Multipole structure of current vectors in curved spacetime

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A method is presented which allows the exact construction of conserved (i.e. divergence-free) current vectors from appropriate sets of multipole moments. Physically, such objects may be taken to represent the flux of particles or electric charge inside some classical extended body. Several applications are discussed. In particular, it is shown how to easily write down the class of all smooth and spatially-bounded currents with a given total charge. This implicitly provides restrictions on the moments arising from the smoothness of physically reasonable vector fields. We also show that requiring all of the moments to be constant in an appropriate sense is often impossible; likely limiting the applicability of the Ehlers-Rudolph-Dixon notion of quasirigid motion. A simple condition is also derived that allows currents to exist in two different spacetimes with identical sets of multipole moments (in a natural sense).

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I. INTRODUCTION

There are a number of instances of “currents” appearing throughout physics as fields which somehow express a conservation law. Here, we will be studying vector fields \( \mathbf{J}(x) \) in 3+1 dimensional spacetimes. In classical physics, a common example of such an object would be the electromagnetic current. Another would be the numerical flux density of some species in a continuum (as long as the total number of such particles isn’t expected to change via chemical or other reactions).

Keeping these examples in mind, we take the support of such currents to be a spatially-bounded region \( W \) which surrounds a timelike worldline \( Z \). Intuitively, this could be thought of as the worldtube of some extended body. The “central” worldline would then be most naturally identified with the body’s center-of-mass.

We may also use any spacelike hypersurface \( S \) which cuts \( W \) to define a charge in the usual way:

\[
q(S) := \int_S \mathbf{J} \cdot d\mathbf{S}. \tag{1.1}
\]

The current would be “conserved” if \( q(S) \) were independent of \( S \). This clearly occurs when

\[
\nabla \cdot \mathbf{J} = 0. \tag{1.2}
\]

Our purpose in this paper is to study vector fields which satisfy this equation and have the given support properties. In a sense, finding such fields is actually quite simple. This can be seen by constructing a 3-form from \( \mathbf{J} \) in the natural way, and then observing that \( [12] \) is equivalent to requiring that this form be closed. Solutions follow immediately. For example, \( [1] \) discusses a common procedure in continuum mechanics whereby the number density flux is essentially the pullback to spacetime of a fixed volume form defined in a three-dimensional “body space.” This method is quite natural when discussing the worldlines of a body’s constituent particles, but may be rather awkward in other cases.

For example, it is often useful to decompose a current into a set of multipole moments. In some cases, moments may be experimentally determined be measuring the asymptotic behavior of the fields sourced a body. If \( \mathbf{J} \) happened to be an electric current, the moments would also appear as coefficients in an expansion for the net force and torque due to externally applied electromagnetic fields \( [2,3,4] \). Even when including the effects of self-fields, some aspects of the multipole decomposition remain useful in a wide range of scenarios \( [2,5] \).

These types of methods also provide an elegant way of separating out different scales when discussing the structure of some extended body. Since the moments (that we will choose) are in fact tensor fields defined along \( Z \), it becomes relatively simple to discuss this structure from the point of view of a single comoving observer. This makes a number of problems more intuitive.

Unfortunately, multipole moments are related to \( \mathbf{J} \) (or the deformation gradient which may be used to construct it) in a rather complicated way. In this paper, we shall take the point of view that the moments themselves are fundamental. Other more commonly-discussed quantities are to be derived from them. It is shown how to explicitly construct \( \mathbf{J} \) in this way in Sec. \( [1] \) which leads to the introduction of two “moment potentials” defined on the tangent bundle restricted to \( Z \). A moment may be derived from these quantities essentially by taking a (three-dimensional) Fourier transform, and then differentiating a sufficient number of times. Importantly, these potentials are not required to obey any evolution equations. Their “time-dependence” is essentially arbitrary.

Not all of them will describe physically reasonable currents, however. Most choices introduce singularities. Sec. \( [1] \) discusses how to avoid such problems. This implicitly shows which restrictions must be placed on the moments for them to represent plausible current vectors.

As a byproduct of this construction, we obtain a simple method of constructing the class of essentially all smooth currents with a given total charge. The situation
is nearly identical to the one in flat spacetime, where the charge density and three-current observed by someone on $Z$ are very simply related to the moment potentials. This could be useful in studying corrections to the Dewitt-Brehme expression [3] for the electromagnetic self-force on a charged test particle in curved spacetime (which is intended to describe all “sufficiently small” charges with a given $q$). The flat spacetime analog of this idea was carried out in [2].

Lastly, we investigate some consequences of imposing simple $a priori$ conditions on the moment set itself. Sec. [V] supposes that the moments are all constant with respect to some frame attached to the central worldline $Z$. This closely mirrors the idea of quasi-rigid motion advanced in [7, 8, 9]. Unfortunately, it is shown that the moment can only remain constant under certain conditions involving the Riemann tensor and the acceleration of $Z$ (roughly speaking).

Sec. [V] then looks at the possibility of two currents—possibly in different spacetimes—having identical sets of multipole moments as viewed in their respective frames. As in the case of setting all of the moments constant, this is not always possible. A restriction on the Riemann tensors on both central worldlines is derived which allows these types of “comparable” currents to exist.

The system of multipole moments adopted here is the one developed by W. G. Dixon [3, 4, 7, 8, 10]. The relevant portions of the formalism are reviewed in appendix B. It has a number of desirable properties without which many of the results in this paper would likely be impossible to reproduce. It also has a very elegant place in the (exact) theory of relativistic mechanics. And perhaps most importantly, the formalism also works for symmetric rank-2 tensors $T$ satisfying $\nabla \cdot T = 0$ (or more generally $\nabla \cdot T = F \cdot J$ with $F$ a closed 2-form; e.g. an electromagnetic field). This paper may therefore be taken as a model for attempting to construct all possible stress-energy tensors with a given center-of-mass and net linear and angular momentum vectors.

The results here also make some use of bitensors. Appendix A reviews the subject, and states some results used in text.

For the remainder of this paper, indices shall always be included on tensor or bitensor quantities. Those which refer to points on $Z$ will be denoted by $a, b, \ldots$. Indices associated with all other points in $W$ will be written as $a', b', \ldots$. Tetrad labels running from 0 to 3 will be written as $I, J, \ldots$. We choose the signature of the metric to be $-+++$. 

II. CONSTRUCTING CURRENTS

At this point, we assume that the reader has at least skinned the appendices to gain familiarity with the notation and basic elements of Dixon’s formalism. These outline how conserved vector fields in spacetime may be represented by the current skeleton $\tilde{J}^a$; a vector-valued distribution defined on the tangent bundle (restricted to a neighborhood of $Z$). Dixon has actually proven that all well-behaved currents may be uniquely represented in this way [4].

The reverse is not true, however. One could pick moments satisfying all of the given constraints, and yet the right-hand side of (19) may not even exist. Then there wouldn’t be any current vector at all associated with that set of moments. Even when the integrals do exist, the resulting $J^a$ may be singular. Such a result would clearly be unphysical.

The main purpose of this paper is to remove this problem. We want to be able to specify a current skeleton in some simple way which automatically guarantees that the associated vector field is both physically reasonable and has a given charge $q$. We do this by constructing $J^a$ explicitly in terms of $\tilde{J}^a$, and then checking that the result is reasonable.

Our first step will be to construct current vectors from all of the skeletons satisfying the constraints described in appendix B. The natural starting point for this is of course (19). It is convenient to convert the integral over $X$ on the right-hand of this equation into one over $x = \exp_x X$. The Jacobian of this transformation is given by (A12), from which it follows that

$$\langle J, \phi \rangle = \int ds \langle \Delta \tilde{J}^a H^a \phi \rangle . \quad (2.1)$$

As in (19), $\phi_a$ is an arbitrary test function. This equation therefore determines the current completely in terms of its skeleton (ignoring irrelevant sets of measure zero). Assuming that the integrals on the right-hand side are absolutely convergent, they may be freely commuted. Taking advantage of this, one finds that

$$\langle J^a - \int ds \Delta \tilde{J}^a H^a, \phi_a \rangle = 0 , \quad (2.2)$$

which allows the identification

$$J^a(x) = \int ds \Delta(x, z) \tilde{J}^a(X, z) H^a_a(x, z) . \quad (2.3)$$

Here, $X^a = -\sigma^a(x, z)$ and $z = z(s)$. In [2], it was found that the constraint equations (B3)–(B5) imply that $\tilde{J}^a$ — defined in the text immediately preceding (B16) — has the form

$$\tilde{J}^a(X, z) = A^a(X, z) \delta(n \cdot X) - h^a_{\nu} \delta(z(n \cdot X) . \quad (2.4)$$

$h^a_{\nu} := \delta^a_{\nu} + n^a n^\nu$ is the standard projection operator, and it is given that $n^\nu \nabla_{\nu} A^a = n^\nu \nabla_{\nu} B = 0$. The only other restrictions on the moments may be written as

$$n_{\alpha} A^a = \nabla_{\alpha} (h^a_{\nu} X^\nu B) , \quad (2.5)$$
$$\nabla_{\alpha} A^a = 0 . \quad (2.6)$$
Since $\mathcal{A}^a$ does not depend on $n \cdot X$, this second equation is really only a 3-divergence. Also note that by definition,

$$\dot{J}^a = Q^a \delta(X) + \dot{\bar{J}}^a. \quad (2.7)$$

Using these expressions, the $s$-integral in (2.3) can be performed explicitly to yield

$$J^{a'}(x) = \int ds \delta(X) Q^{a'} - N^{-1} \left[ \Delta \mathcal{A}^{a'} H^{a'a} \right. \\
+ \left. \nu^c \nabla_c \left( N^{-1} \Delta \mathcal{B} h_a^b X^b H^{a'a} \right) \right], \quad (2.8)$$

where the right-hand side is understood to be evaluated at the (unique) $z$ which sets $n(z) \cdot X(x, z) = 0$. Also,

$$N := -n_a \sigma^b \dot{v}^b + \dot{n}_a X^a. \quad (2.9)$$

Following the notation of (B1), $\dot{n}^a := \delta n^a / ds$. In flat spacetime, this expression reduces to the lapse of the foliation $\{ \Sigma \}$. Although it doesn’t retain this interpretation more generally, we shall still refer to it as a lapse.

Now for $x \notin Z$, note that the integral in (2.8) doesn’t contribute to the current. Temporarily restricting ourselves to such cases, (A5) and (A13) can be used to show that

$$J^{a'}(x) = \Delta N^{-1} H^{a'a} \left[ A^a + X^a \nabla_b (\mathcal{B}/N) \right. \\
+ \left. (\mathcal{B}/N) \left( n_a \dot{h}_b X^b - h_a^b \sigma^c \nu^c \right. \\
- \left. 2 \mathcal{B} h_b^c \nu^c X^a \right) \right]. \quad (2.10)$$

It is natural to split this up into components adapted to an observer moving along $Z$. Define the charge density to be

$$\rho(x) := n_a(z(\tau)) \sigma^a \dot{a'}(x, z(\tau)) J^{a'}(x). \quad (2.11)$$

Similarly, let the three-current be given by

$$j^a(x) := -\dot{h}_b^c \sigma^a \nu^c (x, z(\tau)) J^{b'}(x). \quad (2.12)$$

Substituting into (2.10), and using (2.3) and (2.9), the charge density reduces to

$$\rho = -\Delta \nabla_{a'} \left[ h^a_b X^b (\mathcal{B}/N) \right]. \quad (2.13)$$

With the exception of the overall factor of $\Delta$, this is exactly the same form that occurs in flat spacetime. Such a simple correspondence will not hold for the three-current, however.

Before computing it, note that $\mathcal{B}$ does not appear by itself in (2.13). One instead finds the ratio $\mathcal{B}/N$. This grouping will continue to be useful when expanding the three-current, although it is a little inconvenient as stated. We would like this ratio to have the same arguments as $\mathcal{B}$ itself. Specifically, it should be independent of $n \cdot X$. But $n_a \nabla_a N(X, z) \neq 0$ even when $n \cdot X = 0$. To fix this, define a “flattened” lapse $\tilde{N}$ as

$$\tilde{N}(X, z) := N(h(z) \cdot X, z), \quad (2.14)$$

and a “normalized” moment potential

$$\mathcal{C}(X, z) := \mathcal{B}(X, z) / \tilde{N}(X, z). \quad (2.15)$$

Clearly, $n^a \nabla_a \mathcal{C}(X, z) = 0$, as desired. The projected divergence in (2.13) allows $\mathcal{C}$ to be substituted for $\mathcal{B}/N$ there, so

$$\rho = -\Delta \nabla_{a'} (h^a_b X^b \mathcal{C}). \quad (2.16)$$

This may be further simplified by introducing an orthonormal tetrad $\{ n^a(z), e_i^a(z) \}$ defined on $Z$. Enforcing the orthonormality conditions requires that the spatial triad evolve according to

$$e_i^a \dot{\nu} = v^b \nabla_b e_i^a = n^a h_b e_i^b + \Omega_{ij} e_j^a, \quad (2.17)$$

where $\Omega_{ij}(z) = \Omega_{ij}(z)$ is arbitrary, and represents the angular velocity of the triad. It needn’t have any particular physical significance, however, so we shall leave it unspecified for now.

If the tetrad is assumed known, we can trivially introduce a set of (normalized) Riemann normal coordinates defined by

$$R^A(x, z) := e_i^A(z) X^a(x, z) \quad (2.18)$$

for any fixed $z$. Then an arbitrary bitensor $f(x, z)$ could be rewritten as either (abusing the notation a bit)

$$f(R, z) = \mathcal{C} = \mathcal{C}(R, z), \quad (2.19)$$

where $\mathcal{C}$ is arbitrary, and represents the angular velocity of the triad. It needn’t have any particular physical significance, however, so we shall leave it unspecified for now.

Recall that this expression is only correct away from $Z$. Correcting it requires evaluating the integral in (2.8). This is easily done by noting that $\delta(X) = \delta(R^0)^3 \delta^3(R)$. Then

$$\int ds \delta(X) Q^{a'} = \delta^3(R) Q^{a'}. \quad (2.20)$$

Using (B3), the charge density everywhere is finally given by

$$\rho = \Delta [ q \delta^3(R) - \partial_1 (R^1 \mathcal{C})]. \quad (2.21)$$

It is now useful to define a function $\psi(R, z)$ which satisfies

$$\partial_1 (R^1 \psi) = \delta^3(R). \quad (2.22)$$

This equation will be solved in the following section. But for now, $\psi$ will be left unspecified.

We instead define

$$\varphi(R, z) := q \psi(R, z) - \mathcal{C}(R, z). \quad (2.23)$$

Anywhere in a neighborhood of $W$, the charge density is now given by

$$\rho = \Delta \partial_1 (R^1 \varphi). \quad (2.24)$$
\( \mathcal{B} \) (and therefore \( \mathcal{C} \)) was originally defined to vanish outside of \( W \). But applying Gauss’ theorem to \( \mathcal{E} \) shows that \( \psi \) cannot share this property. \( \varphi \) must therefore be equal to \( q\psi \) everywhere outside of \( W \). This clearly implies that \( \rho = 0 \) in such regions, as required. Beyond this restriction, \( \varphi \) may be specified arbitrarily as long as the charge density which it determines remains nonsingular.

Implicit in these statements is the fact that any function depending on \((R, z)\) may be uniquely identified with another function depending only on \( x \). In a sense, we are simply choosing to work in a special system of coordinates. These happen to be closely related to Fermi normal coordinates, and exactly coincide with them when \( a^a = v^a \) and \( \Omega^{IJ} = 0 \).

In any case, we may now apply a similar analysis to the three-current. Using (2.8), (2.10), and (2.12), the tetrad components of \( j^a \) reduce to

\[
\begin{align*}
j^I &= \Delta N^{-1} \left[ A^I + qR^I \hat{\nabla}^3 (R) + R^I \hat{C} + \partial_J (R^J \mathcal{C}) \right] \\
&\quad \times \left( \Omega^{J} K R^K - e^b_a \delta^a_b \right) - C \left( \Omega^{IJ} R^J \right) \\
&\quad - 2e^b_a H \delta^a_b \mathcal{C} \right), \quad (2.24)
\end{align*}
\]

This uses the notation \( j^I := e^b_a j^a \), \( A^I := e^b_a A^a \), etc. But it can be rewritten more suggestively using (2.22):

\[
\begin{align*}
j^I &= \Delta N^{-1} \left[ A^I + R^I \hat{C} - (\rho/\Delta) (\Omega^{I} J R^J - e^b_a \delta^a_b) \right] \\
&\quad + 2\partial_J \left( R^I \left( \Omega^{J} K R^K - e^b_a \delta^a_b \right) \mathcal{C} \right). \quad (2.25)
\end{align*}
\]

This form is useful because the second line is manifestly (three-) divergence-free. From (2.22), \( A^I \) also has this property, but is otherwise arbitrary (ignoring its support properties for the moment). We may therefore absorb these terms by defining

\[
\mathcal{J}^I := A^I + qR^I \hat{\nabla}^3 (R) + 2\partial_J \left( R^I \left( \Omega^{J} K R^K - e^b_a \delta^a_b \right) \mathcal{C} \right). \quad (2.26)
\]

The three-current then reduces to

\[
\begin{align*}
j^I &= N^{-1} \left[ \Delta (\mathcal{J}^I - R^I \hat{\nabla}^3 \hat{\psi}) + \rho (e^b_a \delta^a_b) \right] \\
&\quad - \Omega^{IJ} R^J \right). \quad (2.27)
\end{align*}
\]

In summary, a conserved current vector may be constructed by choosing “moment potentials” \( \varphi(R, z) \) and \( \mathcal{J}^I(R, z) \). \( \varphi \) must reduce to \( q\psi \) outside of \( W \), but is otherwise arbitrary if we don’t worry about smoothness of \( J^a \). Similarly, \( \mathcal{J}^I \) must reduce to \( qR^I \hat{\nabla}^3 \hat{\psi} \) outside of the worldtube. It is also required to satisfy \( \partial_I \mathcal{J}^I = 0 \) everywhere. Given functions with these properties, the current is straightforward to compute from (2.23), (2.24), and (2.25), and

\[
J^a = H^a \left( n^a \rho + e^b_a j^b \right). \quad (2.28)
\]

This last equation is a direct consequence of (2.11) and (2.12).

### III. CHOOSING \( \varphi \) AND \( \mathcal{J}^I \)

We saw in the previous section that nearly any current may be constructed from potentials \( \varphi \) and \( \mathcal{J}^I \) chosen to satisfy very simple algebraic matching conditions. But many such choices generate singular current vectors. In this section, we show how to exclude this class; allowing one to easily construct physically reasonable divergence-free vector fields.

Since \( \varphi \) must reduce to \( q\psi \) outside of \( W \), our first step is to explicitly solve (2.21). Adopting spherical coordinates \((r, \theta, \phi)\) defined in terms of \( R \) in the standard way, the solutions of this equation are easily seen to be of the form

\[
\psi = \frac{1}{4\pi r^3} \left( 1 + q^{-1} A(\theta, \phi; z) \right), \quad (3.1)
\]

where \( A(\theta, \phi; z) \) is any function with a vanishing monopole moment (in the elementary sense). The factor \( q^{-1} \) is for later convenience.

The arbitrariness of \( A \) simply means that \( \psi \) is not uniquely determined by (2.21), which is to be expected. This ambiguity translates into a freedom in \( \varphi \) as well. The same conclusion may also be reached directly from (2.22), where a given \( \rho \) only determines \( \varphi \) up to terms of the form \( A/r^3 \). But there cannot be a \( 1/r^3 \) divergence in \( \varphi \) (without any angular dependence), as this would imply that the charge density contained an unphysical \( \delta \)-function at the origin. It follows that there always exists a \( \psi \) which completely removes any \( \sigma(r^{-3}) \) singularities in \( \varphi \). From now on, we always choose potentials in this class. This choice actually removes the only polynomial singularity that may exist in \( \varphi \). Others would lead to unphysical charge densities.

Most \( \delta \)-function singularities are excluded from \( \varphi \) on similar grounds. The only exceptions are terms of the form \( a(z)\delta^3(R) \). But these do not affect the current or the moments in any way, so they have no physical relevance. We can therefore set \( a(z) = 0 \) without any loss of generality.

Now choose a \( \varphi \) that is continuous in both \( R \) and \( z \). As long as the boundary of the worldtube can be written in the form \( r = R(\theta, \phi; z) \), where \( R \) is single-valued, specifying \( \varphi \) in some neighborhood of \( W \) determines \( \psi \) everywhere:

\[
q\psi = \left( \frac{R}{r} \right)^3 \varphi(R, \theta, \phi; z). \quad (3.2)
\]

The problem with this approach is that arbitrary choices of \( \varphi \) will often lead to charges that vary over time.

It is much simpler if we look for a \( \varphi \) that fits a given \( \psi \) rather than a \( \psi \) which fits a given \( \varphi \). This has the advantage of allowing one to specify the charge \( a \) priori. It is then trivial to ensure that it remains constant.

Before showing how to do this, we first require that the charge density at \( r = 0 \) does not depend on the angular variables (i.e. that it is single-valued). Define a central density \( \rho_0(z) \) such that

\[
\rho_0(z) := \rho(0, \theta, \phi; z). \quad (3.3)
\]
Now in order to find ϕ, choose q, A, ρ₀, and R(θ, φ; z). For r ≤ R, ϕ may then be taken to have the form
\[
ϕ = \frac{r}{R} \left( \frac{q + A}{4\pi R^3} \right) + (1 - r/R) \times \left( \frac{1}{3} \rho₀ + \frac{r}{R} \bar{ϕ} \right), \tag{3.4}
\]
\(\bar{ϕ}(r, θ, φ; z)\) is any bounded and continuous function in W. As an example, a charge density of \(\rho₀\Delta\) is generated by choosing \(\bar{ϕ} = 0\) and
\[
A = \frac{4\pi}{3} R^3 \rho₀ - q. \tag{3.5}
\]
One sees that A vanishes in this case when the boundary is a sphere.

With very little extra effort, we could also take into account the charge density on the body’s surface, or any number of other boundary conditions. This isn’t particularly necessary, however, and the extensions are straightforward.

The remainder of the current vector is now found by specifying an arbitrary divergence-free “three-vector” \(\mathcal{J}^I(R, z)\) which reduces to \(qR^Iψ\) outside of the body. If we choose a nonrotating triad, this vector field has a fairly straightforward physical interpretation in many cases of physical interest. Suppose that \(n^a\) and \(v^a\) are determined by center-of-mass conditions, and that the forces and torques on the body are not too large. Then \(n^a \approx v^a\), and the components of \(h^J\) would be small relative to the (reciprocal of the) worldtube’s proper diameter. If the spacetime curvature inside W is also small in an appropriate sense, (2.27) will reduce to \(j^J \approx \mathcal{J}^J - R^J \bar{ϕ}\).

So physically, \(\mathcal{J}^J\) may be interpreted as essentially that portion of the 3-current which arises due to the bulk motion “internal” charges. In this approximation, \(\mathcal{J}^J\) should therefore be nonsingular throughout the body. A slightly more detailed look at (2.27) and (3.4) shows that this observation must remain true even in the exact theory.

We are therefore left with the problem of writing down a nonsingular divergence-free three-vector which has a specified form outside of a given boundary. Finding vector fields with these properties may be nontrivial when \(A \neq 0\), so we now illustrate a method that may simplify the process. First write down \(\mathcal{J}^J\) so that it automatically satisfies the appropriate boundary condition:
\[
\mathcal{J}^J = Θ(R - r)\mathcal{J}^J_{(-)} + Θ(r - R)QR^I ψ. \tag{3.6}
\]
Here, \(Θ(\cdot)\) is the Heaviside step function: \(Θ(x) = 1\) if \(x > 0\), but otherwise vanishes.

This expression also introduces the new vector field \(\mathcal{J}^J_{(-)}(r, θ, φ; z)\) which must be both nonsingular and divergence-free. While we could require this to exactly match \(qR^I ψ\) at \(r = R\), doing so would eliminate the possibility of currents flowing on the surface of the body. Moreover, the divergence of (3.6) will vanish everywhere if only the normal component of \(\mathcal{J}^J\) is continuous across the boundary. We therefore suppose only that
\[
(s \cdot \mathcal{J}^J_{(-)})_{r=R} = \frac{\dot{A}}{4\pi R^2}, \tag{3.7}
\]
where
\[
s_I := \partial_I (r - R). \tag{3.8}
\]

The boundary condition (3.7) is difficult to apply directly if we (say) write \(\mathcal{J}^J_{(-)}\) in terms of a vector potential. To fix this, we instead construct it from two scalar potentials \(Q\) and \(P\) in such a way that its divergence will always vanish. To simplify the notation, it is first useful to define the operator
\[
\mathbb{L}_J := ε_{JKL} R^I \partial^K, \tag{3.9}
\]
or “\(\mathbb{L} := R × ∇\)” in a more traditional notation. This is just \((i/\hbar)\) multiplied by the standard angular momentum operator used in quantum mechanics.

We may now write
\[
\mathcal{J}^J_{(-)} = ε_{JKL} \partial_J L_K Q + \mathbb{L}_J P. \tag{3.10}
\]
This expansion is essentially an application of the angular momentum Helmholtz theorem. The scalars may also be interpreted as Debye potentials or the generators of the toroidal and poloidal components of \(\mathcal{J}^J_{(-)}\).

The utility of this expansion can be seen by considering scalar potentials which may be split into angular and (normalized) radial components. For example, let
\[
Q(r, θ, φ; z) = \sum_{ℓ=0}^{∞} Q_ℓ(r/R; z)\hat{Q}_ℓ(θ, φ; z), \tag{3.11}
\]
and break up \(P\) in the same way. We choose the first term to generate the right-hand side of (3.7). For \(ℓ ≥ 1\), the angular functions \(\hat{Q}_ℓ\) could then be taken as appropriate basis functions. The corresponding \(Q_ℓ\) will be chosen to ensure that the \(ℓ\)th term in the sum doesn’t add anything to the normal component of \(\mathcal{J}^J_{(-)}\) at the boundary.

Looking at the \(ℓ = 0\) terms, (3.7) becomes
\[
\frac{\dot{A}}{4π} = \mathbb{L} \cdot \left[ Q_0^R \mathbb{L} Q_0^R - \hat{Q}_0^R \dot{Q}_0^R \mathbb{L} R \right. \left. - \dot{P}_0 R^2 \mathbb{L} R \right], \tag{3.12}
\]
where \(\dot{Q}_0^R\) indicates a derivative of \(Q_0^R\) with respect to \(r/R\). Also note that these radial functions are understood to already be evaluated at \(r/R = 1\) (i.e. \(\mathbb{L}\) does not act on them here).

The solutions to (3.12) are clearly not unique (even taking into account the gauge freedom). Still, we only need to generate a solution to this equation. Supposing
that $\tilde{P}_0 = \hat{P}_0 = 0$, $\tilde{Q}_0(1;z) = 1$, and $\tilde{Q}_0'(1;z) = -1$, we are left with the relatively simple result

$$\tilde{A} = \frac{1}{4\pi} \left( R \tilde{Q}_0 \right). \quad (3.13)$$

This may be solved by expanding $R\tilde{Q}_0$ in spherical harmonics, using the addition theorem to express the series in terms of Legendre polynomials, and then summing the result. One eventually finds that (up to a possible additive function proportional to $1/R$)

$$\tilde{Q}_0(\theta, \phi; z) = -\frac{1}{16\pi^2 R} \int d\Omega' \tilde{A}(\theta', \phi'; z) \times \ln (1 - \cos \alpha). \quad (3.14)$$

$d\Omega'$ is that standard volume element on a sphere, and $\alpha$ is the angle between the point of integration and the point where $\tilde{Q}_0$ is being evaluated.

The radial profile $\tilde{Q}_0$ may now be chosen. This must satisfy the aforementioned conditions $\tilde{Q}_0(1;z) = 1$ and $\tilde{Q}_0'(1;z) = -1$, but we must also require

$$\lim_{r \to 0} \tilde{Q}_0(r/R; z)/r = 0. \quad (3.15)$$

This is necessary to ensure that the three-current is nonsingular and single-valued at the origin. A particularly simple choice is then

$$\tilde{Q}_0(r/R; z) = \left( \frac{r}{R} \right)^2 \left( 4 - 3 \frac{r}{R} \right). \quad (3.16)$$

This completes our solution to (3.12).

But as was already mentioned, the full solution to (3.7) is usually more complicated. If each term in (3.11) may be considered individually, then for all $\ell \geq 1$,

$$\tilde{Q}_\ell RL_\ell \tilde{Q}_\ell = \tilde{Q}_\ell' \tilde{Q}_\ell L_\ell R + \hat{P}_\ell \hat{P}_\ell R^2 \partial_\ell R \quad (3.17)$$

at $r = R$.

If $\partial_\ell R = 0$, this implies that either $\tilde{Q}_\ell(1;z)$ vanishes, or that $\tilde{Q}_\ell$ is independent of $\theta$ and $\phi$. But throughout the body, only the angular derivatives of $\tilde{Q}_\ell$ have any physical relevance, so this case is equivalent to $\tilde{Q}_\ell = 0$. But this is itself indistinguishable from saying that $\tilde{Q}_\ell$ vanishes everywhere. So we may always say that $\tilde{Q}_\ell$ vanishes at $r = R$ when the boundary is locally spherical.

In this case, $J_\ell^{(1)}$ reduces to a sum of terms of the form

$$\tilde{Q}_\ell \left( \frac{R'}{r^2} L^2 \tilde{Q}_\ell - \partial_\ell' \tilde{Q}_\ell \right) + \frac{r}{R} \tilde{Q}_\ell' \partial_\ell' \tilde{Q}_\ell + \hat{P}_\ell \hat{P}_\ell L' \hat{P}_\ell. \quad (3.18)$$

On the boundary, the first group of terms clearly vanishes for all $\ell \geq 1$.

But we must also make sure that (3.18) is everywhere finite. We first require none of the functions $\hat{P}_\ell$ contain any singularities. Choosing $\tilde{Q}$ is slightly more complicated due to the $o(r^{-1})$ divergence in (3.18). But for all $\ell \neq 0$, we may suppose that there exists a nonsingular function $Q_{\ell,\delta}$ such that

$$\tilde{Q}_\ell(r/R; z) = \frac{r}{R} \left( 1 - \frac{r}{R} \right) Q_{\ell,\delta}(r/R; z). \quad (3.19)$$

This guarantees that (3.18) remains finite. As long as the sum over $\ell$ does not diverge, these choices are sufficient to describe a wide variety of functions $J_\ell$ (for angles) where $\partial_\ell R = 0$.

If the boundary radius is changing, the situation is more complicated, and our discussion will not be quite so complete. The full equation (3.17) must be solved in general at $r = R$, although it isn’t obvious how to do this. To make some progress, suppose that $\hat{P}_\ell$ vanishes on the boundary. If $Q_{\ell}(1;z)$ also vanishes, $\tilde{Q}_\ell$ may be anything at all (so long as it remains finite). But more generally, it must have the form

$$\tilde{Q}_\ell \propto R [Q_{\ell}(1;z)/Q_{\ell}(1;1)]. \quad (3.20)$$

The general case where $P_\ell \neq 0$ at $r = R$ does not seem simple to discuss. It is likely that an ansatz better adapted to nonspherical boundaries than (3.10) would be useful for this purpose. The class of currents which we are excluding by neglecting this case does not really seem significant, however.

To summarize, $J_\ell$ is required to compute the three-current. It has the form (3.6), where $J_\ell^{(1)}$ may be expanded using (3.11). The scalar potentials in this expression have the form (3.11). By definition, $P_0 = 0$, and $\tilde{Q}_0$ and $Q_0$ are given by (3.14) and (3.16) respectively. The higher-$\ell$ components of the potentials may be chosen by first writing down any (smooth) sets of functions $\hat{P}_\ell$, $\tilde{P}_\ell$, and $Q_\ell$. Then each $\tilde{Q}_\ell$ should have the form (3.19). Note that while these choices do not allow for every possibility, they are still quite general.

In all of this, the $z$-dependence of $\varphi$ and $J_\ell$ is essentially arbitrary. Since $z(s)$ is a one-to-one function, these potentials may easily be rewritten as functions of $(R, s)$ rather than $(R, z)$. When doing so, it seems sufficient to require that they be $C^4$ in $s$. All of these conditions taken together allow us to easily generate a very wide class of physically reasonable current vectors with a given $q$. Note that the continuity restrictions are sufficient but not necessary. Still, the stated classes seem large enough for almost any practical purposes.

Now that acceptable classes of $\varphi$ and $J_\ell$ have been identified, expressions in Sec. 11 and appendix B may be used to find the which sets of moments correspond to plausible current vectors. Since translating between these two representations involves a Fourier transform, it is likely to be difficult to do this explicitly. We shall not attempt it, but merely state that most of the restrictions on the moments would involve their asymptotic behavior in the “large-$n$” limit.
IV. DYNAMICAL RIGIDITY

In Newtonian mechanics, one often makes use of the concept of rigid bodies. Although such objects do not (and cannot) exist, the idea is still useful. In many cases, the bulk dynamics of a large class of real materials are known to be adequately approximated by the behavior of a suitable rigid body. Just as importantly, rigidity conditions allow equations of motion to be solved uniquely in relatively simple ways. Although it is not at all clear that a similarly useful concept should exist in highly relativistic systems, it is still interesting to search for one that is roughly analogous.

Various definitions of rigidity in curved spacetimes have been introduced over the years. But perhaps the simplest is a generalization of the concept advanced by Born for objects in Minkowski space \([14, 15, 16]\). This essentially requires that the strain tensor throughout a material be Lie-dragged along the worldlines of its constituent particles. Unfortunately, such a restriction fails to be self-consistent in many systems of practical interest \([17]\).

Rather than defining rigidity through local kinematics, it has been conjectured that fixing certain “global” (or quasilocal) quantities may be more fruitful. In particular, one may wish to call a body rigid if its multipole moments evaluated in some particularly natural frame do not change.

Supposing again that we are discussing the mechanics of some extended body, the reference line \(Z\) and the dynamical velocity \(n^a\) of this frame may be chosen using the center-of-mass conditions given in \([8, 9]\) (and rigorously justified in \([18]\)). In doing so, the temporal component of the tetrad would be proportional to the body’s bulk (linear) momentum vector. The triad vectors would not be chosen uniquely in this scheme, although their evolution may be fixed by defining an “average angular velocity” as the vector which must multiply the “inertia tensor” to yield the bulk angular momentum vector \([8, 9]\). This effectively defines \(\Omega^I_J\) in terms of the body’s stress-energy tensor.

These ideas were introduced in \([8]\) under the name of quasirigidity, and supposed that the multipole moments under discussion were those of the object’s stress-energy tensor. Very similar ideas have also been referred to as dynamical rigidity in \([7, 8]\), and we shall use the two terms interchangeably. These latter papers also introduced a scalar defined from the body’s stress-energy tensor (and the metric) which may be interpreted as its total internal energy. It was found that this is in fact a constant of motion for bodies which are dynamically rigid. The concept therefore has an elegant physical interpretation.

Changing definitions slightly, the results of the previous sections may be used to explicitly construct quasirigid bodies. We do so by requiring that the multipole moments of the number density and electric current vectors associated with a given object be fixed in the aforementioned frame (rather than the moments of its stress-energy tensor). This is actually far less restrictive than the definition in \([8, 9]\), as it ignores the stress and energy density distributions inside the body. In doing so, it also appears closer in spirit to the concept of Newtonian rigidity. A downside is that the total internal energy of the body will no longer be conserved in general. It is also somewhat less “dynamical” than the original definition.

There is one potential ambiguity in this: it might be useful to allow the monopole moment to vary even if the higher moments do not. If \(Q^A := e^A_a n_a\) were held constant, \((B6)\) implies that \(q \partial (v^I) / \partial s = 0\), or

\[
q (n^I - \dot{v}^I - \Omega^I_J v^J) = 0 . \tag{4.1}
\]

This is of course satisfied trivially if either \(q = 0\) or \(n^a = v^a\) and \(\Omega^I_J = 0\). But in general, it provides a rather strict restriction on the class of allowed frames.

In a mechanics problem, one would usually derive \(v^a\) from \(n^a\) using the mass, force, torque, and linear and angular momenta of the body \([8, 9]\). In these cases, \((4.1)\) would restrict which combinations of these objects could be allowed, which seems unphysical. In any case, one of the main reasons for adopting rigidity conditions in the first place is to obtain unique evolution equations for all relevant quantities. But changes in \(Q^a\) would already be fixed in most cases of interest. There is no need to force a potentially contradictory condition on top of the already existing one. We therefore define a current to be quasirigid if all of its moments except for (possibly) the monopole are constant.

It is clear from \((B2)\) that this definition implies that \(J^a(X, z) e^A_a(z) = J^A\) is a function purely of \(e^A_a X^a = R^A\). Noting \((2.1)\), this means that \(\mathcal{A}^I\) and \(\mathcal{B}\) must be independent of \(z\) (or equivalently, of \(s\)).

We first consider the effect of holding \(\mathcal{B}\) constant. Supposing that \(\rho\) is at least instantaneously smooth, \((2.15)\) and \((2.20)\) can be used to show that

\[
\dot{\rho} \sim \Delta \left( B / \dot{N} \right) \partial_s \left( R^I \partial_I \ln \dot{N} \right) , \tag{4.2}
\]

where terms that are obviously nonsingular have been suppressed.

In general, \(\mathcal{B}\) diverges at the origin. The charge density may therefore develop a singularity of its own if we are not careful. One necessary (but not sufficient) condition that must be satisfied to avoid this is

\[
(q + A) \lim_{r \to 0} \frac{\partial}{\partial s} \left( R^I / r \right) \partial_I \ln \dot{N} = 0 . \tag{4.3}
\]

Writing this in a more useful form requires an expansion of \(\dot{N}\) in powers of \(R\) or \(X\) around \(r = 0\). But that in turn requires a similar expansion for \(\sigma^a_b\). Such an expression may be found in \([8]\):

\[
\sigma^a_b(X, z) = \delta_b^a - \frac{1}{3} R^a_{\ c b d}(z) X^c X^d + \ldots \ , \tag{4.4}
\]
Here, \( R^a{}_{bcd}(z) \) is the Riemann tensor at the origin. Using this in (2.9), we find that the flattened lapse has an expansion of the form

\[
\tilde{N} \simeq 1 + \hat{n}_I R^I + \frac{1}{3} (R_{0101} + R_{01KJ} u^K) R^I R^J .
\]  

(4.5)

This allows (4.3) to be reduced to

\[
(q + A) \lim_{r \to 0} (R^I / r) (\hat{n}_I + \Omega_I^J \hat{n}_J) = 0 .
\]

(4.6)

There are two ways to satisfy this equation: either \( q = A = 0 \) (i.e. \( q \psi = 0 \)), or

\[
\frac{\partial (\hat{n}_I)}{\partial s} = \hat{n}_I + \Omega_I^J \hat{n}_J = 0 .
\]

(4.7)

(4.3) isn’t the only condition that must be satisfied in order for \( \tilde{B} \) to remain constant. One must also have

\[
(q + A) \lim_{r \to 0} \frac{\partial}{\partial s} \left[ (R^I / r^2) \partial_I \ln \tilde{N} \right] = 0 .
\]

(4.8)

If \( q \psi = 0 \), this is trivially satisfied. Otherwise (4.7) must hold, in which case (4.8) reduces to

\[
\frac{\partial}{\partial s} (R_{0101} + R_{01KJ} u^K) = 0 .
\]

(4.9)

Roughly speaking, (4.7) and (4.9) imply that the inertial and tidal forces felt by a center-of-mass observer must remain constant.

This leads to a very important conclusion: physically acceptable charged currents cannot be dynamically rigid with respect to arbitrary reference frames. Unless the system is particularly simple, no charged rigid current can exist. Although the structure of \( J^a \) will (usually) affect \( \hat{n}_a \) and \( R^a{}_{bcd}(z) \), it seems very unlikely that quasi-rigidity alone could conspire to ensure that they satisfy (4.7) and (4.9) except in a few special cases.

One consequence of this is that the only quasirigid particle fluxes that could exist with generic applied forces must contain zero total particles. Nontrivial particle fluxes therefore cannot be rigid except for special types of motion.

Interesting rigid currents of electric charge always exist, however. This is because although \( q \) still must vanish in general, the charge density may change sign and \( J^a \) needn’t be future-directed timelike. Characterizing such vector fields is relatively simple. We start by making sure that \( \tilde{B} \) is independent of \( s \). Again, setting \( q \psi = 0 \) is sufficient. Then

\[
\varphi = -\tilde{B} / \tilde{N} .
\]

(4.10)

It is obvious from (3.4) that there always exists a large class of \( \tilde{B} \)’s for which this is possible. Note, however, that they all vanish at the edge of the body. This implies that \( \tilde{R} \) cannot change (except perhaps if \( \tilde{B} = \varphi = 0 \)). It is easy to verify that \( \rho_0 \) also remains constant.

Besides requiring that \( \tilde{B} = 0 \), quasirigidity also demands that \( \partial (\tilde{A}^I) / \partial s \). Suppose again that \( q \psi \) vanishes, so \( \varphi \) satisfies (4.10). According to (2.26), (2.27), and (3.1), the evolution equation for the three-current never become singular. There therefore don’t appear to be any particularly interesting restrictions arising from the constancy of \( \tilde{A}^I \).

Note however that if \( \tilde{R} = 0 \) was not already ensured by \( \tilde{B} = 0 \), then it is implied by the time-independence of \( \tilde{A}^I \). Although rigidity requires that the boundary of the object and its central charge density stay constant in the pseudo-Fermi coordinate system we have been using, \( \rho \) and \( \gamma^I \) will still depend on \( s \) in most cases. Their evolution equations are rather lengthy, but easily derived from this discussion (supplemented with some straightforward bitensor identities).

To reiterate, we have shown that charged quasirigid currents often do not exist. As with Born’s original concept of relativistic rigidity, the idea of unchanging multipole moments has significant limitations. Of course, the definitions we’ve adopted are rather strict. Although (4.7) and (4.9) can rarely be satisfied exactly, their left-hand sides will often be very small in realistic physical systems. When this is true, currents may generically exist that are in some sense “nearly rigid.” But even when viewed from afar, the mechanical properties of such objects are likely to be quite different from those of “equivalent” rigid bodies (which must contain singularities). In particular, the computation of gravitational or electromagnetic self-forces in a mechanical system would likely be problematic.

Our definition of dynamical rigidity could also be argued to be overly strict in that it requires all of the current’s multipole moments to be fixed. This has the advantage of producing unique evolution equations, but it isn’t very well-motivated physically. It would usually be more realistic to suppose that only the first few moments are constant (or approximately constant). This is closer in spirit to the type of quasirigidity discussed in (for example) [10], which was shown to have many desirable properties. There does not appear to be any existence problem for objects with only “low order” quasirigidity. But since the evolution of the higher moments is not constrained by such a scheme, the motion of the current cannot be unique.

It should also be stressed that the definition of dynamical rigidity adopted here is different from the one given in [8]. There, the constants of motion were multipole moments of the stress-energy tensor rather than moments of a relevant current vector. This condition appears more restrictive, so we conjecture that a similar non-existence result can also be proven for that case. It would then follow that the total internal energy defined in [8] could never be constant except in particularly simple systems.

V. COMPARABLE CURRENTS

It is often useful to discuss the properties of objects in a highly dynamical spacetime with language borrowed
from something simpler (such as Kerr or Minkowski). For example, one might want to say that a neutron star in some complicated numerical simulation was essentially the same star that had already been studied in isolation. Although defining “same” in this sentence may be quite tricky, there are several ways to proceed depending on the focus of the problem and the amount of effort one is willing to expend on it. This section will introduce one method which is particularly simple from a mathematical perspective, and then study some of its physical consequences.

We define two currents (possibly in different spacetimes) to be “identical” if their multipole moments evaluated in some pair of frames are identical. Distinguishing quantities related to the second current with underlines, we also require these frames to satisfy \( \dot{q}^I(s) = \dot{q}^I(s) \), \( \dot{v}^I = \dot{v}^I \), and \( \dot{\Omega}^I = \dot{\Omega}^I \) for all \( s \). The two currents should also have identical charges: \( q = q \). These conditions automatically imply that the frame components of the monopole moments of both currents are identical.

Equality of the higher moments implies that \( B(R; s) = B(R; s) \) and \( A^I(R; s) = A^I(R; s) \). Finding the consequences of these relations will use arguments very similar to those given in the previous section.

As a consequence of both \( B \)'s being equal, (2.15) and (2.20) imply that

\[
BR^I \partial_I \ln \tilde{N}/\tilde{N} = (q - \rho) \tilde{N}/\Delta - (q - \rho) \tilde{N}/\Delta .
\]

In physically reasonable systems, the right-hand side of this equation is clearly nonsingular. But \( B \) generally involves an \( o(r^{-3}) \) singularity. This divergence is softened by the fact that \( R^I \partial_I \ln \tilde{N}/\tilde{N} \) decreases at least as fast as \( r^{-2} \) near the origin. But this is not enough to allow \( B \) to be chosen arbitrarily (using only the constraints discussed thus far). In order to remain consistent, it must be true that

\[
(q + A) \lim_{r \to 0} \left( R^I/r^2 \right) \partial_I \ln \tilde{N}/\tilde{N} = 0 .
\]

Using (5.5), this is equivalent to either \( q \psi = 0 \) or

\[
R_{0 I J} + R_{0(I[K]J)} v^K = R_{0 I J} + R_{0(I[K]J)} v^K .
\]

As before, these Riemann tensors are evaluated at the respective origins of each frame. This equation essentially states that physically reasonable charged currents can only be comparable if observers attached to the center of each frame would experience identical tidal forces. If two charged currents were “comparable” (as defined above), but did not satisfy (5.3), at least one must be singular.

We may now construct comparable currents satisfying \( q \psi = 0 \). In this case,

\[
\varphi = \left( \tilde{N}/\tilde{N} \right) \varphi .
\]

It follows that \( \rho_0 = \rho_0 \) and \( R = R \) (at least if \( B \neq 0 \)).

Continuing, \( A^I \) must be equal to \( A^I \) if two current are to be comparable. Suppose again that \( q \psi = 0 \). If \( B \) also vanishes, the two current potentials would become equal: \( J^I = J^I \). This would imply that the frame components of the two three-currents would be directly proportional to one another:

\[
j^I = \left( \frac{\Delta N^{-1}}{\Delta N^{-1}} \right) J^I .
\]

In any case, the situation here is similar to the one obtained in the previous section. The idea of having two objects where all of the relevant moments are identical can only be generic if they both carry zero charge.

VI. DISCUSSION

We have shown how to construct a vector field \( J^a' \) satisfying (4.2) from a set of multipole moments defined with respect to a given reference frame. In so doing, we have implicitly found restrictions on the moments which ensure that the resulting current will be smooth and bounded. This effectively completes the program Dixon initiated in \( \mathbb{R}^3 \) to study the multipole moments of nonsingular conserved vector fields in curved spacetime.

An immediate application of these constructions is a method to easily construct currents with a given total charge \( q \). This is described in detail in Sec. III where a few free functions with relatively intuitive interpretations are used to build any smooth \( J^a' \).

In combination with the equations of motion derived in \( \mathbb{R}^3 \), one could (for example) derive the equations of motion for all charged particles with a given charge in some approximation scheme. This was carried out in \( \mathbb{R}^3 \) in flat spacetime in part to determine precisely when the Lorentz-Dirac equation is valid. The results derived in this paper would then allow similar methods to be used to study the validity of its generalization, the Dewitt-Brehme equation [6].

The methods used here may also be seen as a prototype for a similar analysis of the multipole moments of the stress-energy tensor (as defined in \( \mathbb{R}^3 \)). This would allow one to directly construct all bodies having prescribed linear and angular momenta at a given center-of-mass position.

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APPENDIX A: BITENSORS

This appendix briefly reviews the theory of bitensors. The detailed properties of such objects have been discussed in detail by several authors [6, 8, 20, 21], so we...
shall include only a few relevant definitions and identities.

Our discussion will focus on the world function biscal-

cular \( \sigma \). Given two points \( x \) and \( y \) in a spacetime, it is

first assumed that there exists exactly one geodesic con-
necting them. In this paper, these points will always be
spacelike-separated, so this is a rather mild restriction.

Regardless, \( \sigma(x, y) \) is defined to be one half of the square
of the geodesic distance between its arguments.

The derivatives of \( \sigma \) arise in many expressions, so it is
standard to define a special notation for them. In

particular, let

\[
\sigma_a(x, z) := \nabla_a \sigma(x, z), \quad (A1) \\
\sigma_a'(x, z) := \nabla_a' \sigma(x, z). \quad (A2)
\]

Here, we have assumed that \( z \in Z \), and used the
previously-mentioned convention that indices referring
to points on \( Z \) are not primed. This notation trivially gen-
eralizes to expressions involving any number of deriva-
tives. For example, \( \sigma_{a'b'c'} = \nabla_{a'b'c'} \sigma \). Derivatives at \( x \)
always commute with derivatives at \( z \), although the or-
der of derivatives at a particular point cannot generally
be interchanged (as is usual for covariant derivatives).

It may be shown that these vectors are tangent to the
geodesic connecting \( x \) and \( z \), and that their magnitudes
are just the squared geodesic distance \([6, 8, 20, 21]\):

\[
\sigma^a \sigma_a = \sigma^{a'} \sigma_{a'} = 2\sigma. \quad (A3)
\]

This allows one to naturally interpret \( \sigma_a \) and \( \sigma_{a'} \) as
generalized displacement vectors between \( x \) and \( z \).

Differentiating \((A3)\) an appropriate number of times
also allows one to generate a number of useful identities.
The simplest of these are \( \sigma_a \sigma^a_b = \sigma_b \), \( \sigma_a \sigma^{a'}_{a'} = \sigma_a \), etc. It
follows that \( \sigma_{a'b'}^{a} = \delta^{a'}_b \) in flat spacetime, or in general
when \( x = z \). Similarly, \( \sigma_{a'}^{a'} = -\delta^a_{a'} \) in these cases.

The matrix inverse of \( -\sigma_{a'}^{a'} \) appears a number of times
in Dixon’s formalism, so it is convenient to define

\[
H_{a'}^{a}(x, z) := [-\sigma_{a'}^{a}(x, z)]^{-1}. \quad (A4)
\]

This object also appears naturally in solutions of the
geodesic deviation (Jacobi) and Killing transport equa-
tions \([7, 8]\). Its derivatives have the form

\[
\nabla_{b'} H_{a'}^{a}(x, z) = H_{a'}^{a_b} H_{c'}^{c_a} \sigma'_{b'c'}. \quad (A5)
\]

Another bitensor with similar properties is the parallel
propagator \( g_{a'}^{a}(x, z) \). This does exactly what its name
implies: parallel propagates vectors from \( z \) to \( x \) along
the geodesic connecting them. The properties of this object
may be found in \([6, 20]\). For our purposes, note that
\( g_{a'}^{a}(x, z) = g_{a'}^{a}(z, x) \). From its physical inter-
pretation, it also follows that

\[
g_{a'}^{a} \sigma_{b'}^{b'c'} = \delta_{b'}^{b'}, \quad (A6) \\
g_{a'}^{a} \sigma_{b'}^{c'} = \delta_{b'}^{b'}. \quad (A7)
\]

An important biscalar built from \( g_{a'}^{a} \) is known as the
van Vleck determinant \( \Delta(x, z) \). It is defined to be

\[
\Delta(x, z) := \det \left( -g_{a}^{a'} \sigma_{b'}^{b'} \right). \quad (A8)
\]

In \([6]\), it is shown that

\[
\det \left( g_{a}^{a'} \right) = \sqrt{\frac{g(z)}{g(x)}}, \quad (A9)
\]

so

\[
\Delta(x, z) = \sqrt{\frac{g(z)}{g(x)}} \det \left( -\sigma_{a}^{a'} \right). \quad (A10)
\]

The van Vleck determinant plays a fundamental role in
the focusing of geodesic congruences \([6, 22]\). But here, it
will appear as an important factor in the equations giv-
ing the current vector in terms of quantities constructed
from the moments. It arises essentially from a coordinate
transformation performed in the derivation.

In particular, a number of relevant functions that we
deal with are defined on the tangent bundle \( TM \)
restricted to \( Z \). Occasionally, a base point \( z \) will be fixed,
in which case we can switch freely between considering
these objects to be functions on \( TM \) (really just the tan-
gent space \( T_z M \)) or on the manifold itself (at least in a
neighborhood of \( z \)). This identification will be made via
the exponential map, so abusing the notation slightly, we
make identifications of the form \( f(X, z) = f(\exp_z X) \) for
any function \( f \) and vector \( X \in T_z M \).

It is then useful to transform integrals over spacetime
into integrals over \( T_z M \). The inverse of the exponential
map is just \( \sigma \). So define

\[
X^a(x, z) := -\sigma^a(x, z). \quad (A11)
\]

This acts as a “radial vector” between \( x \) and \( z \), and from
\((A10)\), the Jacobian of the transformation \( x \rightarrow X \) is

\[
\left| \frac{\partial X}{\partial x} \right| = \sqrt{\frac{g(z)}{g(x)}} \Delta. \quad (A12)
\]

Lastly, we note that directly differentiating \((A8)\) yields

\[
\nabla_a \ln \Delta = -H^b_{b'} \sigma'^{a} \sigma'^{b'}. \quad (A13)
\]

Other properties of the van Vleck determinant may be
found in \([6, 22]\).

**APPENDIX B: DIXON’S FORMALISM**

In the present context, Dixon’s formalism \([3, 4, 7, 8, 9, 10]\)
defines a set of multipole moments for spatially-
bounded vectors satisfying \([1, 22]\). This set is complete
in the sense that specifying all of the moments allows
one to completely reconstruct \( J \). If this vector field is
interpreted as the electromagnetic current of a charged test body, it is possible to show that these moments also have the standard interpretation in terms of expansions for the electromagnetic force and torque in successively higher derivatives of the applied field \[ ].

In most other formalisms of this type (e.g. [23]), a conservation law like \[1.2\] imposes differential constraints on each moment. There are often algebraic couplings between the various moments as well. This leads one to a system of equations which are impossible to solve without assuming \textit{a priori} that only some finite subset of the moments is important to problem at hand. While adequate for many purposes, this removes any possibility of reconstructing the current vector from its moments. For example, if only a finite number of the moments were nonzero, the associated vector field would involve the Dirac distribution and its derivatives. Such current densities are clearly unphysical.

Dixon’s moments avoid this type of problem by being chosen such that there is exactly one differential equation imposed on them as a consequence of \[1.2\]. This equation is trivial, and only states that the total charge does not change. The algebraic constraints on the moments are also simple solve in closed form, so it becomes possible to discuss the entire set of multipole moments.

To see how these simplifications work, first note that the moments of some source would usually be defined by integrating it against a suitable number of “radial vectors” referred to some arbitrarily chosen origin. The moments should also be allowed to vary in time, which is most naturally allowed by requiring that their defining integrals be carried out only over spacelike hypersurfaces spanning \(W\). Before defining anything, we therefore need a reference frame which foliates the worldtube with hypersurfaces \(\Sigma(s)\). Each of these must also be given an origin \(z(s) \in \Sigma(s)\). Perhaps imagining that these preferred points could represent the position of a physical observer, we assume that they form a continuous timelike worldline \(Z \subset W\).

To be more specific, start by fixing the the reference frame for the moments to be the center-of-mass frame of a body with worldtube \(W\). In that case, \(n^a\) would point in the direction of the body’s bulk momentum vector, while \(v^a\) would be the 4-velocity of its center-of-mass line. These two objects are physically distinct, and would only coincide in special cases.

Regardless, specifying \(z(s)\) and \(n^a(s)\) is sufficient to fix a frame in which the multipole moments may be defined. All we need now is some notion of a “radial vector” between \(x \in \Sigma(s)\) and \(z(s)\). A particularly natural choice for this is the \(X^a(x, z)\) defined in \[A11\], which takes the form of a vector at \(z\). Accepting this, a multipole moment should then be defined by integrating the current against a suitable number of these vectors.

But that cannot be done directly, as \(J^a(x) \notin T_z M\) unless \(x = z\). It will be convenient to parameterize all possible replacements for the current by a vector-valued distribution \(\hat{J}^a(X, z)\). We call this the current skeleton (although our definition of it will differ slightly from Dixon’s). Its precise relation to \(\hat{J}^a\) will be given later, but for now, we simply assume that it vanishes outside of some finite neighborhood of \(X^a = 0\). The 2\(n\)-pole moment of the current is then defined to be

\[
Q^{b_1 \cdots b_n}^a(z) := \int DX X^{b_1} \cdots X^{b_n} \hat{J}^a(X, z) ,
\]

where \(DX := \sqrt{-g(z)} d^4X\). Note the moments are tensors on \(Z\). The skeleton is defined on the tangent bundle \(TM\), and the moments at \(z\) are integrals over \(T_z M\).

It might appear that the moments defined in this way depend on the current at all times, but this is not true. The support of the skeleton will be found to lie on the surface defined by \(n^a X_a = 0\), which is simply \(\Sigma(s)\). So one of the integrals in \[12\] effectively drops out, and we fall back on the expected three-dimensional definition of a multipole moment at some chosen time.

Note that \[12\] trivially implies that for all \(n \geq 1\),

\[
Q^{(b_1 \cdots b_n)}^a = Q^{[b_1 \cdots b_n]}^a .
\]

This is the first of three index symmetries that must be satisfied by the moments. The others are not so trivial, and form the main content of Dixon’s formalism. We simply state that for all \(n \geq 2\),

\[
n_{b_1} Q^{b_1 \cdots b_n-1(a}^a b_{n]} = 0 ,
\]

\[
Q^{(b_1 \cdots b_n)} = 0 .
\]

The monopole moment also has the special form

\[
Q^a(z) := q v^a(z) ,
\]

where \(q\) is the total charge defined by \[11\].

We now need to relate \(\hat{J}^a\) to \(J^a\). In order to do so, it will be convenient to think of the current skeleton as
a linear functional on the space of all $C^\infty$ test functions with compact support (as is typical in distribution theory). Given some $\Phi^a(X, z)$ related to a $\phi^{a'}(x)$ in this class, we want to know the behavior of

$$\int DX \hat{J}^a(X, z) \phi_a(X, z)$$  \hfill (B7) 

for all possible forms of $\phi^{a'}$. This functional should be related in some simple way to

$$\langle J, \phi \rangle := \int d^4x \sqrt{-g(x)} J^{a'} \phi_{a'}$$  \hfill (B8) 

which determines $J^{a'}$. Note that the (B7) depends on $z$, while $\langle J, \phi \rangle$ does not. It is therefore simplest to link the two functionals together by simply integrating out the $z$-dependence:

$$\langle J, \phi \rangle = \int ds \int DX \hat{J}^a(X, z) \phi_a(X, z)$$  \hfill (B9) 

Explicitly relating $\phi^{a'}(x)$ and $\Phi^a(X, z)$ would now complete the definition of the skeleton in terms of $J^{a'}$. It is natural to suppose that $x = \exp_z X$, but there must also be a propagator which maps vectors at $z$ into vectors at $x$. The convenient choice for this happens to be the identity if the $X^a$ are used as coordinates (with a fixed $z$). In general, through

$$\Phi_a(X, z) := H^{a'}_{a}(\exp_z X, z) \phi^{a'}(\exp_z X)$$  \hfill (B10) 

Taken together, the preceding equations completely define the moments (or equivalently the skeleton) of the current vector. To see that they automatically imply (B7), choose a scalar test function $\varphi$, and then set $\phi^{a'} = \nabla_a' \varphi$. With this definition,

$$\Phi_a(X, z) = \frac{\partial}{\partial X^a} \varphi(\exp_z X)$$  \hfill (B11) 

For any object depending on both $X^a$ and $z$, operator $\partial/\partial X^a$ actually has a very natural interpretation in the theory of vector bundles. It is called a vertical covariant derivative in $[4, 8]$, and was given the special symbol $\nabla_a$. Using this notation, (B11) may be rewritten as

$$\Phi_a = \nabla_a \varphi$$  \hfill (B12) 

Combining this with (B9),

$$\langle \nabla \cdot J, \varphi \rangle = \int ds \int DX \left( \nabla_a \hat{J}^a \right) \varphi$$  \hfill (B13) 

The divergence on the right-hand of this equation may be evaluated by defining the Fourier transform of the current skeleton. Let

$$\cal{F} \hat{J}^a(k, z) := \int DX \hat{J}^a(X, z) e^{-ik_a X^a}$$  \hfill (B14) 

where $k^a \in T_z M$. It then follows immediately from (B2) that

$$\cal{F} \hat{J}^a(k, z) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} k_{b_1} \cdots k_{b_n} Q_{b_1 \cdots b_n}^a(\varphi)$$  \hfill (B15) 

So the Fourier transform of the current skeleton is a generating function for the moments.

It will be convenient to define a reduced current skeleton $\hat{J}^a(X, z)$ that drops the first (monopole) term in this series: $\cal{F} \hat{J}^a = \cal{F} \hat{J}^a - Q^a$. Noting that $\cal{F} \nabla_a A^a = ik_a F[A]^a$ for any $A^a(X, z)$, (B15) immediately implies that

$$\nabla_a \hat{J}^a = 0$$  \hfill (B16) 

The divergence of the full current skeleton therefore depends only on the monopole moment. Using the identity $\cal{F} [\delta(X)] = 1$, we see that

$$\nabla_a \hat{J}^a = Q^a \nabla_a \delta(X)$$  \hfill (B17) 

Substituting this result into (B13), and using (B16) finally shows that $\langle \nabla_{a'} J^{a'}, \varphi \rangle = 0$ for any $\varphi$. Hence, (B2) is always satisfied, as claimed.


