i 0$, which is a natural consequence of the Lorentz breaking in the theory [9].

As a final observation, it is interesting that $\Delta L$ given by Eq. (29) can be generated as a quantum correction by conveniently coupling to a fermion field instead of modifying the Poisson bracket. Indeed, let us consider the action of linearized gravity coupled to a Dirac field. The corresponding action looks like [10]:

\[ S[h, \bar{\psi}, \psi] = \int d^4x \left( \frac{1}{2} i \bar{\psi} \Gamma^\mu \partial_\mu \psi + \bar{\psi} h_{\mu\nu} \Gamma_{\mu\nu} \psi - \bar{\psi} b_{\mu} \gamma^\mu \gamma^5 \psi \right) + S_{\text{free}}, \] (47)

with $\Gamma^\mu = \gamma^\mu - \frac{1}{2} h^{\mu\nu} \gamma_\nu$ and $\Gamma^{\mu\nu} = \frac{1}{2} b^{\mu} \gamma^\nu - \frac{1}{16} (\partial_\rho h_{\alpha\beta}) \eta_{\beta\nu} \Gamma^{\rho\mu\alpha}$. Really, the terms proportional to $\Gamma^{\mu\nu}$ will not contribute to the $\Delta L$, Eq. (29). The two-point vertex function of the graviton field receives contributions from the graphs in Fig. 1.

The sum of these contributions, expanded up to the leading order in derivatives of the metric fluctuation, generates the following one-loop correction to the two-point vertex function of $h_{\mu\nu}$:

\[ \Delta \tilde{L} = -i \epsilon^{\lambda\nu\sigma} \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\alpha}{(k^2 - m^2)^3} \left[ b_\mu (k^2 + 3m^2) - 4k_\rho (b \cdot k) \right] h_{\mu\nu} \partial_\sigma h_{\alpha\lambda}. \] (48)

After the explicit integration carried out with the use of the dimensional regularization we get

\[ \Delta \tilde{L} = \frac{m^2}{128\pi^2} \left( 1 + \frac{1}{\varepsilon} \right) \epsilon^{\lambda\nu\rho} b_\rho h_{\mu\nu} \partial_\sigma h^{\mu}_{\lambda}, \] (49)

which reproduces the structure of Eq. (29) generated by deformation of the canonical variables algebra. This term, however, is divergent which is due to the non-renormalizability of the theory given by Eq. (47).

In this paper, we have developed a noncommutative fields approach to the linearized Einstein gravity. The crucial observation is that such a quantization modifies the Lagrangian of the theory by a Lorentz-violating term, which in the case of gravity is shown to be a torsion term. We found that such a term possesses a restricted gauge invariance and modifies the dispersion relations. Again, just as in [18], we found that the noncommutative fields mechanism does not generate Lorentz symmetry breaking when applied to three-dimensional theories. Also, we can suggest that
the noncommutative fields formalism \cite{7,8} has the following interpretation: the noncommutative deformation of the canonical variables algebra for the gauge theories is an efficient equivalent description of coupling of the gauge fields to some extra fields, since both descriptions imply in similar results. We note that in principle this approach can be generalized to obtain terms with higher derivatives, f.e. the gravitational Chern-Simons term \cite{10}. The manifest results will be presented elsewhere.

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The Eqs. (34, 35) are valid in the usual case where the antisymmetric part of the vielbein fluctuation and the (background) contortion tensor vanish.
Figure 1: One-loop contributions to the graviton two-point function. The cross in the fermion lines denotes a $b^\mu$ insertion.